EXISTENCE AND UNIQUENESS THEOREM FOR A HAMMERSTEIN NONLINEAR INTEGRAL EQUATION

A.Kh. Khachatryan, Kh.A. Khachatryan

Abstract. The existence of a solution, as well as some properties of the obtained solution for a Hammerstein type nonlinear integral equation have been investigated. For a certain class of functions the uniqueness theorem has also been proved.

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1. INTRODUCTION

Let us consider the following class of Hammerstein type nonlinear integral equations

$$\varphi(x) = \int_{0}^{+\infty} K(x-t)\varphi^\alpha(t)dt, \quad x \in (0, +\infty), \quad \alpha \in (0, 1),$$

with respect to an unknown function $\varphi(x) \geq 0$. The kernel $K(x) \geq 0$ is an integrable function on $(-\infty, +\infty)$ such that

$$\int_{-\infty}^{+\infty} K(t)dt = 1, \quad \nu = \nu_+ - \nu_- < 0,$$

where $\nu_+ = \int_{0}^{\infty} tK(t)dt < +\infty$ and $\nu_- = \int_{-\infty}^{0} tK(-t)dt < +\infty$.

In the present paper we prove the existence of a positive, monotonic increasing and bounded solution $\varphi(x) \leq 1$. Moreover, we show that $\lim_{x \to +\infty} \varphi(x) = 1$. We also prove that, by putting an additional condition on the kernel, the obtained solution is continuous on $[0, +\infty)$ and unique in a certain class of functions.

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2. PRELIMINARIES

Let $E$ be one of the following Banach spaces: $L^p(0, +\infty)$ for $p \geq 1$, $M[0, +\infty)$, $C_M[0, +\infty)$, $C[0, +\infty)$, where $M[0, +\infty)$ is the space of bounded functions on $[0, +\infty)$, $C_M[0, +\infty)$ is the space of continuous and bounded functions on $[0, +\infty)$, and finally $C_0[0, +\infty)$ is the space of continuous functions, possessing zero limit at infinity.

We denote by $K$ the Wiener-Hopf type integral operator with the kernel $K(x)$

$$
(Kf)(x) = \int_0^\infty K(x-t)f(t)dt, \quad x > 0, \quad f \in E, \quad K : E \to E. \quad (2.1)
$$

It is known (see [1, §1, Theorem 1.1]) that given condition (1.2) the operator $I - K$ permits the following Volterra factorization

$$
I - K = (I - V_-)(I - V_+) \quad (2.2)
$$

as an equality of operators acting in space $E$. Here

$$
(V_-f)(x) = \int_x^\infty v_-(t-x)f(t)dt, \quad (V_+f)(x) = \int_0^x v_+(x-t)f(t)dt, \quad (2.3)
$$

where $0 \leq v_\pm \in L_1(0, +\infty)$, and

$$
\gamma_- = \int_0^\infty v_-(x)dx = 1, \quad \gamma_+ = \int_0^\infty v_+(x)dx < 1. \quad (2.4)
$$

The existence of the solution of the corresponding linear equation

$$
S(x) = \int_0^\infty K(x-t)S(t)dt, \quad x > 0 \quad (2.5)
$$

was proved in [3]. Using factorization (2.2), it was proved that the problem (2.5), such that (1.2) holds, has a positive solution, possessing the following properties (see [1, §3, p. 188]):

(a) $1 \leq S(x) \leq (1 - \gamma_+)^{-1}, \quad x > 0$,
(b) $S(x) \uparrow$ by $x$ on $[0, +\infty)$, i.e. $S(x)$ is increasing on $[0, +\infty)$,
(c) $\lim_{x \to +\infty} S(x) = (1 - \gamma_+)^{-1}$. 

3. BASIC RESULT

We introduce the following iteration for equation (1.1):

$$
\varphi_{n+1}(x) = \int_0^{+\infty} K(x-t)\varphi^n_{\alpha}(t)dt, \quad x > 0, \quad \alpha \in (0,1), \quad n = 0, 1, 2, \ldots, \quad (3.1)
$$

$$
\varphi_0(x) \equiv 1, \quad x > 0.
$$

By induction, it is easy to check that the following statements are true:

**j1)** $\varphi_n(x)$ ↓ by $n$,

**j2)** $\varphi_n(x) \geq (1 - \gamma_+)S(x), \quad n = 0, 1, 2, \ldots$

**j3)** $\varphi_n(x)$ ↑ by $x$ on $[0, +\infty), \quad n = 0, 1, 2, \ldots$

For example, we prove **j2** and **j3**. When $n = 0$, inequality **j2** immediately follows from the double inequality $1 \leq S(x) \leq (1 - \gamma_+)^{-1}$. Assuming that $\varphi_n(x) \geq (1 - \gamma_+)S(x)$ we have

$$
\varphi_{n+1}(x) \geq (1 - \gamma_+)^\alpha \int_0^{+\infty} K(x-t)S^n(t)dt \geq (1 - \gamma_+) \int_0^{+\infty} K(x-t)S(t)dt = (1 - \gamma_+)S(x),
$$

because $\alpha \in (0,1)$ and $0 < (1 - \gamma_+)S(x) \leq 1$.

Now we prove statement **j3**. Let $x_1, x_2 \in [0, +\infty)$ be arbitrary numbers such that $x_1 > x_2$. We may rewrite iteration (3.1) in the following form:

$$
\varphi_{n+1}(x) = \int_{-\infty}^{x} K(\tau)\varphi^n_{\alpha}(x-\tau)d\tau, \quad n = 0, 1, 2, \ldots, \quad \varphi_0(x) \equiv 1,
$$

It is obvious that $\varphi_0(x)$ is increasing by $x$. Assuming that $\varphi_n(x)$ is an increasing function by $x$ we have

$$
\varphi_{n+1}(x_1) - \varphi_{n+1}(x_2) = \int_{-\infty}^{x_1} K(t)\varphi^n_{\alpha}(x_1-t)dt - \int_{-\infty}^{x_2} K(t)\varphi^n_{\alpha}(x_2-t)dt \geq \int_{-\infty}^{x_1} K(t)\varphi^n_{\alpha}(x_2-t)dt - \int_{-\infty}^{x_2} K(t)\varphi^n_{\alpha}(x_2-t)dt = \int_{x_1}^{x_2} K(t)\varphi^n_{\alpha}(x_2-t)dt \geq 0.
$$

We proved that **j3** holds.

It follows from **j1** and **j2** that the sequence of functions $\{\varphi_n(x)\}_{n=0}^{\infty}$ has the pointwise limit

$$
\lim_{n \to \infty} \varphi_n(x) = \varphi(x) \leq 1. \quad (3.2)
$$
From B. Levi’s theorem (see [2]) we deduce that the limit function satisfies equation (1.1). It follows from (3.3) that

$$\phi(x) \uparrow \text{by } x \text{ on } (0, +\infty).$$

(3.3)

Taking into account $j_{2}$ and (3.2) we obtain the following double inequalities:

$$1 - \gamma_{+} \leq (1 - \gamma_{+})S(x) \leq \phi(x) \leq 1,$$

(3.4)

$$\lim_{x \to \infty} \phi(x) = 1.$$ 

(3.5)

Now we prove that if

$$0 < \gamma_{+} < 1 - \frac{1}{e},$$

(3.6)

then $\phi \in C[0, +\infty)$ and a solution of equation (1.1) in the following class of functions

$$\mathfrak{N} = \{ f \in M[0, +\infty) : f(x) \geq 1 - \gamma_{+} \text{ for all } x \in [0, +\infty) \}$$

(3.7)

is unique.

First we show the continuity of the obtained solution assuming that condition (3.6) is fulfilled. By induction, we show that the following inequality holds

$$|\phi_{n+1}(x) - \phi_{n}(x)| \leq (\alpha e^{1-\alpha})^{n}, \quad n = 0, 1, 2, \ldots.$$ 

(3.8)

In the case of $n = 0$ the inequality is obvious, because

$$|\phi_{1}(x) - \phi_{0}(x)| = 1 - \int_{-\infty}^{x} K(t)dt \leq 1.$$ 

Assume that (3.8) is true for any $n = p \in \mathbb{N}$. Taking into account the inequality

$$|x_{1}^\alpha - x_{2}^\alpha| \leq \alpha \left( \frac{1}{1 - \gamma_{+}} \right)^{1-\alpha} |x_{1} - x_{2}| \quad \text{for all } x_{1}, x_{2} \in [1 - \gamma_{+}, +\infty)$$

(3.9)

we obtain from (3.1) that

$$|\phi_{p+2}(x) - \phi_{p+1}(x)| \leq \int_{0}^{+\infty} K(x-t)|\phi_{p+1}^\alpha(t) - \phi_{p}^\alpha(t)|dt \leq$$

$$\leq \alpha \left( \frac{1}{1 - \gamma_{+}} \right)^{1-\alpha} \int_{0}^{+\infty} K(x-t)|\phi_{p+1}(t) - \phi_{p}(t)|dt \leq$$

$$\leq \alpha \left( \frac{1}{1 - \gamma_{+}} \right)^{1-\alpha} \alpha^{p} e^{p-\alpha p} \int_{-\infty}^{x} K(t)dt \leq \alpha^{(p+1)} e^{(1-\alpha)(p+1)}.$$
As $e^{\alpha-1} > \alpha$, $\alpha \in (0, 1)$, then $q = \alpha e^{1-\alpha} \in (0, 1)$. Therefore, in accordance with the Weierstrass theorem, from (3.8) it follows that the convergence of sequences of functions $\{\varphi_n(x)\}_{n=0}^{\infty}$ is uniform. By induction, the reader may easily convince himself that $\varphi_n(x) \in C[0, +\infty)$. Thus, from the Dini inverse theorem it follows that the limit function $\varphi$ is continuous.

Now we prove uniqueness of a solution of equation (1.1) in the class $M$. We assume that equation (1.1) has two different solutions $\varphi$ and $\varphi^*$, which belong to $M$. Then from (1.1), (3.6) and (3.9) we have

$$|\varphi(x) - \varphi^*(x)| \leq \alpha e^{1-\alpha} \int_0^{+\infty} K(x-t)|\varphi(t) - \varphi^*(t)|dt.$$  \hspace{1cm} (3.10)

We set

$$\delta = \sup_{x \in \mathbb{R}^+} |\varphi(x) - \varphi^*(x)|.$$  

Then from (3.10) we infer that $\delta \leq q\delta$ or $\delta = 0$. Therefore, $\varphi(x) = \varphi^*(x)$. In this way we prove that the following theorem holds.

**Theorem 3.1.** Assume that condition (1.2) is fulfilled. Then equation (1.1) has a positive, monotonic increasing and bounded solution $\varphi(x)$ such that $\lim_{x \to +\infty} \varphi(x) = 1$. Moreover, if condition (3.6) holds then the obtained solution is continuous and unique in the class $M$.

**Example 3.2.** Assume that $K(x)$ has the following form:

$$K(x) = \begin{cases} \beta e^{-x}; & x > 0 \\ (1 - \beta)e^x; & x < 0 \end{cases} \quad \beta \in \left(0, \frac{1}{2}\right).$$  \hspace{1cm} (3.11)

Opening brackets in (2.2), from operator equality we come to Yengibaryan’s nonlinear factorization equation (see [1]).

$$v_{\pm}(x) = K(\pm x) + \int_0^{+\infty} v_{\mp}(t)v_{\pm}(x+t)dt, \quad x > 0.$$  \hspace{1cm} (3.12)

From (3.11) and (3.12) it follows that $v_+ = 2\beta e^{-x}$ ($x > 0$), $v_- = e^x$ ($x < 0$), i.e. $\gamma_+ = 2\beta$, $\gamma_- = 1$. If $\beta \in \left(0, \frac{1}{2}\left(1 - \frac{1}{2}\right)\right)$, then both conditions (1.2) and (3.6) are fulfilled. Equation (1.1) with kernel (3.11) is reduced to the following ordinary differential equation

$$\varphi''(x) + (1 - 2\beta)\alpha \varphi^{1-\alpha}(x)\varphi'(x) - \varphi(x) = 0.$$  \hspace{1cm} (3.13)

From the proof it follows that equation (3.13) possesses positive, bounded and monotonic increasing solution, which tends to 1 when $x \to +\infty$.

**Remark 3.3.** It should be noted that if we assume a weaker condition $0 < \gamma_+ < \left(1 - \frac{1}{2}\right)^{1/\alpha}$ instead of (3.6) then the assertion of the theorem remains true.
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REFERENCES


Aghavard Khachatryan
Aghavard@hotbox.ru

National Academy of Sciences Republic of Armenia
Institute of Mathematics
Marshal Bagramyan str. 24b
0019 Yerevan, Republic of Armenia

Khachatur Khachatryan
Khach82@rambler.ru

National Academy of Sciences Republic of Armenia
Institute of Mathematics
Marshal Bagramyan str. 24b
0019 Yerevan, Republic of Armenia

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