GLOBAL OFFENSIVE $k$-ALLIANCE IN BIPARTITE GRAPHS

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Abstract. Let $k \geq 0$ be an integer. A set $S$ of vertices of a graph $G = (V(G), E(G))$ is called a global offensive $k$-alliance if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$, where $0 \leq k \leq \Delta$ and $\Delta$ is the maximum degree of $G$. The global offensive $k$-alliance number $\gamma^k_o(G)$ is the minimum cardinality of a global offensive $k$-alliance in $G$. We show that for every bipartite graph $G$ and every integer $k \geq 2$, $\gamma^k_o(G) \leq n(G) + |L_k(G)|$, where $L_k(G)$ is the set of vertices of degree at most $k - 1$. Moreover, extremal trees attaining this upper bound are characterized.

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1. INTRODUCTION

We begin with some terminology. For a vertex $v$ of a graph $G = (V, E) = (V(G), E(G))$, the open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. The degree of $v$, denoted by $\deg_G(v)$, is $|N(v)|$. By $n(G)$ and $\Delta(G) = \Delta$ we denote the order and the maximum degree of the graph $G$, respectively. Specifically, for a vertex $v$ in a rooted tree $T$, we denote by $C(v)$ and $D(v)$ the set of children and descendants, respectively, of $v$, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$.

In [3] Kristiansen, Hedetniemi, and Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of offensive alliances, namely global offensive $k$-alliances given by Shafique and Dutton [4, 5]. Let $k \geq 0$ be an integer. A set $S$ of vertices of a graph $G$ is called a global offensive $k$-alliance if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$ for $0 \leq k \leq \Delta$. The global offensive $k$-alliance number $\gamma^k_o(G)$ is the minimum cardinality of a global offensive $k$-alliance in $G$. If $S$ is a global offensive $k$-alliance of $G$ and $|S| = \gamma^k_o(G)$, then we say that $S$ is a $\gamma^k_o(G)$-set. A global offensive
1-alliance is a global offensive alliance and a global offensive 2-alliance is a global strong offensive alliance.

In this paper, we show that for every bipartite graph $G$ and every integer $k \geq 1$, $\gamma^k_o(G) \leq \frac{n(G) + |L_k(G)|}{2}$, where $L_k(G) = \{x \in V(G) : \deg_G(x) \leq k - 1\}$. Moreover, extremal trees attaining the upper bound are characterized for $k \geq 2$.

2. MAIN RESULTS

**Theorem 2.1.** Let $k \geq 1$ be an integer. If $G$ is a bipartite graph, then

$$\gamma^k_o(G) \leq \frac{n(G) + |L_k(G)|}{2}. \quad \Box$$

**Proof.** Let $G$ be a bipartite graph. Clearly, $L_k(G)$ is contained in every $\gamma^k_o(G)$-set. Let $H$ be the graph obtained from $G$ by removing $L_k(G)$. If $H$ is empty, then the result is valid. Thus we assume now that $n(H) \geq 1$, and so $H$ admits a bipartition $A, B$, where $A = \emptyset$ or $B = \emptyset$ is possible. Every vertex of $A$ (resp., $B$) has at least $k$ neighbors in $B \cup L_k(G)$ (resp., $A \cup L_k(G)$). It follows that each of $A \cup L_k(G)$ and $B \cup L_k(G)$ is a global offensive $k$-alliance of $G$ and so

$$\gamma^k_o(G) \leq \min\{|A \cup L_k(G)|, |B \cup L_k(G)|\} \leq \frac{n(G) - |L_k(G)|}{2} + |L_k(G)| = \frac{n(G) + |L_k(G)|}{2}. \quad \Box$$

The case $k = 2$ in Theorem 2.1 leads to the next result.

**Corollary 2.2 ([2]).** If $G$ is a bipartite graph, then

$$\gamma^2_o(G) \leq \frac{n(G) + |L_2(G)|}{2}. \quad \Box$$

For a positive integer $k$, a set of vertices $D$ in a graph $G$ is said to be a $k$-dominating set if each vertex of $G$ not contained in $D$ has at least $k$ neighbors in $D$. The order of a smallest $k$-dominating set of $G$ is called the $k$-domination number, and it is denoted by $\gamma_k(G)$. Clearly, if $S$ is any $\gamma^k_o(G)$-set, then every vertex of $V(G) - S$ has at least $k$ neighbors in $S$. Thus $S$ is a $k$-dominating set of $G$, and hence $\gamma_k(G) \leq \gamma^k_o(G)$. Using this fact, Theorem 2.1 implies the following corollary.

**Corollary 2.3 ([1]).** Let $k \geq 1$ be an integer. If $G$ is a bipartite graph, then

$$\gamma_k(G) \leq \frac{n(G) + |L_k(G)|}{2}. \quad \Box$$

In [1], Bilidia, Chellali and Volkmann defined the following trees. For a positive integer $p$, a nontrivial tree $T$ is called an $N_p$-tree if $T$ contains a vertex, say $w$, of degree at least $p - 1$ and $\deg_T(x) \leq p - 1$ for every vertex of $x \in V(T) - \{w\}$. We will call $w$ the special vertex of $T$. An $N_p$-tree with special vertex $w$ is called exact if $\deg_T(w) = p - 1$. The subdivided star $K_{1,p}$ ($p \geq 1$) is an example of an $N_p$-tree.
In order to characterize extremal trees achieving equality in Theorem 2.1 we define the family $\mathcal{F}_k$ of all trees $T$ that can be obtained from a sequence $T_1, T_2, \ldots, T_p$ ($p \geq 1$) of trees, where $T_1$ is an exact $N_k$-tree, $T = T_p$, and, if $p \geq 2$, $T_{i+1}$ can be obtained recursively from $T_i$ by one of the two operations listed below.

- Operation $O_1$: Attach an $N_k$-tree of special vertex $w$ of degree at least $k + 1$ by adding an edge from $w$ to a vertex $u$ of $T_i$ of degree exactly $k - 1$, and adding at most one new tree, all vertices of degree at most $k - 1$ and join a vertex of degree at most $k - 2$ with $u$ by an edge.
- Operation $O_2$: Attach an $N_k$-tree of special vertex $w$ of degree $k$ or $k - 1$ by adding an edge from $w$ to a vertex $u$ of $T_i$ of degree exactly $k - 1$, and adding $t$ ($t \geq 0$) new trees, all vertices of degree at most $k - 1$ and join a vertex of degree at most $k - 2$ of each new tree with $u$ by an edge.

We state a lemma.

**Lemma 2.4.** If $T \in \mathcal{F}_k$, then $\gamma_k^b(T) = \left( n(T) + |L_k(T)| \right) / 2$.

**Proof.** Assume that $T \in \mathcal{F}_k$. Clearly, $\Delta(T) \geq k - 1$ and $T$ is obtained from a sequence $T_1, T_2, \ldots, T_p$ ($p \geq 1$) of trees, where $T_1$ is an exact $N_k$-tree, $T = T_p$, and, if $p \geq 2$, $T_{i+1}$ can be obtained recursively from $T_i$ by one of the two operations defined above. We will use an induction on $p$. If $p = 1$, then $T$ is an exact $N_k$-tree where $\gamma_k^b(T) = |L_k(T)| = n(T)$ and so $\gamma_k^b(T) = \left( n(T) + |L_k(T)| \right) / 2$.

Assume now that $p \geq 2$ and that the result holds for all trees $T \in \mathcal{F}_k$ that can be constructed from a sequence of length at most $p - 1$, and let $T' = T_{p-1}$. By the inductive hypothesis on $T' \in \mathcal{F}_k$, we have $\gamma_k^b(T') = \left( n(T') + |L_k(T')| \right) / 2$. Let $T$ be a tree obtained from $T'$ and $S$ a $\gamma_k^b(T)$-set. We consider the following two cases.

**Case 1.** $T$ is obtained from $T'$ by using operation $O_1$.

Let $H$ be the $N_k$-tree of special vertex $w$ of degree at least $k + 1$ added to $T'$ and let $Q$ be the new tree of maximum degree at most $k - 1$ that can possibly be added to $T'$. Clearly $n(T) = n(T') + n(H) + n(Q)$ and $|L_k(T)| = |L_k(T')| + |V(H)| + |V(Q)| - 2$. Then $S$ contains all vertices of $Q$, $H$ except possibly $w$. If $w \in S$, then $u \notin S$ otherwise $S - \{u\}$ is a global offensive $k$-alliance of $T$, contradicting the minimality of $S$, but then $\{u\} \cup S - \{w\}$ is a $\gamma_k^b(T)$-set that contains $u$ and not $w$. Now if $w \notin S$, then $u \in S$ otherwise since $k \leq \deg_T(u) \leq k + 1$, $k \geq |N(u) \cap S| \geq |N(u) - S| + k \geq 1 + k$, which is impossible. Thus we may assume without loss of generality that $u \in S$ and $w \notin S$. Now let $S' = S \cap V(T')$. Since $S$ is a $\gamma_k^b(T)$-set, every vertex of $z \in V(T') - S'$ satisfies $|N(z) \cap S'| \geq |N(z) - S'| + k$ and hence $S'$ is a global offensive $k$-alliance of $T'$, implying that $\gamma_k^b(T') \leq \gamma_k^b(T) - |V(H)| - |V(Q)| + 1$. Now since $\deg_T(u) = k - 1$, $u$ is in every $\gamma_k^b(T')$-set, and such a set can be extended to a global offensive $k$-alliance of $T$ by adding $(V(H) - \{w\}) \cup V(Q)$; and so $\gamma_k^b(T) \leq \gamma_k^b(T') + |V(H)| + |V(Q)| - 1$. It follows that $\gamma_k^b(T) = \gamma_k^b(T') + |V(H)| + |V(Q)| - 1$. Using induction on $T'$, we obtain $\gamma_k^b(T) = \left( n(T) + |L_k(T)| \right) / 2$. 

Case 2. \( T \) is obtained from \( T' \) by using operation \( O_2 \).

Let \( H \) be the \( N_k \)-tree of special vertex \( w \) of degree \( k - 1 \) or \( k \) added to \( T' \) and let \( Q_1, Q_2, \ldots, Q_t \) be the \( t \geq 0 \) new trees that can possibly be added to \( T' \), each one of maximum degree at most \( k - 1 \). Then

\[
n(T) = n(T') + n(H) + \sum_{j=1}^{t} |V(Q_j)|,
\]

and

\[
|L_k(T)| = |L_k(T')| - 1 + |V(H) - \{w\}| + \sum_{j=1}^{t} |V(Q_j)|.
\]

Every \( \gamma_k^h(T') \)-set contains \( u \) and can be extended to a global offensive \( k \)-alliance of \( T \) by adding the set \( V(H) - \{w\} \) and all the vertices of \( Q_j \) for every \( j \), so

\[
\gamma_k^h(T) \leq \gamma_k^h(T') + |V(H)| - 1 + \sum_{j=1}^{t} |V(Q_j)|.
\]

On the other hand, \( V(Q_j) \subseteq S \) for every \( j \), \((V(H) - \{w\}) \subseteq S \) and \( S \) must contain one of \( w \) or \( u \), otherwise \( S \) would not be a global offensive \( k \)-alliance of \( T \) since \(|N(w) \cap S| = k < k + 1 = |N(w) - S| + k \). Thus we may assume that \( u \in S \), and hence \( S \) minus the sets \( V(H) - \{w\} \) and \( V(Q_j) \) for every \( j \) is a global offensive \( k \)-alliance of \( T' \) implying that

\[
\gamma_k^h(T') \leq \gamma_k^h(T) - |V(H)| + 1 - \sum_{j=1}^{t} |V(Q_j)|,
\]

and so

\[
\gamma_k^h(T) = \gamma_k^h(T') + |V(H)| - 1 + \sum_{j=1}^{t} |V(Q_j)|.
\]

Using the induction on \( T' \), we obtain \( \gamma_k^h(T) = (n(T) + |L_k(T)|)/2 \).

We now give a constructive characterization of the trees \( T \) with the property that \( \gamma_k(T) = (n(T) + |L_k(T)|)/2 \) for every integer \( k \geq 2 \).

**Theorem 2.5.** Let \( k \geq 2 \) be an integer. A tree \( T \) satisfies \( \gamma_k(T) = (n(T) + |L_k(T)|)/2 \) if and only if either \( \Delta(T) \leq k - 2 \) or \( T \in \mathcal{F}_k \).

**Proof.** Clearly, if \( T \) is a tree with \( \Delta(T) \leq k - 2 \), then \( |L_k(T)| = n(T) \) and so \( \gamma_k(T) = n(T) = (n(T) + |L_k(T)|)/2 \). By Lemma 2.4, if \( T \in \mathcal{F}_k \), then \( \gamma_k(T) = (n(T) + |L_k(T)|)/2 \).

Let us prove the necessity. Let \( T \) be a tree with \( \gamma_k^h(T) = (n(T) + |L_k(T)|)/2 \) for a positive integer \( k \geq 2 \). Suppose that \( \Delta(T) \geq k - 1 \) and let \( Z(T) = \{x \in V(T) : \)
\( \deg_T(x) \geq k - 1 \). We use an induction on the size of \( Z(T) \), where \(|Z(T)| \geq 1\).

If \(|Z(T)| = 1\) then \( T \) is an exact \( N_k \)-tree and hence \( T \in \mathcal{F}_k \), because otherwise \( \gamma_k^h(T) = n(T) - 1 < n(T) = \frac{n(T) + |L_k(T)|}{2} \).

Let \(|Z(T)| \geq 2\) and assume that every tree \( T' \) with \(|Z(T')| < |Z(T)|\) such that \( \gamma_k^h(T') = (n(T') + |L_k(T')|)/2 \) is in \( \mathcal{F}_k \).

Note that we have seen in the proof of Theorem 2.1 that \( A \cup L_k(T) \) and \( B \cup L_k(T) \) are two global offensive \( k \)-alliances of \( T \), where \( \min\{|A \cup L_k(T)|, |B \cup L_k(T)|\} \leq \frac{n(T) - |L_k(T)|}{2} \). It follows that if \( \gamma_k^h(T) = \frac{n(T) + |L_k(T)|}{2} \), then \( A \cup L_k(T) \) and \( B \cup L_k(T) \) are two \( \gamma_k^h(T) \)-sets.

Let \( T \) be a tree with \( \gamma_k^h(T) = (n(T) + |L_k(T)|)/2 \) and \( S \) a \( \gamma_k^h(T) \)-set. If every vertex of \( T \) has degree at most \( k - 1 \) then \( T \) is an exact \( N_k \)-tree. So assume that \( \Delta(T) \geq k \).

Then \( T \) has at least two vertices of degree at least \( k \) for otherwise \( \gamma_k^h(T) = n - 1 \neq \frac{n(T) + |L_k(T)|}{2} \), a contradiction.

We now root \( T \) at a vertex \( r \) of maximum eccentricity. Let \( w \) be a vertex of degree at least \( k \) at maximum distance from \( r \). Such a vertex exists since \( \Delta(T) \geq k \). Clearly \( w \neq r \) and \( T_w \) is an \( N_k \)-tree. Let \( u \) be the parent of \( w \) in the rooted tree. Assume that \( \deg_T(u) < k \). Without loss of generality we may assume that \( w \in A \). Then \( u \in L_k(T) \) and every descendant of \( w \) is in \( L_k(T) \). As seen above \( A \cup L_k(T) \) is a \( \gamma_k^h(T) \)-set but then \( (A - \{w\}) \cup L_k(T) \) is a global offensive \( k \)-alliance of \( T \), a contradiction. Thus \( \deg_T(u) \geq k \). Likewise if \( u \) has a child \( b \neq w \) of degree at least \( k \), then \( w, b \in A \), and so \( (A - \{w, b\}) \cup \{w\} \cup L_k(T) \) is a global offensive \( k \)-alliance of \( T \) of size \( \frac{n(T) - |L_k(T)|}{2} \) which leads to a contradiction too. Thus every child of \( u \) besides \( w \) has degree at most \( k - 1 \) and so every vertex of \( D(u) - \{w\} \) has degree at most \( k - 1 \). We distinguish between two cases:

**Case 1.** Assume that \( \deg_T(w) \geq k + 2 \). Assume that \( \deg_T(u) \geq k + 2 \). Then every neighbor of \( u \) is in \( L_k(T) \) or in \( A \) (\( w \) and possibly the parent of \( u \)). It follows that \( (A - \{u\}) \cup L_k(T) \) is a global offensive \( k \)-alliance of \( T \), a contradiction.

It remains the case that \( k \leq \deg_T(u) \leq k + 1 \). Now consider the subtree \( T' = T - (T_w \cup T_b) \), where \( T_b \) is any subtree rooted at a child \( b \neq w \) of \( u \) if \( \deg_T(u) = k + 1 \) and \( V(T_b) = \emptyset \) if \( \deg_T(u) = k \). Thus in both cases \( u \) has degree \( k - 1 \) in \( T' \) and \( b \) has degree at most \( k - 2 \) in \( T_b \). Then every \( \gamma_k^h(T') \)-set contains \( u \) and such a set can be extended to a global offensive \( k \)-alliance of \( T \) by adding \( (V(T_w) - \{w\}) \cup V(T_b) \), and so \( \gamma_k^h(T) \leq \gamma_k^h(T') + |D(w)| + |D[b]| \). The equality is obtained by the fact that \( (B \cup L_k(T)) \setminus (D(w) \cup D[b]) \) is a global offensive \( k \)-alliance of \( T' \). Since \( w \) is a vertex of degree at least \( k \) at maximum distance from \( r \), we deduce that \(|L_k(T)| = |L_k(T')| + |D(w)| + |D[b]| - 1 \). It follows that

\[
\frac{n(T) + |L_k(T)|}{2} = \gamma_k^h(T) = \gamma_k^h(T') + |D(w)| + |D[b]|
\]

and therefore \( \frac{n(T') + |L_k(T')|}{2} = \gamma_k^h(T') \). Since \(|Z(T')| < |Z(T)|\), by induction on \( T' \), we have \( T' \in \mathcal{F}_k \). Because \( T \) is obtained from \( T' \) by using Operation \( \mathcal{O}_k \), \( T \in \mathcal{F}_k \).

**Case 2.** Assume that \( k \leq \deg_T(w) \leq k + 1 \). Let \( C(u) = \{w, y_1, \ldots, y_p\} \) where \( p = \deg_T(u) - 2 \). Recall that every vertex of \( C(u) - \{w\} \) has degree at most \( k - 1 \). Let
$T' = T - T_w - \bigcup_{j=1}^{p+2-k} T_{y_j}$. Then $T'$ is nontrivial and $\deg_{T'}(u) = k - 1$. It can be seen that

$$\gamma^k_0(T) = \gamma^k_0(T') + \left| D(w) \cup \left( \bigcup_{j=1}^{p+2-k} D[y_j] \right) \right|,$$

$$n(T) = n(T') + \left| D(w) \cup \left( \bigcup_{j=1}^{p+2-k} D[y_j] \right) \right| + 1$$

and

$$L_k(T) = L_k(T') + \left| D(w) \cup \left( \bigcup_{j=1}^{p+2-k} D[y_j] \right) \right| - 1,$$

implying that $\gamma^k_0(T') = (n(T') + |L_k(T')|)/2$ with $|Z(T')| < |Z(T)|$. By the inductive hypothesis on $T'$, we have $T' \in \mathcal{F}_k$. Thus $T \in \mathcal{F}_k$ because it is obtained from $T'$ by using Operation $O_2$.

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