NEIGHBOURHOOD TOTAL DOMINATION
IN GRAPHS

S. Arumugam, C. Sivagnanam

Abstract. Let \( G = (V,E) \) be a graph without isolated vertices. A dominating set \( S \) of \( G \) is called a neighbourhood total dominating set (ntd-set) if the induced subgraph \( \langle N(S) \rangle \) has no isolated vertices. The minimum cardinality of a ntd-set of \( G \) is called the neighbourhood total domination number of \( G \) and is denoted by \( \gamma_{ntd}(G) \). The maximum order of a partition of \( V \) into ntd-sets is called the neighbourhood total domatic number of \( G \) and is denoted by \( d_{ntd}(G) \). In this paper we initiate a study of these parameters.

Keywords: neighbourhood total domination, total domination, connected domination, paired domination, neighbourhood total domatic number.

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1. INTRODUCTION

By a graph \( G = (V,E) \) we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of \( G \) are denoted by \( n \) and \( m \) respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [3].

Let \( G = (V,E) \) be a graph and let \( v \in V \). The open neighbourhood and the closed neighbourhood of \( v \) are denoted by \( N(v) \) and \( N[v] = N(v) \cup \{v\} \) respectively. If \( S \subseteq V \), then \( N(S) = \bigcup_{v \in S} N(v) \) and \( N[S] = N(S) \cup S \). If \( S \subseteq V \) and \( u \in S \), then the private neighbour set of \( u \) with respect to \( S \) is defined by \( pn[u,S] = \{v : N[v] \cap S = \{u\}\} \).

A subset \( S \) of \( V \) is called a dominating set of \( G \) if \( N[S] = V \). The minimum (maximum) cardinality of a minimal dominating set of \( G \) is called the domination number (upper domination number) of \( G \) and is denoted by \( \gamma(G) \) (\( \Gamma(G) \)). An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [6]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [7].

Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the Appendix of Haynes et al. [6].
Sampathkumar and Walikar [9] introduced the concept of connected domination in graphs. A dominating set $S$ of a connected graph $G$ is called a connected dominating set if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected dominating set of $G$ is called the connected domination number of $G$ and is denoted by $\gamma_c(G)$. Cockayne et al. [4] introduced the concept of total domination in graphs. A dominating set $S$ of a graph $G$ without isolated vertices is called a total dominating set if $\langle S \rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of $G$ is called the total domination number of $G$ and is denoted by $\gamma_t(G)$. Haynes and Slater [5] introduced the concept of paired domination in graphs. A dominating set $S$ of a graph $G$ without isolated vertices is called a paired dominating set if $\langle S \rangle$ has a perfect matching. The minimum cardinality of a paired dominating set of $G$ is called the paired domination number of $G$ and is denoted by $\gamma_{pr}(G)$.

For a dominating set $S$ of $G$ it is natural to look at how $N(S)$ behaves. For example, for the cycle $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$, $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_1, v_2, v_4\}$ are dominating sets, $\langle N(S_1) \rangle$ is not connected and $\langle N(S_2) \rangle$ is connected. Motivated by this example, in [1] we have introduced the concept of neighbourhood connected domination in graphs.

**Definition 1.1** ([1]). A dominating set $S$ of a connected graph $G$ is called a neighbourhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S) \rangle$ is connected. A ncd-set $S$ is said to be minimal if no proper subset of $S$ is a ncd-set. The minimum cardinality of a ncd-set of $G$ is called the neighbourhood connected domination number of $G$ and is denoted by $\gamma_{nc}(G)$.

For the path $P_{10} = (v_1, v_2, \ldots, v_{10})$, $S_1 = \{v_2, v_5, v_7, v_9\}$ and $S_2 = \{v_1, v_4, v_6, v_7, v_{10}\}$ are dominating sets, $\langle N(S_1) \rangle$ has isolates and $\langle N(S_2) \rangle$ has no isolates. Motivated by this example, in this paper we introduce the concept of neighbourhood total domination and initiate a study of neighbourhood total domination number and neighbourhood total domatic number.

We need the following theorems.

**Theorem 1.2** ([8]). Let $G$ be a nontrivial connected graph. Then $\gamma_c(G) + \kappa(G) = n$ if and only if $G = C_n$ or $K_n$ or $K_{2a} - X$ where $a \geq 3$ and $X$ is a $1$-factor of $K_{2a}$.

**Theorem 1.3** ([1]). Let $G$ be any graph such that both $G$ and $\overline{G}$ are connected. Then

$$\gamma_{nc}(G) + \gamma_{nc}(\overline{G}) \leq \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 2 & \text{if diam } G \geq 3, \\ \left\lceil \frac{n}{2} \right\rceil + 3 & \text{if diam } G = 2. \end{cases}$$

**Theorem 1.4** ([1]). Let $T$ be any tree with $n > 2$. Then $\gamma_{nc}(T) = n - \Delta$ if and only if $T$ can be obtained from a star by subdividing $k$ of its edges, $k \geq 1$, once or by subdividing exactly one edge twice.
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2. MAIN RESULTS

We assume throughout that $G$ is a graph without isolated vertices.

Definition 2.1. A dominating set $S$ of a graph $G$ is called a neighbourhood total dominating set (ntd-set) if the induced subgraph $\langle N(S) \rangle$ contains no isolated vertices. A ntd-set $S$ is said to be minimal if no proper subset of $S$ is a ntd-set. The minimum cardinality of a ntd-set of $G$ is called the neighbourhood total domination number of $G$ and is denoted by $\gamma_{nt}(G)$.

Remark 2.2. (i) Let $S$ be a ntd-set of $G$. Since $\langle N(S) \rangle$ has no isolated vertices, it follows that $|N(S)| \geq 2$.

(ii) Clearly $\gamma_{nt} \geq \gamma$. Further if $S$ is a total dominating set or a paired dominating set or a connected dominating set with $|S| > 1$, then $N(S) = V$ and hence $\gamma_{nt} \leq \gamma_{nt}, \gamma_{nt} \leq \gamma_{pr}$ and $\gamma_{nt} \leq \gamma_{c}$ if $\gamma_{c} > 1$.

(iii) For any connected graph $G$, $\gamma_{nt} = 1$ if and only if there exists a vertex $v \in V(G)$ such that $\deg v = n - 1$ and $G - v$ has no isolated vertices.

Theorem 2.3. For any connected graph $G$, $\gamma(G) \leq \gamma_{nt}(G) \leq \gamma_{nc}(G) \leq 2\gamma(G)$. Further given three positive integers $a, b$ and $c$ with $a \leq b \leq c \leq 2a$, there exists a graph $G$ with $\gamma(G) = a$, $\gamma_{nt}(G) = b$ and $\gamma_{nc}(G) = c$.

Proof. We have $\gamma(G) \leq \gamma_{nt}(G) \leq \gamma_{nc}(G) \leq \gamma_{pr}(G) \leq 2\gamma(G)$. Now, let $a, b$ and $c$ be positive integers with $a \leq b \leq c \leq 2a$. Let $b = a + r, 0 \leq r \leq a, c = a + k, r \leq k \leq 2a - r$. Consider the corona $K_{a} \circ K_{1}$ with $V(K_{a}) = \{v_{1}, v_{2}, \ldots, v_{a}\}$ and let $u_{i}$ be the pendant vertex adjacent to $v_{i}$. Take $r$ copies $H_{1}, H_{2}, \ldots, H_{r}$ of $K_{2}$ and $k - r$ copies $G_{r+1}, G_{r+2}, \ldots, G_{n}$ of $P_{3}$. Let $G$ be the graph obtained from $K_{a} \circ K_{1}$ by joining $u_{i}$ to all the vertices of $H_{i}$ where $1 \leq i \leq r$ and by joining $u_{r+j}$ to all the vertices of $G_{r+j}$ where $1 \leq j \leq k - r$. Then $\gamma(G) = a$, $\gamma_{nt}(G) = a + r = b$ and $\gamma_{nc}(G) = a + k = c$.

Theorem 2.4. For the path $P_{n}$,

$\gamma_{nt}(P_{n}) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } n \equiv 1(\text{mod } 3), \\ \left\lceil \frac{n}{3} \right\rceil + 1 & \text{otherwise.} \end{cases}$

Proof. Let $P_{n} = (v_{1}, v_{2}, \ldots, v_{n})$. If $n \equiv 1(\text{mod } 3)$, then $S = \{v_{i} : i = 3k + 1, k = 0, 1, 2, \ldots\}$ is a ntd-set of $P_{n}$. If $n \equiv 2(\text{mod } 3)$, then $S \cup \{v_{n}\}$ is a ntd-set of $P_{n}$. If $n \equiv 0(\text{mod } 3)$, then $S \cup \{v_{n-1}\}$ is a ntd-set of $P_{n}$. Hence

$\gamma_{nt}(P_{n}) \leq \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } n \equiv 1(\text{mod } 3), \\ \left\lceil \frac{n}{3} \right\rceil + 1 & \text{otherwise.} \end{cases}$

Now, $\gamma_{nt}(P_{n}) \geq \gamma(P_{n}) = \left\lceil \frac{n}{3} \right\rceil$. Further if $n \not\equiv 1(\text{mod } 3)$, then for any $\gamma$-set $S$ of $P_{n}$, $\langle N(S) \rangle$ has at least one isolated vertex and hence $\gamma_{nt}(P_{n}) \geq \left\lceil \frac{n}{3} \right\rceil + 1$. Hence the result follows.
Corollary 2.5. For any nontrivial path $P_n$,

(i) $\gamma_{nt}(P_n) = \gamma(P_n)$ if and only if $n \equiv 1 (\mod 3)$.

(ii) $\gamma_{nt}(P_n) = \gamma_c(P_n)$ if and only if $n = 4$ or 5.

(iii) $\gamma_{nt}(P_n) = \gamma_t(P_n)$ if and only if $n = 2, 3, 4, 5$ or 8.

(iv) $\gamma_{nt}(P_n) = \gamma_{nc}(P_n)$ if and only if $n = 3, 4, 5, 6$ or 8.

Proof. Since $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$, $\gamma_c(P_n) = n - 2$,

$$\gamma_t(P_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 0 (\mod 4), \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{otherwise} \end{cases}$$

and $\gamma_{nc}(P_n) = \left\lceil \frac{n}{3} \right\rceil$ the corollary follows.

\square

Theorem 2.6. For the cycle $C_n$,

$$\gamma_{nt}(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n \equiv 2 (\mod 3), \\ \left\lceil \frac{n}{3} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $n = 3k + r$, where $0 \leq r \leq 2$.

Let $S = \{v_i : i = 3j + 1, 0 \leq j \leq k\}$.

Let $S_1 = \begin{cases} S \cup \{v_n\} & \text{if } n \equiv 2 (\mod 3), \\ S & \text{otherwise.} \end{cases}$

Then $S_1$ is a ntd-set of $C_n$ and hence

$$\gamma_{nt}(C_n) \leq \left\lceil \frac{n}{3} \right\rceil + 1$$

if $n \equiv 2 (\mod 3)$, otherwise.

Now, $\gamma_{nt}(C_n) \geq \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$. Further if $n \equiv 2 (\mod 3)$, then for any $\gamma$-set of $S$ of $C_n$, $\langle N(S) \rangle$ has at least one isolated vertex and hence $\gamma_{nt}(C_n) \geq \left\lceil \frac{n}{3} \right\rceil + 1$. Hence the result follows.

\square

Corollary 2.7. (i) $\gamma_{nt}(C_n) = \gamma(C_n)$ if and only if $n \not\equiv 2 (\mod 3)$.

(ii) $\gamma_{nt}(C_n) = \gamma_c(C_n)$ if and only if $n = 3, 4$ or 5.

(iii) $\gamma_{nt}(C_n) = \gamma_t(C_n)$ if and only if $n = 4, 5$ or 8.

(iv) $\gamma_{nt}(C_n) = \gamma_{nc}(C_n)$ if and only if $n = 3, 4, 5, 7$.

Proof. Since $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$, $\gamma_c(C_n) = n - 2$,

$$\gamma_t(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 2 (\mod 4), \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{otherwise,} \end{cases}$$

and

$$\gamma_{nc}(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } n \not\equiv 3 (\mod 4), \\ \left\lfloor \frac{n}{3} \right\rfloor & \text{if } n \equiv 3 (\mod 4) \end{cases}$$

the result follows.

\square
Clearly, if a vertex $u$ is not adjacent to a vertex of degree $1$, then $u$ is a support vertex.

**Remark 2.11.** A dominating set $\gamma$ of $G$ is called a support vertex.

**Theorem 2.14.** A graph $G$ is isomorphic to $H$ if and only if $\gamma_{\text{dom}}(G) = \gamma_{\text{dom}}(H)$.

Now, let $G$ be any graph with $\gamma_{\text{dom}}(G) = \gamma_{\text{dom}}(H)$. Then $G$ and $H$ are isomorphic.

**Remark 2.12.** A dominating set $S$ of $G$ is isomorphic to $H$ if and only if $\gamma_{\text{dom}}(S) = \gamma_{\text{dom}}(H)$.

**Theorem 2.15.** A graph $G$ is isomorphic to $H$ if and only if $\gamma_{\text{dom}}(G) = \gamma_{\text{dom}}(H)$.

We now proceed to obtain a characterization of minimal ntd-sets.

**Lemma 2.8.** A super set of a ntd-set is a ntd-set.

**Theorem 2.16.** A graph $G$ is isomorphic to $H$ if and only if $\gamma_{\text{dom}}(G) = \gamma_{\text{dom}}(H)$.

**Remark 2.13.** A graph $G$ is isomorphic to $H$ if and only if $\gamma_{\text{dom}}(G) = \gamma_{\text{dom}}(H)$.

**Theorem 2.17.** A graph $G$ is isomorphic to $H$ if and only if $\gamma_{\text{dom}}(G) = \gamma_{\text{dom}}(H)$.

We now proceed to obtain a characterization of minimal ntd-sets.

**Remark 2.14.** A graph $G$ is isomorphic to $H$ if and only if $\gamma_{\text{dom}}(G) = \gamma_{\text{dom}}(H)$.

**Lemma 2.9.** A ntd-set $S$ of a graph $G$ is a minimal ntd-set if and only if for every vertex $v \in S$, one of the following holds:

(i) $\gamma_{\text{dom}}(S) = \emptyset$.

(ii) $\gamma_{\text{dom}}(S) =\{v\}$.

(iii) $\gamma_{\text{dom}}(S) =\{v, u\}$, where $v \in V - S$.

Clearly, if a vertex $v$ is not adjacent to a vertex of degree $1$, then $v$ is a support vertex. Suppose there exists an isolated vertex in $S$. Then $S$ is not a minimal ntd-set and hence $v$ is an isolated vertex in $\gamma_{\text{dom}}(S)$, which is a contradiction. Hence, $S$ has no isolated vertices and $S$ is a minimal ntd-set.
then \( \{v\} \) is a ntd-set of \( G \) and hence \( \gamma_{nt}(G) = 1 \), which is a contradiction. Hence \( \deg u = 1 \) for some \( u \in N(v) \), so that \( v \) is a support vertex of \( H \).

**Case ii.** \( G \) is disconnected.

Let \( G_1, G_2, \ldots, G_k \) be the components of \( G \) and let \( |V(G_i)| = n_i \). If \( \Delta = 1 \), then \( \gamma_{nt} = n \) and each \( G_i \) is isomorphic to \( K_2 \). Suppose \( \Delta \geq 2 \). Let \( v \in V(G_1) \) be such that \( \deg v = \Delta \). Since \( \gamma_{nt}(G) = n - \Delta + 1 \) it follows that \( \gamma_{nt}(G_1) = n_1 - \Delta + 1 \) and \( \gamma_{nt}(G_i) = n_i \) for all \( i \geq 2 \). Hence by Case i, \( G_1 \) is isomorphic to \( H \) where \( H \) is any graph having a support vertex \( v \) with \( \deg v = |V(H)| - 1 \) and \( G_1 \) is isomorphic to \( K_2 \) for all \( i \geq 2 \).

**Theorem 2.15.** Let \( G \) be a connected graph with \( \Delta < n - 1 \). Then \( \gamma_{nt}(G) \leq n - \Delta \).

Further, for a tree \( T \) with \( \Delta < n - 1 \) the following are equivalent.

(i) \( \gamma_{nt}(T) = n - \Delta \).

(ii) \( \gamma_{nc}(T) = n - \Delta \).

(iii) \( T \) can be obtained from a star by subdividing \( k \) of its edges, \( k \geq 1 \) once or by subdividing exactly one edge twice.

**Proof.** Let \( v \in V(G) \) and \( \deg v = \Delta \). Since \( G \) is connected and \( \Delta < n - 1 \), there exist two adjacent vertices \( u \) and \( w \) such that \( u \in N(v) \) and \( w \notin N[v] \). Let \( S = (N(v) - \{u\}) \cup \{w\} \). Then \( V-S \) is a ntd-set of \( G \) and hence \( \gamma_{nt}(G) \leq n - \Delta \).

Now, let \( T \) be a tree with \( \Delta < n - 1 \). Suppose \( \gamma_{nt}(T) = n - \Delta \). Then \( n - \Delta = \gamma_{nt}(T) \leq \gamma_{nc}(T) \leq n - \Delta \). Hence \( \gamma_{nc}(T) = n - \Delta \), so that (i) implies (ii).

It follows from Theorem 1.4 that (ii) implies (iii). We now prove (iii) implies (i). Consider the star \( K_{1,\Delta} \), where \( V(K_{1,\Delta}) = \{v, v_1, v_2, \ldots, v_{\Delta}\} \) with \( \deg v = \Delta \).

**Case i.** \( T \) is obtained from \( K_{1,\Delta} \) by subdividing the \( k \) edges \( vv_1, vv_2, \ldots, vv_k \). Let \( v_i \) be the vertex subdividing \( vv_i \), \( 1 \leq i \leq k \). Clearly, \( n - \Delta = k + 1 \). Also any ntd-set \( S \) of \( T \) contains either \( u_i \) or \( v_i \) for each \( i, 1 \leq i \leq k \) and also contains the vertex \( v \). Hence it follows that \( |S| \geq k + 1 = n - \Delta \) and \( \gamma_{nt}(T) = n - \Delta \).

**Case ii.** \( T \) is obtained from \( K_{1,\Delta} \) by subdividing the edge \( vv_1 \) twice.

Let \( u_1, u_2 \) be the vertices subdividing \( vv_1 \). Then \( n - \Delta = 3 \) and \( S = \{v, u_1, u_2\} \) is a minimum ntd-set of \( T \). Thus \( \gamma_{nt}(T) = n - \Delta \).

**Corollary 2.16.** For a forest \( G \), \( \gamma_{nt}(G) = n - \Delta \) if and only if \( G \) is isomorphic to \( K_2 \cup T \), where \( T \) is a tree with \( \gamma_{nt}(T) = |V(T)| - |\Delta(T)| \).

**Theorem 2.17.** For each \( \gamma_{nt}\)-set \( S \) of a connected graph \( G \), let \( t_{S} \) denote the number of vertices \( v \) such that \( v \) is not a pendant vertex of \( G \) and \( v \) is isolated in \( (S) \). Let \( t = \min\{t_S : S \text{ is a } \gamma_{nt}\text{-set of } G\} \). Then \( \gamma_{nc}(G) \leq \gamma_{nt}(G) + t \).

**Proof.** Let \( S \) be a \( \gamma_{nt}\)-set of \( G \) such that the number of vertices in \( S \) which are non-pendant vertices of \( G \) and are isolated in \( (S) \) is \( t \).

Let \( X = \{v \in S : d(v) = 0 \text{ in } (S) \text{ and } d(v) > 1 \text{ in } G\} \) so that \( |X| = t \). For each \( v \in X \), choose a vertex \( f(v) \in V(G) \) which is adjacent to \( v \). Then \( S_1 = S \cup \{f(v) : v \in X\} \) is a ncs-set of \( G \) and hence \( \gamma_{nc}(G) \leq |S_1| \leq \gamma_{nt}(G) + t \).

**Theorem 2.18.** Let \( G \) be a connected graph with \( \text{diam} \geq 2 \). Then \( \gamma_{nt}(G) \leq 1 + \delta(G) \) and the bound is sharp.
Theorem 2.21. Let $G$ be a connected graph with $\delta(G) = 2$ and $\gamma_{nt}(G) = 1 + \delta(G)$. Then for every vertex $v \in V(G)$ with $\deg v = \delta(G)$, $N(v)$ is an independent set and for all $u \in N(v)$ there exists a vertex $w \notin N(v)$ such that $w$ is adjacent only to $u$.

Proof. If $v \in V(G)$ and $\deg v = \delta$, then $N[v]$ is an ntd-set of $G$ and hence the result follows. The bound is attained for $K_{1,n}$ and $C_5$. \hfill \square

Theorem 2.19. Let $G$ be a connected graph with $\text{diam} G = 2$ and $\gamma_{nt}(G) = 1 + \delta(G)$. Then for every vertex $v \in V(G)$ with $\deg v = \delta(G)$, $N(v)$ is an independent set and for all $u \in N(v)$ there exists a vertex $w \notin N(v)$ such that $w$ is adjacent only to $u$.

Proof. Let $S_1 = N(v)$. Clearly $S_1$ is a dominating set of $G$. Now, suppose $N(v)$ is not an independent set. Then $\langle N(v) \rangle$ contains an edge $e = xy$. Hence $v$ is not isolated in $\langle N(S_1) \rangle$ and since $\text{diam} G = 2$, every vertex $w \notin N[v]$ is adjacent to either $x$ or a neighbour of $x$. Thus $w$ is not isolated in $\langle N(S_1) \rangle$. Hence $S_1$ is a ntd-set of $G$ and $\gamma_{nt}(G) \leq \delta(G)$ which is a contradiction. Thus $N(v)$ is an independent set.

Now, suppose there exists a vertex $u \in N(v)$ such that $u$ has no private neighbour in $V - N[v]$. Then $N[v] - \{u\}$ is a ntd-set of $G$ with cardinality $\delta(G)$ which is a contradiction. Hence the result follows. \hfill \square

Remark 2.20. The converse of Theorem 2.19 is not true. Consider the graph $G$ given in Figure 1.

Here $\delta(G) = 2$ and $\gamma_{nt}(G) = 2$. However, the unique vertex $v$ with $\deg v = \delta = 2$ satisfies the conditions given in Theorem 2.19.

Theorem 2.21. Let $G$ be a graph such that both $G$ and $\overline{G}$ have no isolated vertices. Then $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq n + 2$. Further, equality holds if and only if $G$ or $\overline{G}$ is isomorphic to $sK_2$, where $s > 1$.

Proof. If $G$ and $\overline{G}$ are both connected, then $\gamma_{nt}(G) \leq \gamma_{nt}(G) \leq \left\lceil \frac{n}{2} \right\rceil$ and $\gamma_{nt}(\overline{G}) \leq \left\lceil \frac{n}{2} \right\rceil$, so that $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq n + 1$.

If $G$ is disconnected, then $\gamma_{nt}(G) = 2$ and hence $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq n + 2$.

Now, let $G$ be any graph with $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = n + 2$. Then $G$ or $\overline{G}$ is disconnected. Suppose $G$ is disconnected. Then $\gamma_{nt}(G) = n$ and $\gamma_{nt}(\overline{G}) = 2$ and hence $G$ is isomorphic to $sK_2$ where $s > 1$. The converse is obvious. \hfill \square

The bound given by Theorem 2.21 can be substantially improved when $G$ and $\overline{G}$ are both connected, as shown in the following theorem.

Theorem 2.22. Let $G$ be any graph such that both $G$ and $\overline{G}$ are connected. Then

$$\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 2 & \text{if } \text{diam } G \geq 3, \\ \left\lceil \frac{n}{2} \right\rceil + 3 & \text{if } \text{diam } G = 2. \end{cases}$$
Proof. Since $\gamma_{nt} \leq \gamma_{nc}$ the result follows from Theorem 1.3

Remark 2.23. The bounds given in Theorem 2.22 are sharp. The graph $G = C_5$ has diameter 2; $\gamma_{nt}(G) = \gamma_{nt}(\overline{G}) = 3$ and $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = 6 = \left\lceil \frac{n}{2}\right\rceil + 3$. For the graph $G = C_k \circ K_1$ diam $G \geq 3$ and $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = \left\lceil \frac{n}{2}\right\rceil + 2$.

Problem 2.24. Characterize graphs which attain the bounds given in Theorem 2.22.

Theorem 2.25. For any connected graph $G$, $\gamma_{nt}(G) + \kappa(G) \leq n - \Delta + \delta + 1$ and equality holds if and only if $G$ contains a support vertex $v$ with deg $v = n - 1$.

Proof. We have $\gamma_{nt} \leq n - \Delta + 1$ and $\kappa \leq \delta$. Hence $\gamma_{nt} + \kappa \leq n - \Delta + \delta + 1$.

Let $G$ be a connected graph and let $\gamma_{nt}(G) + \kappa(G) = n - \Delta + \delta + 1$. Then $\gamma_{nt}(G) = n - \Delta + 1$ and $\kappa = \delta$ and the result follows from Theorem 2.14.

Theorem 2.26. For any graph $G$, $\gamma_{nt}(G) + \kappa(G) = n$ if and only if $G$ is isomorphic to one of the graphs $sK_2$, $s > 1$, $P_3$ or $C_5$ or $K_n$ or $K_{2a}$ if $a \geq 3$ and $X$ is a 1-factor of $K_{2a}$.

Proof. Let $G$ be a graph with $\gamma_{nt}(G) + \kappa(G) = n$.

Case i. $G$ is connected.

Suppose $\Delta = n - 1$. Then $\gamma_{nt} = 1$ or 2. If $\gamma_{nt} = 1$, then $\kappa = n - 1$ and hence $G$ is isomorphic to $K_n$. If $\gamma_{nt} = 2$ then $G$ contains a support vertex of degree $n - 1$ and hence $\kappa = 1$, $n = 3$. Hence $G$ is isomorphic to $P_3$.

Suppose $\Delta < n - 1$. Then $\gamma_{nt} \leq \gamma_c$ and $\gamma_{nt} + \kappa \leq \gamma_c + \kappa$ so that $\gamma_c + \kappa \geq n$. Since $\gamma_c + \kappa \leq n$ we get $\gamma_c = n - 1$ and $\gamma_{nt} = \gamma_c$. Therefore by Theorem 1.2 $G$ is isomorphic to $C_5$ or $K_{2a} - X$ where $X$ is a 1-factor in $K_{2a}$.

Case ii. $G$ is disconnected.

Then $\kappa = 0$. Hence $\gamma_{nt} = n$ so that $G$ is isomorphic to $sK_2$, $s > 1$. The converse is obvious.

3. NEIGHBOURHOOD TOTAL DOMATIC NUMBER

The maximum order of a partition of the vertex set $V$ of a graph $G$ into dominating sets is called the domatic number of $G$ and is denoted by $d(G)$. For a survey of results on domatic number and their variants we refer to Zelinka [10]. In [2] we have initiated a study of the neighbourhood connected domatic number of a graph. In this section we present a few basic results on the neighbourhood total domatic number of a graph.

Definition 3.1. Let $G$ be a graph without isolated vertices. A neighbourhood total domatic partition (nt-domatic partition) of $G$ is a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ in which each $V_i$ is a nt-d-set of $G$. The maximum order of an nt-domatic partition of $G$ is called the neighbourhood total domatic number (nt-domatic number) of $G$ and is denoted by $d_{nt}(G)$.

Observation 3.2. Since any domatic partition of $K_n$, where $n \geq 3$, is also a nt-domatic partition, we have $d_{nt}(K_n) = d(K_n) = n$. Similarly $d_{nt}(K_{r,s}) = d(K_{r,s}) = \min\{r, s\}$. Also for the wheel $W_n$, $d_{nt}(W_n) = d(W_n) = \begin{cases} 4 & \text{if } n \equiv 1(\mod 3), \\ 3 & \text{otherwise.} \end{cases}$
Observation 3.3. Since any total domatic partition of $G$ is a nt-domatic partition and any nc-domatic partition is a nt-domatic partition, we have $d_t(G) \leq d_{nc}(G) \leq d_{nt}(G) \leq d(G)$.

Observation 3.4. Let $v \in V(G)$ and $\deg v = \delta$. Since any ntd-set of $G$ must contain either $v$ or a neighbour of $v$, it follows that $d_{nt}(G) \leq \delta(G) + 1$.

Definition 3.5. A graph $G$ is called nt-domatically full if $d_{nt}(G) = \delta(G) + 1$.

Example 3.6. The graph $G$ given in Figure 2 is nt-domatically full. In fact $\{\{v_1\}, \{v_2, v_4, v_6, v_8\}, \{v_3, v_5, v_7, v_9\}\}$ is a nt-domatic partition of $G$ of maximum order and $d_{nt}(G) = 3 = 1 + \delta(G)$.

Fig. 2. nt-domatically full graph

Observation 3.7. Given two positive integers $n$ and $k$ with $n \geq 4$ and $1 \leq k \leq n$, there exists a graph $G$ with $n$ vertices such that $d_{nt}(G) = k$. We take

$$G = \begin{cases} K_n & \text{if } k = n, n \geq 3, \\ K_{1,n-1} & \text{if } k = 1, \\ B(n_1, n - 2 - n_1) & \text{if } k = 2, \\ K_{k-1} + K_{n-k+1} & \text{otherwise.} \end{cases}$$

Theorem 3.8. For the path $P_n, n \geq 2$, we have

$$d_{nt}(P_n) = \begin{cases} 1 & \text{if } n = 2, 3 \text{ or } 5, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. The result is trivial for $n = 2, 3$ or 5. Suppose $n \neq 2, 3, 5$. It follows from Observation 3.4 that $d_{nt}(P_n) \leq 2$. Now let $S = \{v_i : i \equiv 1(\text{mod } 3)\}$ and let

$$V_1 = \begin{cases} S & \text{if } n \equiv 1(\text{mod } 3), \\ S \cup \{v_{n-2}\} & \text{if } n \equiv 2(\text{mod } 3), \\ S \cup \{v_{n-1}\} & \text{if } n \equiv 0(\text{mod } 3). \end{cases}$$
Then \( \{V_1, V - V_1\} \) is a nt-domatic partition of \( P_n \) and hence \( d_{nt}(P_n) = 2 \). \( \square \)

**Theorem 3.9.** For the cycle \( C_n \) with \( n \geq 4 \) we have

\[
d_{nt}(C_n) = \begin{cases} 
1 & \text{if } n = 5, \\
3 & \text{if } n \equiv 0 \pmod{3}, \\
2 & \text{otherwise.}
\end{cases}
\]

**Proof.** Let \( C_n = (v_0, v_1, \ldots, v_{n-1}, v_0) \). The result is trivial for \( n = 5 \). Suppose \( n \neq 5 \).

It follows from Observation 3.4 that \( d_{nt}(C_n) \leq 3 \). If \( n \equiv 0 \pmod{3} \), let \( n = 3k \) and let \( S_i = \{v_j : 0 \leq j \leq n - 1 \text{ and } j \equiv i \pmod{3}\} \), \( i = 0, 1, 2 \). Then \( \{S_0, S_1, S_2\} \) is a nt-domatic partition of \( C_n \) and hence \( d_{nt}(C_n) = 3 \). Now, suppose \( n \not\equiv 0 \pmod{3} \). Let

\[
n = 3k + r \text{ where } r = 1 \text{ or } 2.
\]

Let \( S_1 = \left\{ v_i : i \equiv i \pmod{3} \right\} \) if \( n \equiv 1 \pmod{3} \), \( \{v_i : i \equiv 2 \text{ or } 3 \pmod{4}\} \) if \( n \equiv 2 \pmod{3} \).

Then \( \{S_1, V - S_1\} \) is a nt-domatic partition of \( C_n \) and hence \( d_{nt}(C_n) \geq 2 \). Also it follows from Theorem 2.6 that \( d_{nt}(C_n) \leq 2 \) and hence \( d_{nt}(C_n) = 2 \). \( \square \)

**Observation 3.10.** If \( \{V_1, V_2, \ldots, V_{d_{nt}}\} \) is a nt-domatic partition of \( G \), then \( |V_i| \geq \gamma_{nt} \) for each \( i \) and hence \( \gamma_{nt}(G)d_{nt}(G) \leq n \).

**Example 3.11.** (i) If \( G \cong sK_r \), \( r \geq 3 \), \( s \geq 1 \), then \( d_{nt}(G) = r \) and \( \gamma_{nt}(G) = s \) and hence \( d_{nt}(G)\gamma_{nt}(G) = sr = n \).

(ii) If \( G \cong sK_{r,r} \), \( r \geq 2 \), \( s \geq 1 \), then \( d_{nt}(G) = r \), \( \gamma_{nt}(G) = 2s \) and hence \( d_{nt}(G)\gamma_{nt}(G) = 2sr = n \).

(iii) If \( G \cong G_1 \circ K_1 \) where \( G_1 \) is any connected graph, then \( d_{nt}(G) = 2 \) and \( \gamma_{nt}(G) = \frac{n}{2} \) and hence \( d_{nt}(G)\gamma_{nt}(G) = n \).

**Problem 3.12.** Characterize the class of graphs for which \( d_{nt}(G)\gamma_{nt}(G) = n \).

**Theorem 3.13.** Let \( G \) be a graph of order \( n \geq 5 \) with \( \Delta = n - 1 \) and let \( k \) denote the number of vertices of degree \( n - 1 \). Then \( d_{nt}(G) \leq \frac{1}{2}(n + k) \) if and only if \( G = K_k + H \) where either \( H \) is isomorphic to \( 2K_{\frac{n - k}{2}} \) or \( H \) is a connected graph with \( V(H) = X_1 \cup X_2 \cup \cdots \cup X_r \), \( r = \frac{n - k}{2} \), \( |X_i| = 2 \), \( X_i \cap X_j = \emptyset \) for all \( i \neq j \) and the subgraph induced by the edges of \( H \) with one end in \( X_i \) and the other end in \( X_j \) has a perfect matching.

**Proof.** Let \( \{V_1, V_2, \ldots, V_k\} \) be any nt-domatic partition of \( G \) with \( |V_i| = 1, 1 \leq i \leq k \). Since \( |V_j| \geq 2 \) for all \( j \) with \( k + 1 \leq j \leq s \), it follows that \( s \leq k + \frac{n - k}{2} = \frac{n + k}{2} \). Hence \( d_{nt}(G) \leq \frac{1}{2}(n + k) \).

Now, let \( G \) be a graph with \( d_{nt}(G) = \frac{1}{2}(n + k) \). Then there exists a nt-domatic partition \( \{V_1, V_2, \ldots, V_k, V_{k+1}, \ldots, V_{2n+k}\} \) such that \( |V_i| = 1 \) if \( 1 \leq i \leq k \) and \( |V_j| = 2 \) if \( k + 1 \leq j \leq \frac{n + k}{2} \). Clearly, \( \{V_1 \cup V_2 \cup \cdots \cup V_k\} \cong K_k \). Let \( H = \left\{V_{k+1} \cup \cdots \cup V_{\frac{n + k}{2}}\right\} \).

**Case i.** \( H \) is disconnected.

Since \( |V_j| = 2 \) for all \( j \) with \( k + 1 \leq j \leq \frac{n + k}{2} \), it follows that \( H \) has exactly two components \( H_1, H_2 \) and each \( V_j \) contains one vertex from \( H_1 \) and one vertex from \( H_2 \). Since \( V_j \) is a ntd-set of \( G \), it follows that \( H_1 \) and \( H_2 \) are complete graphs and
\[ |V(H_1)| = |V(H_2)| = \frac{n+k}{2}. \] Hence \( H \) is isomorphic to \( 2K_{\frac{n+k}{2}} \). If \( k = 1 \), then each \( H_1 \) and \( H_2 \) must contain at least two vertices. Hence \( n \geq 5 \).

Case ii. \( H \) is connected.

Let \( X_i = V_{k+1}, 1 \leq i \leq \frac{n-k}{2} \). Then \( V(H) = X_1 \cup X_2 \cup \cdots \cup X_r \) and \( X_i \cap X_j = \emptyset \) when \( i \neq j \). Now, since each \( X_i \) is a dominating set of \( G \), it follows that the subgraph induced by the edges of \( H \) with one end in \( X_i \) and the other end in \( X_j \) has a perfect matching.

Conversely, suppose \( G \) is of the form given in the theorem. Let \( u_1, u_2, \ldots, u_k \) be the vertices of \( G \) with \( \text{deg } u_i = n-1, 1 \leq i \leq k \).

Suppose \( G = K_k + H \) where \( H \) is isomorphic to \( 2K_{\frac{n-k}{2}} \) with \( n \geq 5 \) when \( k = 1 \).

Let \( H_1 \) and \( H_2 \) be the two components of \( H \) with \( V(H_1) = \{ x_i : k+1 \leq i \leq \frac{n+k}{2} \} \) and \( V(H_2) = \{ y_i : k+1 \leq i \leq \frac{n+k}{2} \} \). Let

\[
V_i = \begin{cases} 
\{ u_i \} & \text{if } 1 \leq i \leq k, \\
\{ x_i, y_i \} & \text{where } x_i \in V(H_1) \text{ and } y_i \in V(H_2), \text{ if } k+1 \leq i \leq \frac{n+k}{2}.
\end{cases}
\]

Then \( \{ V_1, V_2, \ldots, V_{\frac{n+k}{2}} \} \) is a ndt-domatic partition of \( G \). Also if \( G = K_k + H \), where \( H \) is a connected graph satisfying the conditions stated in the theorem, then \( \{ \{ u_1 \}, \{ u_2 \}, \ldots, \{ u_k \}, X_1, X_2, \ldots, X_r \} \) is a ndt-domatic partition of \( G \). Thus \( d_{nt}(G) \geq k + r = \frac{n+k}{2} \) and hence \( d_{nt}(G) = \frac{n+k}{2} \).

**Corollary 3.14.** Let \( G \) be a graph with \( \Delta < n-1 \). Then \( d_{nt}(G) \leq \frac{n}{2} \). Further \( d_{nt}(G) = \frac{n}{2} \) if and only if \( V = X_1 \cup X_2 \cup \cdots \cup X_2 \), where \( |X_i| = 2 \) for all \( i \), \( X_i \cap X_j = \emptyset \) if \( i \neq j \), the subgraph induced by the edges of \( G \) with one end in \( X_i \) and the other end in \( X_j \) has a perfect matching and \( (V - X_i) \) has no isolated vertex if \( X_i \) is independent.

**Theorem 3.15.** Let \( G \) be any graph such that both \( G \) and \( \overline{G} \) are connected. Then \( d_{nt}(G) + d_{nt}(\overline{G}) \leq n \). Further equality holds if and only if \( V(G) = X_1 \cup X_2 \cup \cdots \cup X_2 \), where \( X_i \cap X_j = \emptyset \) and \( (X_i \cup X_j) \) is \( C_4 \) or \( P_4 \) or \( 2K_2 \) for all \( i \neq j \).

**Proof.** Since both \( G \) and \( \overline{G} \) are connected, it follows that \( \Delta < n-1 \). Hence \( d_{nt}(G) \leq \frac{n}{2} \) and \( d_{nt}(\overline{G}) \leq \frac{n}{2} \), so that \( d_{nt}(G) + d_{nt}(\overline{G}) \leq n \).

Now, suppose \( d_{nt}(G) + d_{nt}(\overline{G}) = n \). Then \( d_{nt}(G) = \frac{n}{2} \) and \( d_{nt}(\overline{G}) = \frac{n}{2} \). Since \( d_{nt}(G) \leq \delta(G) + 1 \), it follows that \( \delta(G) \geq \frac{n}{2} - 1 \) and \( \delta(\overline{G}) \geq \frac{n}{2} - 1 \) and hence \( \text{deg } v = \frac{n}{2} - 1 \) or \( \frac{n}{2} \) for all \( v \in V(G) \).

Now, let \( V = X_1 \cup X_2 \cup \cdots \cup X_2 \) be a ndt-domatic partition of \( G \). Then the subgraph induced by the edges of \( G \) with one end in \( X_i \) and the other end in \( X_j \) has a perfect matching. Further, if \( (X_i \cup X_j) \) has at least four edges, then at least one vertex \( v \) of \( (X_i \cup X_j) \) has degree at least 3. Since there are \( \frac{n}{2} - 2 \) ndt-sets other than \( X_i \) and \( X_j \), \( \text{deg } v \geq \frac{n}{2} + 1 \) which is a contradiction. Thus \( (X_i \cup X_j) \) contains at most four edges and hence is isomorphic to \( C_4 \) or \( P_4 \) or \( 2K_2 \). The converse is obvious. \( \square \)
4. CONCLUSION AND SCOPE

In this paper we have introduced a new type of domination, namely, neighbourhood total domination. We have also discussed the corresponding neighbour total domatic partition. The following are some interesting problems for further investigation.

**Problem 4.1.** Characterize the class of graphs for which $\gamma_{nt}(G) = n - \Delta$.

**Problem 4.2.** Characterize graphs for which $\gamma_{nt}(G) = \left\lceil \frac{n}{2} \right\rceil$.

**Problem 4.3.** Characterize the class of graphs for which $\gamma_{nt}(G) = n - 1$ or $n - 2$.

**Problem 4.4.** Characterize nt-domatically full graphs.

**Problem 4.5.** Characterize graphs for which $d_{nt}(G) = d(G)$.

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S. Arumugam
s.arumugam.klu@gmail.com

Kalasalingam University
National Centre for Advanced Research in Discrete Mathematics (n-CARDMATH)
Anand Nagar, Krishnankoil-626190, India

The University of Newcastle
School of Electrical Engineering and Computer Science
NSW 2308, Australia

C. Sivagnanam
choshi71@gmail.com

St. Joseph’s College of Engineering
Department of Mathematics
Chennai-600119, India

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