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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR HÉNON TYPE EQUATIONS

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Abstract. This paper is concerned with ground state solutions for the Hénon type equation $-\Delta u(x) = |y|^\alpha u^{p-1}(x)$ in Ω, where $\Omega = B^k(0,1) \times B^{n-k}(0,1) \subset \mathbb{R}^n$ and $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. We study the existence of cylindrically symmetric and non-cylindrically symmetric ground state solutions for the problem. We also investigate asymptotic behavior of the ground state solution when p tends to the critical exponent $2^* = \frac{2n}{n-2}$ if $n \geq 3$.

Keywords: Hénon equation, cylindrical symmetry, non-cylindrical symmetry, asymptotic behavior.

Mathematics Subject Classification: 35J50, 35J55, 35J60.

1. INTRODUCTION

In this paper, we consider the following problem

$$-\Delta u = |y|^\alpha u^{p-1}, \quad u > 0, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega, \tag{1.1}$$

where $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $\Omega = B^k(0,1) \times B^{n-k}(0,1) \subset \mathbb{R}^n$, $B^m(0,1)$ denotes the unit ball in $\mathbb{R}^m$ centered at the origin, and $k \geq 2,\ n \geq 3,\ \alpha > 0,\ 2 < p < 2^*$.

In [8], M. Hénon proposed the following problem

$$-\Delta u = |x|^\alpha u^{p-1}, \quad u > 0, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega, \tag{1.2}$$

where Ω is the unit ball in $\mathbb{R}^n$ with $n \geq 3$, $\alpha > 0$ and $p > 1$, which stems from the study of rotating stellar structures and is called the Hénon equation. Such a problem has been extensively studied, see for instance [4, 11, 12] and [13] etc. Interesting phenomenon concerning problem (1.2) that was revealed recently includes, among other things, that the exponent α affects the critical exponent for the existence of solutions. Precisely, it was shown in [11] that for $p \in (2, \frac{2n+2\alpha}{n-2})$, problem (1.2) admits at least one radial solution. One also notices that the moving plane method in [6] can not be applied to (1.2) since the weight function r^α is increasing. So it can be expected that problem (1.2) possesses non-radial solutions. Such solutions were found in [13] for $2 < p < \frac{2n}{n-2}$ and in [12] for $p = \frac{2n}{n-2}$. Furthermore, it was proved in [4] that the maximum point of the ground state solution u_p of (1.2) approaches a boundary point of $\partial\Omega$ as $p \to 2^*$.

On the other hand, the following Hardy-Sobolev-Mazya equation

$$-\Delta u - \lambda \frac{u}{|y|^2} = \frac{|u|^{p_t-1}u}{|y|^t}, \quad u > 0, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega \tag{1.3}$$

was considered in [1–3, 5] and references therein, where $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}, 2 \le k < n$, and a point $x \in \mathbb{R}^n$ is denoted as $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. Various existence results were obtained in these papers.

Motivated by above mentioned works, in this paper we study problem (1.1). First of all, we show that all solutions of (1.1) are symmetric in z.

Theorem 1.1. *Let u be a solution of* (1.1)*. Then $u(y, z) = u(y, |z|)$ for $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$.*

Next, we show that there exists a non-cylindrically symmetric solution.

Theorem 1.2. *Assume $k \ge 3$. For $2 < p < 2^*$, there exists $\alpha^* > 0$ such that problem* (1.1) *possesses at least two solutions, one is cylindrically symmetric and another one is non-cylindrically symmetric provided $\alpha > \alpha^*$.*

Using the concentration-compactness principle, we have

Theorem 1.3. *Suppose $p \in (2, 2^*)$ and $\alpha > 0$. Then there exists $x_0 \in \partial\Omega$ such that the ground state solution u_p satisfies, up to a subsequence, that:*

(a) $|\nabla u_p|^2 \to \mu\delta_{x_0}$ *as $p \to 2^*$ in the sense of measure,*
(b) $|u_p|^{2^*} \to \nu\delta_{x_0}$ *as $p \to 2^*$ in the sense of measure,*

where $\mu > 0, \nu > 0$ satisfy $\mu \ge S\nu^{2/2^*}$, δ_x is the Dirac mass at x.

Finally, we investigate the limiting behavior of the ground state solution u_p of problem (1.1). By the continuity of u_p, there exists $x_p = (y_p, z_p) \in \bar{\Omega}$ such that

$$u_p(x_p) = \sup_{x \in \bar{\Omega}} u_p(x) = M_p.$$

By Theorem 1.1, $u_p(y_p, z_p) = u_p(y_p, |z_p|)$. Let $\lambda_p > 0$ be such that $\lambda_p M_p^{\frac{2}{n-2}} = 1$. We know from section 4 that $\lambda_p \to 0$ when $p \to 2^*$.

Theorem 1.4. *There holds:*

(a) *$dist(x_p, \partial\Omega) \to 0$ and $dist(x_p, \partial\Omega)/\lambda_p \to \infty$ as $p \to 2^*$. Moreover, x_p is unique for p close to 2^* and $(|y_p|, |z_p|) \to (1, 0)$ when $p \to 2^*$.*
(b) $\lim\limits_{p \to 2^*} \int_\Omega |\nabla(u_p - U_{\lambda_p, x_p})|^2 = 0.$

In section 2, we prove Theorem 1.1 by the moving plane method. Theorem 1.2, Theorem 1.3 and Theorem 1.4 are proved in section 3 and section 4, respectively.

2. PROOF OF THEOREM 1.1

We use the moving plane method developed in [6] to prove the result. It suffices to show that u is symmetric in the z_1, the other directions can be done in the same way. Let $\Omega_\lambda = \{(y,z) \in \Omega : z_1 > \lambda\}$, $0 < \lambda < 1$ and define

$$x_\lambda = (y, 2\lambda - z_1, \cdots, z_{n-k}), \quad w_\lambda(x) = u(x) - u(x_\lambda), \quad x \in \Omega_\lambda.$$

Then, by the mean value theorem, we have

$$\Delta w_\lambda(x) = c(x,\lambda) w_\lambda \quad \text{in} \quad \Omega_\lambda, \quad w_\lambda \le 0, \quad w_\lambda(x) \not\equiv 0 \quad \text{on} \quad \partial\Omega_\lambda, \tag{2.1}$$

where

$$|c(x,\lambda)| = \left| |y|^\alpha \frac{u^{p-1} - u_\lambda^{p-1}}{u - u_\lambda} \right| = \big| |y|^\alpha (p-1) u^{p-2}(\xi) \big| \le C.$$

Now, let us show that $w_\lambda < 0$ in Ω_λ for any $\lambda \in (0,1)$. This implies in particular that w_λ assumes along $\partial\Omega_\lambda \cap \Omega$ its maximum in Ω_λ.

For λ close to 1, the maximum principle for a domain with small volume [9, Theorem 2.32] yields $w_\lambda \le 0$ in Ω_λ. By the strong maximum principle [9, Theorem 2.10], we obtain $w_\lambda < 0$ in Ω_λ. Let $\lambda_0 = \inf\{\lambda : w_\lambda < 0 \text{ in } \Omega_\lambda\}$. We will show that $\lambda_0 = 0$. Suppose on the contrary that $\lambda_0 > 0$, by continuity, $w_{\lambda_0} \le 0$ in Ω_{λ_0} and $w_{\lambda_0} \not\equiv 0$ on $\partial\Omega_{\lambda_0}$. The strong maximum principle implies $w_{\lambda_0} < 0$. We will show now that there is $\varepsilon > 0$ small enough such that

$$w_{\lambda_0 - \varepsilon} < 0 \quad \text{in} \quad \Omega_{\lambda_0 - \varepsilon}. \tag{2.2}$$

This is a contradiction to the choice of λ_0, the assertion $\lambda_0 = 0$ then follows.

Let $\delta > 0$ to be determined and let K be a closed subset in Ω_{λ_0} such that $|\Omega_{\lambda_0} \setminus K| < \frac{\delta}{2}$. Noticing that $w_{\lambda_0} < 0$ in Ω_{λ_0}, we deduce

$$w_{\lambda_0}(x) \le -\eta < 0 \quad \text{for all} \quad x \in K.$$

By continuity, we obtain

$$w_{\lambda_0 - \varepsilon}(x) \le 0 \quad \text{in} \quad K.$$

For $\varepsilon > 0$ sufficient small, $|\Omega_{\lambda_0 - \varepsilon} \setminus K| < \delta$. We choose $\delta > 0$ in such a way that we can apply [9, Theorem 2.32] to $w_{\lambda_0 - \varepsilon}$ in $|\Omega_{\lambda_0 - \varepsilon} \setminus K|$. Hence, we obtain

$$w_{\lambda_0 - \varepsilon}(x) \le 0 \quad \text{in} \quad \Omega_{\lambda_0 - \varepsilon} \setminus K,$$

and then by [9, Theorem 2.10], we obtain

$$w_{\lambda_0 - \varepsilon} < 0 \quad \text{in} \quad \Omega_{\lambda_0 - \varepsilon} \setminus K.$$

Therefore, (2.2) holds. The proof is complete. □

3. NON-CYLINDRICALLY SYMMETRIC SOLUTION

Let $H^1_{0,cs}(\Omega) = \{u(y,z) \in H^1_0(\Omega) : u(y,z) = u(|y|,z)\}$ be the space of cylindrically symmetric functions in $H^1_0(\Omega)$ with the norm $\|u\|_{H^1_{0,cs}(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$. Consider the variational problem

$$S_{\alpha p} = \inf_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |y|^\alpha |u|^p dx)^{2/p}}. \tag{3.1}$$

One can verify that $S_{\alpha p}$ is achieved by a positive function, which corresponds to a ground state solution of (1.1), that is the least energy solution. Since (1.1) and (3.1) are rotation invariant with respect to y, it is natural to consider the problem

$$S^C_{\alpha p} := \inf_{u\in H^1_{0,cs}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |y|^\alpha |u|^p dx)^{2/p}}. \tag{3.2}$$

We may verify that $S^C_{\alpha p}$ is also achieved by a positive function, which is a cylindrically symmetric solution of (1.1). Obviously, $S_{\alpha p} \le S^C_{\alpha p}$. Now, we prove

$$S_{\alpha p} < S^C_{\alpha p}, \tag{3.3}$$

which means that equation (1.1) has at least two positive solutions, one is cylindrically symmetric and another one is non-cylindrically symmetric. Hence, this proves Theorem 1.2. We define on $H^1_0(\Omega)$ the Raleigh quotient

$$R(u) = \frac{Z(u)}{N(u)} := \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |y|^\alpha |u|^p dx)^{2/p}}. \tag{3.4}$$

Lemma 3.1. *Any cylindrically symmetric local minimizer u of R satisfies*

$$\int_\Omega |\nabla u|^2 dx \le \frac{k-1}{p-2} \int_\Omega \frac{u^2}{|y|^2} dx. \tag{3.5}$$

Proof. Let $f(t) = R(u+th)$ for $h \in H^1_0(\Omega)$. Since $f'(0) = 0$, $f''(0) \ge 0$, we deduce as [13] that

$$\begin{aligned} &\int_\Omega |\nabla u|^2 dx \Big[(2-p)\Big(\int_\Omega |y|^\alpha |u|^{p-2} uh dx\Big)^2 + (p-1)\int_\Omega |y|^\alpha |u|^p dx \int_\Omega |y|^\alpha |u|^{p-2} h^2 dx\Big] \le \\ &\le \int_\Omega |\nabla h|^2 dx \Big(\int_\Omega |y|^\alpha |u|^p dx\Big)^2. \end{aligned} \tag{3.6}$$

We may write $h = u(r,z) f(\theta)$, where f is a smooth function defined on the sphere S^{k-1} with zero mean. Since

$$|\nabla h|^2 = \left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] f^2 + \frac{1}{r^2} u^2 |\nabla_\theta f|^2,$$

we obtain

$$\int_\Omega |\nabla u|^2 dx \Big[(2-p)\Big(\int_\Omega r^\alpha |u|^p f r^{k-1}\omega(\theta) dr dz d\theta\Big)^2 +$$

$$+ (p-1)\int_\Omega |y|^\alpha |u|^p dx \int_\Omega r^\alpha |u|^p f^2 r^{k-1}\omega(\theta) dr dz d\theta\Big] \le$$

$$\le \Big\{\int_\Omega \Big[\Big(\frac{\partial u}{\partial r}\Big)^2 + \Big(\frac{\partial u}{\partial z}\Big)^2\Big] f^2 r^{k-1}\omega(\theta) + \frac{1}{r^2}u^2 |\nabla_\theta f|^2 r^{k-1}\omega(\theta) dr dz d\theta\Big\} \times$$

$$\times \Big(\int_\Omega |y|^\alpha |u|^p dx\Big)^2,$$

which yields

$$(p-1)\int_\Omega |\nabla u|^2 dx \left(\int_\Omega |y|^\alpha |u|^p dx\right)^2 \int_{S^{k-1}} f^2 d\theta \le$$

$$\le \left\{\int_\Omega |\nabla u|^2 dx \int f^2 d\theta + \int_\Omega \frac{1}{|y|^2} u^2 dx \int_{S^{k-1}} |\nabla_\theta f|^2 d\theta\right\}\left(\int_\Omega |y|^\alpha |u|^p dx\right)^2,$$

i.e.

$$(p-2)\int_\Omega |\nabla u|^2 dx \int_{S^{k-1}} f^2 d\theta \le \int_\Omega \frac{u^2}{|y|^2} dx \int_{S^{k-1}} |\nabla_\theta f|^2 d\theta.$$

Note that

$$\inf_{f\in\mathscr{E}} \frac{\int_{S^{k-1}} |\nabla_\theta f|^2 d\theta}{\int_{S^{k-1}} f^2 d\theta} = k-1,$$

where

$$\mathscr{E} = \Big\{f | f \in H^1(S^{k-1}), \int_{S^{k-1}} f d\theta = 0\Big\},$$

inequality (3.5) follows. □

Next, let $\alpha > 0$ be fixed and u_α be a minimizer of problem (3.2) such that

$$\int_\Omega |\nabla u_\alpha|^2 dx = 1.$$

Lemma 3.2. *For* $0 < R < 1$,

$$\lim_{\alpha\to+\infty} \int_{B^k(0,R)\times B^{n-k}(0,1)} |\nabla u_\alpha|^2 dx = 0.$$

Proof. It follows from

$$-\triangle u_\alpha = \Big(\int_\Omega |y|^\alpha u_\alpha^p dx\Big)^{-1} |y|^\alpha u_\alpha^{p-1}$$

that for $r > 0$,

$$\begin{aligned}&\int_{B^k(0,r)\times B^{n-k}(0,1)} |\nabla u_\alpha|^2 dx = \\ &= \int_{\partial B^k(0,r)\times B^{n-k}(0,1)} u_\alpha \frac{\partial u_\alpha}{\partial \eta} d\sigma + \Big(\int_\Omega |y|^\alpha u_\alpha^p dx\Big)^{-1} \int_{B^k(0,r)\times B^{n-k}(0,1)} |y|^\alpha u_\alpha^p dx.\end{aligned}$$

By the Hopf lemma and the fact that u_α is decreasing with respect to $|y|$,

$$\int_{B^k(0,r)\times B^{n-k}(0,1)} |\nabla u|^2 dx \le \frac{\int_{B^k(0,r)\times B^{n-k}(0,1)} |y|^\alpha u_\alpha^p dx}{\int_\Omega |y|^\alpha u_\alpha^p dx}. \tag{3.7}$$

For $\varepsilon > 0$, let $m \in N$ satisfy $2R^{m-1} < \varepsilon$, and

$$v_\alpha(|y|, z) = u_\alpha(|y|^\gamma, z), \quad \gamma = 1 + \frac{m}{\alpha + k - m},$$

we may verify that

$$\int_\Omega |y|^\alpha v_\alpha^p dx = \int_\Omega |y|^\alpha u_\alpha^p(|y|^\gamma, z) dx = \frac{1}{\gamma}\int_\Omega |y|^{\alpha-m} u_\alpha^p dx, \tag{3.8}$$

and

$$\int_\Omega |\nabla v_\alpha|^2 dx = \int_\Omega |\nabla u_\alpha(|y|^\gamma, z)|^2 dx \le \gamma \int_\Omega |y|^{-\frac{mk}{\alpha+k}} |\nabla u_\alpha|^2 dx. \tag{3.9}$$

Equations (3.8) and (3.9) lead, with help of Hölder's inequality, to

$$R(v_\alpha)^{-\frac{p}{2}} \ge \gamma^{-1-\frac{p}{2}} \frac{\int_\Omega |y|^{\alpha-m} u_\alpha^p dx}{\Big(\int_\Omega |y|^{-\frac{mk}{\alpha+k}} |\nabla u_\alpha|^2 dx\Big)^{\frac{p}{2}}} \ge \gamma^{-1-\frac{p}{2}} \frac{\int_\Omega |y|^{\alpha-m} u_\alpha^p dx}{\int_\Omega |y|^{-\frac{mkp}{2(\alpha+k)}} |\nabla u_\alpha|^2 dx}. \tag{3.10}$$

Noticing that $\beta := \frac{mkp}{2(\alpha+k)} = o(1)$ as $\alpha \to +\infty$, we have

$$
\begin{aligned}
\int\limits_{\Omega} |y|^{-\beta} |\nabla u_\alpha|^2 dx &= \omega_{k-1} \int\limits_0^1 \int\limits_{B^{n-k}(0,1)} r^{-\beta} |\nabla u_\alpha(r,z)|^2 r^{k-1} dr dz = \\
&= \omega_{k-1} \int_0^{+\infty} \int\limits_0^{g(s)} \int\limits_{B^{n-k}(0,1)} |\nabla u_\alpha(r,z)|^2 r^{k-1} dz dr ds \le \\
&\le \omega_{k-1} \int\limits_0^{+\infty} \int\limits_0^{g(s)} \int\limits_{B^{n-k}(0,1)} r^\alpha u_\alpha^p r^{k-1} dz dr ds \Big(\int\limits_\Omega |y|^\alpha u_\alpha^p dx \Big)^{-1} = \\
&= \int\limits_\Omega |y|^{\alpha-\beta} u_\alpha^p dx \Big(\int\limits_\Omega |y|^\alpha u_\alpha^p dx \Big)^{-1},
\end{aligned}
\tag{3.11}
$$

where

$$
g(s) := \begin{cases} 1, & s \le 1, \\ s^{-1/\beta}, & s \ge 1. \end{cases}
$$

Since $R(v_\alpha) \ge R(u_\alpha)$, by (3.10) and (3.11),

$$
\gamma^{-1-\frac{p}{2}} \frac{\int_\Omega |y|^{\alpha-m} u_\alpha^p dx}{\int_\Omega |y|^{\alpha-\beta} u_\alpha^p dx} \le 1,
$$

that is,

$$
\gamma^{-1-\frac{p}{2}} R^{-m+\beta} \frac{\int_{B^k(0,r)\times B^{n-k}(0,1)} |y|^{\alpha-\beta} u_\alpha^p dx}{\int_\Omega |y|^{\alpha-\beta} u_\alpha^p dx} \le 1. \tag{3.12}
$$

Set $A^k(R,1) = B^k(0,1) \setminus B^k(0,R)$, we have

$$
\begin{aligned}
&\int\limits_{B^k(0,R)\times B^{n-k}(0,1)} |y|^{\alpha-\beta} u_\alpha^p dx \int\limits_\Omega |y|^\alpha u_\alpha^p dx - \int\limits_\Omega |y|^{\alpha-\beta} u_\alpha^p dx \int\limits_{B^k(0,R)\times B^{n-k}(0,1)} |y|^\alpha u_\alpha^p dx \ge \\
&\ge \int\limits_{B^k(0,R)\times B^{n-k}(0,1)} |y|^\alpha u_\alpha^p dx \Big(\int\limits_{A^k(R,1)\times B^{n-k}(0,1)} (|y|^\alpha - |y|^{\alpha-\beta}) u_\alpha^p dx \Big) = \\
&= o\Big(\int\limits_\Omega |y|^\alpha u_\alpha^p dx \int\limits_\Omega |y|^{\alpha-\beta} u_\alpha^p dx \Big).
\end{aligned}
\tag{3.13}
$$

Hence, (3.12) yields

$$
\frac{\int_{B^k(0,R)\times B^{n-k}(0,1)} |y|^\alpha u_\alpha^p dx}{\int_\Omega |y|^\alpha u_\alpha^p dx} \le 2\gamma^{1+\frac{p}{2}} R^{m-\beta} \le \varepsilon \text{ if } \alpha \to +\infty. \tag{3.14}
$$

The proof is then completed by (3.7). $\square$

Lemma 3.3. *If $\alpha \to +\infty$, we have*

$$\int_\Omega \frac{u_\alpha^2}{|y|^2} dx \to 0. \tag{3.15}$$

Proof. For $\varepsilon > 0$, there exists $\frac{1}{2} < R < 1$ independent of α such that $u_\alpha(R, z) < \varepsilon$. Indeed,

$$|u_\alpha(R,z)|^2 \le (1-R) \int_R^1 \left(\frac{\partial u_\alpha}{\partial r}\right)^2 dr \le \frac{C(1-R)}{R^{k-1}} \int_\Omega |\nabla u_\alpha|^2 dx \le \frac{C(1-R)}{R^{k-1}} < \varepsilon$$

if R close to 1. Let $\tilde{u}_\alpha := u_\alpha - u_\alpha(R, z)$. We deduce

$$\begin{aligned} &\int_\Omega \frac{u_\alpha^2}{|y|^2} dx = \int_{B^k(0,R)\times B^{n-k}(0,1)} \frac{u_\alpha^2}{|y|^2} dx + \int_{A^k(R,1)\times B^{n-k}(0,1)} \frac{u_\alpha^2}{|y|^2} dx \le \\ &\le 2 \int_{B^k(0,R)\times B^{n-k}(0,1)} \frac{\tilde{u}_\alpha^2}{|y|^2} dx + 2 \int_{B^k(0,R)\times B^{n-k}(0,1)} \frac{u_\alpha^2(R,z)}{|y|^2} dx + \int_{A^k(R,1)\times B^{n-k}(0,1)} \frac{u_\alpha^2}{|y|^2} dx := \\ &:= I_1 + I_2 + I_3, \end{aligned} \tag{3.16}$$

As $\tilde{u}_\alpha \in H_0^1(B^k(0,r) \times B^{n-k}(0,1))$, by the Hardy inequality,

$$\int_{B^k(0,r)\times B^{n-k}(0,1)} \frac{\tilde{u}_\alpha^2}{|y|^2} dx \le C \int_{B^k(0,r)\times B^{n-k}(0,1)} |\nabla u_\alpha|^2 dx.$$

By Lemma 3.2 and Lemma 3.3, $\lim_{\alpha\to+\infty} I_1 = 0$. Apparently, $I_2 \le C\varepsilon^2$. For I_3, recalling that $u_\alpha(R) < \varepsilon$ and u_α is decreasing with respect to $|y|$, we obtain

$$I_3 \le \int_{A^k(R,1)\times B^{n-k}(0,1)} \frac{u_\alpha^2(R,z)}{|y|^2} dx \le C\frac{\varepsilon^2}{R^2} \le C\varepsilon^2.$$

The proof is complete. □

Proof of Theorem 1.2. As a consequence of Lemma 3.1 and Lemma 3.3, $S_{\alpha p} < S_{\alpha p}^C$ if $\alpha > 0$ large. It implies the results in Theorem 1.2. □

4. ASYMPTOTIC BEHAVIOR OF GROUND STATE SOLUTIONS

Let u_p be a minimizer of

$$S_{\alpha p} := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |y|^\alpha |u|^p dx)^{2/p}}.$$

In this section, we study the asymptotic behavior of u_p, and the location of the maximum point of u_p as $p \to 2^*$. Denote by $S = S_{0,2^*}$ the best Sobolev constant. It is well known that S can only be achieved in $\mathbb{R}^n$ by

$$U(x) = \frac{[n(n-2)]^{(n-2)/4}}{(1+|x|^2)^{(n-2)/2}}.$$

For $\varepsilon > 0, x' \in \mathbb{R}^n$, set

$$U_{\varepsilon,x'}(x) = \varepsilon^{-(n-2)/2} U\Big(\frac{x-x'}{\varepsilon}\Big).$$

By Hölder's inequality, we readily have

Lemma 4.1.

$$\frac{\int_\Omega |\nabla u_p|^2}{(\int_\Omega |u_p|^p)^{2/p}} \geq \frac{\int_\Omega |\nabla u_p|^2}{(\int_\Omega |u_p|^{2^*})^{2/2^*}} + O(1), \quad p \to 2^*.$$

For $\varepsilon > 0$ small enough, let

$$x_0 = (y_0, 0) = (1 - 1/|\ln\varepsilon|, 0, \dots, 0) \in \mathbb{R}^n, \ y_0 = (1 - 1/|\ln\varepsilon|, 0, \dots, 0) \in \mathbb{R}^k$$

and $\varphi \in C_0^\infty(\Omega)$ be a cut-off function satisfying

$$\varphi(x) = \begin{cases} 1, & x \in \Omega_\varepsilon \\ 0, & x \in \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases} \qquad 0 \leq \varphi(x) \leq 1, \ |\nabla\varphi(x)| \leq C|\ln\varepsilon|, \quad \forall x \in \mathbb{R}^n, \tag{4.1}$$

where $\Omega_\varepsilon = B_k(y_0, 1/(2|\ln\varepsilon|)) \times B_{n-k}(0, 1/(2|\ln\varepsilon|))$, C is a constant independent of ε. Set $u_\varepsilon = \varphi U_{\varepsilon,x_0}$. Then, $u_\varepsilon \in H_0^1(\Omega)$.

Lemma 4.2. *There holds*

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2}{(\int_\Omega |y|^\alpha |u_\varepsilon|^p)^{2/p}} = S_{0,2^*} + K(\varepsilon), \quad p \to 2^*,$$

where $K(\varepsilon) \to 0$, *when* $\varepsilon \to 0$.

Proof. It is standard to verify that

$$|u_\varepsilon|_p^2 = |U|_p^2 \varepsilon^{n/p-(n-2)} + CK_1(\varepsilon)|U|_p^{2-p} \varepsilon^{(n-2)p/2-n/2+n/p-(n-2)} + o(1), \tag{4.2}$$

$$\left| \int_\Omega (|\nabla u_\varepsilon|^2 - \varepsilon^{-\frac{n-2}{2}} |\nabla U|^2)\, dx \right| \leq C|\ln\varepsilon|^n \tag{4.3}$$

and

$$\int_\Omega |y|^\alpha |u_\varepsilon|^p dx \geq \Big(1 - \frac{2}{|\ln\varepsilon|}\Big)^\alpha \int_\Omega \frac{\varphi^p [n(n-2)]^{(n-2)p/4}}{(\varepsilon + |x-x_0|^2)^{\frac{(n-2)p}{2}}}, \tag{4.4}$$

where $o(1) \to 0$ if $p \to 2^*$ and $K_1(\varepsilon) = C|\ln\varepsilon|^{(n-2)p-n}$. Equations (4.2), (4.3), and (4.4) lead to

$$\begin{aligned}
&\lim_{p\to 2^*} \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\int_\Omega (|y|^\alpha |u_\varepsilon|^p)^{2/p}} \le \\
&\le \lim_{p\to 2^*} \frac{1}{(1-\frac{2}{|\ln\varepsilon|})^{2\alpha/p}} \frac{\|\nabla U\|_2^2 \varepsilon^{-\frac{n-2}{2}} + C|\ln\varepsilon|^n}{|U|_p^2 \varepsilon^{n/p-(n-2)} + CK_1(\varepsilon)|U|_p^{2-p} \varepsilon^{(n-2)p/2-n/2+n/p-(n-2)}} = \\
&= \frac{1}{(1-\frac{2}{|\ln\varepsilon|})^{2\alpha/2^*}} \frac{\|\nabla U\|_2^2 + C|\ln\varepsilon|^n \varepsilon^{\frac{n-2}{2}}}{\|U\|_p^2 + C|\ln\varepsilon|^n \varepsilon^{\frac{n}{2}}} = \\
&= \frac{\|\nabla U\|_2^2}{\|U\|_{2^*}^2} + K(\varepsilon).
\end{aligned} \tag{4.5}$$

Similarly,

$$\lim_{p\to 2^*} \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\int_\Omega (|u_\varepsilon|^p)^{2/p}} \ge \frac{\|\nabla U\|_2^2}{\|U\|_{2^*}^2} + K(\varepsilon). \tag{4.6}$$

The assertion follows since

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2}{\int_\Omega (|y|^\alpha |u_\varepsilon|^p)^{2/p}} \ge \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\int_\Omega (|u_\varepsilon|^p)^{2/p}}. \qquad \square$$

Lemma 4.3. *We have*

$$\lim_{p\to 2^*} \frac{\int_\Omega |\nabla u_p|^2}{\int_\Omega (|y|^\alpha |u_p|^p)^{2/p}} = \lim_{p\to 2^*} \frac{\int_\Omega |\nabla u_p|^2}{\int_\Omega (|u_p|^{2^*})^{2/2^*}} = S_{0,2^*}. \tag{4.7}$$

Proof. Suppose that $S_{\alpha p}$ is achieved by u_p and since $|y| \le 1$, by Lemma 4.2,

$$\begin{aligned}
S_{0,2^*} &\le \frac{\int_\Omega |\nabla u_p|^2}{\int_\Omega (|u_p|^{2^*})^{2/2^*}} \le \frac{\int_\Omega |\nabla u_p|^2}{\int_\Omega (|u_p|^p)^{2/p}} \le \\
&\le \frac{\int_\Omega |\nabla u_p|^2}{\int_\Omega (|y|^\alpha |u_p|^p)^{2/p}} \le \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\int_\Omega (|y|^\alpha |u_\varepsilon|^p)^{2/p}} = S_{0,2^*} + K(\varepsilon),
\end{aligned}$$

which implies the result. $\square$

As a result of Lemma 4.3, we have

Corollary 4.4. *As $p \to 2^*$,*

$$\int_\Omega |\nabla u_p|^2 \to S^{n/2}.$$

Corollary 4.5. *In the case of $p = 2^*$,* (1.1) *does not possess any ground state solution.*

Proof of Theorem 1.3. The proof is similar to that of [4, Theorem 1.1] by the concentration-compactness principle, we omit the details. $\square$

Let

$$M_p = \sup_{x \in \overline{\Omega}} u_p(x) = u_p(x_p), \quad x_p \in \overline{\Omega}.$$

Lemma 4.6. *$M_p \to +\infty$ as $p \to 2^*$.*

Proof. We argue indirectly. Suppose that there exists a constant $C > 0$ and a sequence $\{p_i\}$ with $\lim\limits_{i \to +\infty} p_i = 2^*$ such that

$$M_{p_i} \le C, \quad i = 1, 2, \dots.$$

It follows from Theorem 1.3 that $u_{p_i} \rightharpoonup 0$ in $H_0^1(\Omega)$. Then, by the Sobolev imbedding theorem, we have $u_{p_i} \rightharpoonup 0$ in $L^{2^*}(\Omega)$. So, for $\sigma > 0$ small, due to the compactness of $L^{2^*}(\Omega) \hookrightarrow L^{2^*-\sigma}(\Omega)$, we have, up to a subsequence, that

$$1 = \int_\Omega |u_{p_i}|^{2^*} \le |u_{p_i}|^\sigma_{L^\infty(\Omega)} \int_\Omega |u_{p_i}|^{2^*-\sigma} \le C^\sigma \int_\Omega |u_{p_i}|^{2^*-\sigma} \to 0 \quad (i \to \infty),$$

which is impossible. □

Proof of Theorem 1.4. The proof is based on the blow up argument in [7]. Suppose $p \to 2^*$, $x_p \to x_0 \in \overline{\Omega}$. Let λ_p be a sequence of positive numbers defined by $\lambda_p^{\frac{n-2}{2}} M_p = 1$ and $x' = (x - x_p)/\lambda_p$, where $x' = (y', z')$, $x = (y, z)$. Let

$$v_p(x') = \lambda_p^{\frac{n-2}{2}} u_p(x), \quad \Omega_p = \{x' \in R^n | \lambda_p x' + x_p \in \Omega\}. \tag{4.8}$$

Then, $v_p(x')$ satisfies

$$\begin{cases} -\Delta v_p(x') = |y'\lambda_p + y_p|^\alpha \lambda_p^{\frac{(n-2)(2^*-p)}{2}} v_p^{p-1}, & x' \in \Omega_p, \\ v_p = 0, & x' \in \partial\Omega_p, \\ 0 < v_p \le 1, v_p(0) = 1. \end{cases} \tag{4.9}$$

Apparently, $0 \le \lambda_p \le 1$ and $\lambda_p \to 0$ if $p_k \to 2^*$. Let $L(p) := \lambda_p^{\frac{(n-2)(2^*-p)}{2}}$ and $L(2^*) := \lim\limits_{p \to 2^*} L(p)$, we have $0 \le L(2^*) \le 1$. We claim that $x_0 \in \partial\Omega$. Indeed, if this is not true, we have $x_0 \in \Omega$, and then $d := (2dist\{x_0, \partial\Omega\})^{-1} > 0$. For $p \to 2^*$, $v_p(x')$ is well defined in $B^k(0, d/\lambda_p) \times B^{n-k}(0, d/\lambda_p)$, and

$$\sup_{x' \in B^k(0,d/\lambda_p) \times B^{n-k}(0,d/\lambda_p)} v_p(x') = v_p(0) = 1.$$

For $l > 0$, we have for p close to 2^* that

$$B^k(0, 2l) \times B^{n-k}(0, 2l) \subseteq B^k(0, d/\lambda_p) \times B^{n-k}(0, d/\lambda_p).$$

By the L^p and Schauder estimates in the theory of elliptic equations [10], we find $\|v_p\|_{W^{2,r}(B^k(0,2l) \times B^{n-k}(0,2l))}$ $(r > n)$, and then $\|v_p\|_{C^{1,\theta}(B^k(0,2l) \times B^{n-k}(0,2l))}, 0 < \theta < 1,$

are uniformly bounded for p sufficiently close to 2^*. Hence, there is a subsequence $\{v_{p_k}\}$ of $\{v_p\}$ such that $v_{p_k} \to v$ in $C^{1,\theta}_{loc}$ as $p_k \to 2^*$. By a diagonal process, we derive that v satisfies

$$-\triangle v = |y_0|^\alpha L(2^*) v^{2^*-1} \quad \text{in } \mathbb{R}^n \tag{4.10}$$

and $v(0) = 1$.

If $y_0 = 0$ or $L(2^*) = 0$, we would have $\triangle v = 0$ in $\mathbb{R}^n$, i.e., $v \equiv const$. Thus $v \equiv 0$ since $v \in L^r(\mathbb{R}^n)$. This contradicts $v(0) = 1$. If $0 < L(2^*) \le 1$, we have

$$-\Delta v = cv^{2^*-1} \text{ in } \mathbb{R}^n, \quad 0 < v \le 1, \quad v(0) = 1, \tag{4.11}$$

where $0 < c := |y_0|^\alpha L(2^*) < 1$ since $0 < |y_0| < 1$. Let $w = c^{1/(2^*-2)} v$. We see that

$$-\Delta w = w^{2^*-1} \text{ in } \mathbb{R}^n, \quad 0 < w \le c^{1/(2^*-2)}, \quad w(0) = c^{1/(2^*-2)}. \tag{4.12}$$

It follows that $w(x) = \xi^{(2-n)/2} U(x/\xi)$, where ξ is determined by c. By Corollary 4.4 and the Fatou Lemma, we deduce for $p \to 2^*$ that

$$\begin{aligned} S^{n/2} &= \int_{\mathbb{R}^n} |\nabla w|^2 = c^{2/(2^*-2)} \int_{R^n} |\nabla v|^2 \le c^{2/(2^*-2)} \lim_{p\to 2^*} \int_{\Omega_p} |\nabla v_p|^2 = \\ &= c^{2/(2^*-2)} \lim_{p \to 2^*} \int_\Omega |\nabla u_p|^2 = c^{2/(2^*-2)} S^{n/2} < S^{n/2}, \end{aligned} \tag{4.13}$$

which is impossible.

We conclude that $x_0 \in \partial\Omega$. Denote $x_0 = (y_0, z_0)$. Next, we claim that $|y_0| = 1$ and $|z_0| = 0$. In fact, by Theorem 1.1, $u(y, z) = u(y, |z|)$ if u is a solution of (1.1). Then, by Theorem 1.3, $x_0 \in \partial\Omega$, then $|y_0| = 1, |z_0| = 0$.

Now we straighten $\partial\Omega$ in a neighborhood of x_0 by a non-singular C^1 change of coordinates as in [7]. Let

$$y_k = \psi(y'),\ y' = (y_1, y_2, \dots, y_{k-1}),\ \psi \in C^1$$

be the equation of $\partial B_k(0, l)$. Define a new coordinate system:

$$x'_i = x_i, \quad (i = 1, \dots, k-1, k+1, \dots, n), \quad x'_k = x_k - \psi(y') = y_k - \psi(y').$$

Then, u_p is again a solution of (1.1) and $\partial\Omega$ is contained in the hyperplane $x_k = y_k = 0$. Let d_p be the distance from x_p to $\partial\Omega$ (*i.e.* $d_p = x_p \cdot e_k$). Note that for $p \to 2^*$, v_p is well defined in $B_k(0, \delta/\lambda_p) \times B_{n-k}(0, \delta/\lambda_p) \cap \{y_k > d_p/\lambda_p\}$ for some $\delta > 0$ and satisfies (4.9). Moreover, $\sup v_p(y) = v_p(0) = 1$.

We assert that $\frac{d_p}{\lambda_p} \to +\infty$ as $p \to 2^*$. In fact, suppose on the contrary that $\frac{d_p}{\lambda_p}$ is uniformly bounded. Then, we may assume $\frac{d_p}{\lambda_p} \to s$ with $s \ge 0$. Arguing as above and noting that $|y_0| = 1$, we have that $v_p \to v$ and v satisfies

$$\begin{cases} -\Delta v = L(2^*) v^{2^*-1}, & x' \in \mathbb{R}^n_s = \{x' = (y'_1, \dots, y'_k, z'_1, \dots, z'_{n-k}) | y'_k \ge -s\}, \\ v = 0, & x' \in \partial\mathbb{R}^n_s, \\ 0 < v \le 1, v(0) = 1, & x' \in \mathbb{R}^n_s. \end{cases} \tag{4.14}$$

Since

$$\begin{cases} -\Delta v = cv^{2^*-1}, & x' \in \mathbb{R}^n_+ = \{x' = (y'_1, \ldots, y'_k, z'_1, \ldots, z'_{n-k}) | y'_k \geq 0\}, \\ v(x') = 0, & x' \in \partial \mathbb{R}^n_+ \end{cases}$$

has only a trivial solution, it implies $v \equiv 0$, which contradicts $v(0) = 1$.

Consequently, $\frac{d_p}{\lambda_p} \to +\infty$ as $p \to 2^*$. Now, we show that $L(2^*) = 1$. Indeed, we may show as before that $v_p \to v$ as $p_k \to 2^*$ and v satisfies

$$-\Delta v = L(2^*) v^{2^*-1} \quad \text{in} \quad \mathbb{R}^n, \quad 0 < v \leq 1, \quad v(0) = 1. \tag{4.15}$$

If $0 \leq L(2^*) < 1$, we may derive a contradiction either as (4.13) or $v \equiv 0$ respectively. So $L(2^*) = 1$.

Finally, we may show as [4, Theorem 1.2] that

$$\lim_{p \to 2^*} \int_\Omega |\nabla (u_p - U_{\lambda_p, x_p})|^2 = 0. \qquad \square$$

Corollary 4.7. *For p close to* 2^**, the ground state solution of* (1.1) *is not cylindrically symmetric.*

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REFERENCES

[1] M. Bhakta, K. Sandeep, *Hardy-Sobolev-Maz'ya type equations in bounded domains*, J. Differential Equations **247** (2009), 119–139.

[2] D. Castorina, I. Fabbri, G. Mancini, K. Sandeep, *Hardy-Sobolev inequalities, hyperbolic symmetry and the Webster scalar curvature problem*, J. Differential Equations **246** (2009), 1187–1206.

[3] D. Castorina, I. Fabbri, G. Mancini, K. Sandeep, *Hardy-Sobolev inequalities and hyperbolic symmetry*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **19** (2008) 3, 189–197.

[4] D. Cao, S. Peng, *The asymptotic behavior of the ground state solutions for Hénon equation*, J. Math. Anal. Appl. **278** (2003), 1–17.

[5] I. Fabbri, G. Mancini, K. Sandeep, *Classification of solutions of a critical Hardy-Sobolev operator*, J. Differential Equations **224** (2006) 2, 258–276.

[6] B. Gidas, W.N. Ni, L. Nirenberg, *Symmetries and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[7] B. Gidas, J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations **8** (1981), 883–901.

[8] M. Hénon, *Numerical experiments on the stability of spherical stellar systems*, Astronom. Astrophys. **24** (1973), 229–238.

[9] Q. Han, F.H. Lin, *Elliptic Partial Differential Equations*, AMS Providence, Rhode Island, 2000.

[10] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, Berlin, 1983.

[11] W.M. Ni, *A nonlinear Dirichlet problem on the unit ball and its applications*, Indiana Univ. Math. J **31** (1982), 801–807.

[12] E. Serra, *Non radial positive solutions for the Hénon equation with critical growth*, Calc. Var. Partial Differential Equations **23** (2005) 3, 301–326.

[13] D. Smets, J.B. Su, M. Willem, *Non-radial ground states for the Hénon equation*, Commun. Contemp. Math. **4** (2002), 467–480.

[14] M. Willem, *Minimax Theorems*, Birkhäuser, Basel, 1996.

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