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MONOTONE ITERATIVE TECHNIQUE FOR FINITE SYSTEMS OF NONLINEAR RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Comparison results of the nonlinear scalar Riemann-Liouville fractional differential equation of order q, $0 < q \leq 1$, are presented without requiring Hölder continuity assumption. Monotone method is developed for finite systems of fractional differential equations of order q, using coupled upper and lower solutions. Existence of minimal and maximal solutions of the nonlinear fractional differential system is proved.

Keywords: fractional differential systems, coupled lower and upper solutions, mixed quasi-monotone property.

Mathematics Subject Classification: 34A08, 24A34.

1. INTRODUCTION

Fractional differential equations or fractional differential systems have numerous applications in diverse and widespread fields of science and engineering. See [2, 5, 6] for details.The approach to obtain existence and uniqueness of solutions for the nonlinear fractional differential systems in general has been through fixed point theorem methods. In this paper we develop monotone method combined with the method of coupled upper and lower solutions for fractional differential systems with initial conditions.

The monotone method is useful for nonlinear equations and systems because it reduces the problem to sequences of linear equations. Specifically, if the nonlinear system is unwieldy, either too difficult or impossible to solve explicitly, then the monotone method may be beneficial. If one can find upper and lower solutions to the original system that are less unwieldy and satisfy the particular requirements, then the monotone method implements a technique for constructing sequences from these upper and lower solutions. These sequences are solutions to linear equations and converge uniformly and monotonically to either a unique solution, maximal and minimal solutions, or coupled maximal and minimal solutions. Practically, one could numerically approximate solutions to within a desired tolerance with this iterative method.

The nature of fractional calculus complicates this method somewhat, as does the generalization of the nonlinear systems that we consider. Here the nonlinear terms satisfy the mixed quasimonotone property, which generalizes results related to the quasimonotone property. Further, we do not get direct uniform convergence when considering the fractional derivative. Instead, using coupled lower and upper solutions v_0 and w_0, we develop sequences $\{t^p v_n\}$ and $\{t^p w_n\}$ which converge uniformly and monotonically to $t^p v$ and $t^p w$. Here v and w are the coupled extremal solutions of the nonlinear system satisfying the initial condition.

In order to develop our main result we modify the existing comparison result. The modification in our comparison result has the advantage of not requiring Hölder continuity of the order $\lambda > q$ (where q is the fractional order of the system) for the coupled upper and lower solutions. It is to be noted, that in general we cannot prove that the iterates we develop in the monotone method possess this Hölder continuity property. We prove the comparison result by using the C_p continuity property of the coupled lower and upper solutions. Also we have recalled and modified the proof for the existence of a solution of the scalar fractional differential equation.

2. PRELIMINARY RESULTS

In this section we consider results for the Riemann-Liouville (R-L) fractional integral of order $q \in \mathbb{R}^+$. We then consider results for the R-L differential equation of order q, where $0 < q \leq 1$. In the next sections we will apply the results to finite systems of fractional differential equations of order q. In the direction of proving these results we will need the following definitions. Note for simplicity we will only consider results on an interval $(0, T]$, where $T > 0$.

Definition 2.1. Let $q \in \mathbb{R}^+$ and let $p = m - q$ where $m = \text{ceil}(q) = [q] + 1$. Let $J = (0, T]$. Then a function $\phi(t) \in C(J, \mathbb{R})$ is a C_p function if $t^p \phi(t) \in C(\bar{J}, \mathbb{R})$. The set of C_p functions is denoted $C_p(J, \mathbb{R})$. Further, given a function $\phi(t) \in C_p(J, \mathbb{R})$ we call the function $t^p \phi(t)$ the continuous extension of $\phi(t)$.

Definition 2.2. Let $\phi \in C_p(J, \mathbb{R})$, then $D_t^{-q} \phi(t)$ is the q-th R-L integral of ϕ with respect to t defined as

$$D_t^{-q} \phi(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \phi(s)\, ds.$$

Definition 2.3. Let $\phi \in C_p(J, \mathbb{R})$, then $D_t^q \phi(t)$ is the q-th R-L derivative of ϕ with respect to t defined as

$$D_t^q \phi(t) = \frac{1}{\Gamma(p)} \frac{d^m}{dt^m} \int_0^t (t - s)^{-q} \phi(s)\, ds.$$

Note that in cases where the initial value may be different, or ambiguous, we will write out the definition explicitly. The next definition is related to the solution of linear R-L fractional differential equations and is also of great importance in the study of the R-L derivative.

Definition 2.4. The Mittag-Leffler function with parameters α and β, denoted $E_{\alpha,\beta}$, is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

which is entire for $\alpha, \beta > 0$.

The next result gives us that the q-th R-L integral of a C_p function is also a C_p function. This result will give us that the solutions of R-L differential equations are also C_p continuous.

Lemma 2.5. *Let $q \in \mathbb{R}^+$, then $f \in C_p(J, \mathbb{R})$ implies that $D_t^{-q} f(t) \in C_p(J, \mathbb{R})$.*

Proof. Since f is C_p continuous we have that $|t^p f(t)|$ is bounded by some $M > 0$ for all $t \in \bar{J}$. Let $t, \tau \in \bar{J}$ and, without loss of generality, suppose that $t \geq \tau > 0$. Then we have that

$$\int_\tau^t (t-s)^{q-1} s^{q-m}\, ds \leq \frac{1}{\tau^p} \int_\tau^t (t-s)^{q-1}\, ds.$$

Now, we will consider the case where $q \geq 1$. Note in this case

$$t^p (t-s)^{q-1} - \tau^p (\tau - s)^{q-1} \geq 0,$$

where $\tau > s$. Then we obtain

$$\begin{aligned}
&\left| t^p D_t^{-q} f(t) - \tau^p D_\tau^{-q} f(\tau) \right| \leq \\
&\leq \frac{t^p}{\Gamma(q)} \int_\tau^t (t-s)^{q-1} \left| f(s) \right| ds + \frac{1}{\Gamma(q)} \int_0^\tau \left| t^p (t-s)^{q-1} - \tau^p (\tau - s)^{q-1} \right| \left| f(s) \right| ds = \\
&= \frac{t^p}{\Gamma(q)} \int_\tau^t (t-s)^{q-1} \left| f(s) \right| s^p s^{-p}\, ds + \\
&\quad + \frac{1}{\Gamma(q)} \int_0^\tau [t^p (t-s)^{q-1} - \tau^p (\tau - s)^{q-1}] \left| f(s) \right| s^p s^{-p}\, ds \leq \\
&\leq \frac{M}{\Gamma(q)} \frac{t^p}{\tau^p} \int_\tau^t (t-s)^{q-1}\, ds + \frac{M}{\Gamma(q)} \int_0^\tau [t^p (t-s)^{q-1} - \tau^p (\tau - s)^{q-1}] s^{q-m}\, ds = \\
&= \frac{M}{\Gamma(q+1)} \frac{t^p}{\tau^p} (t-\tau)^q - \frac{M \Gamma(q-m+1)}{\Gamma(2q-m+1)} \tau^q + \frac{M}{\Gamma(q)} t^q \int_0^{\frac{\tau}{t}} (1-\alpha)^{q-1} \alpha^{q-m}\, d\alpha,
\end{aligned}$$

where $\alpha = \frac{s}{t}$. Now note that as $t \to \tau$,

$$\frac{M}{\Gamma(q)} t^q \int_0^{\frac{\tau}{t}} (1-\alpha)^{q-1} \alpha^{q-1} \, d\alpha \to \frac{M\Gamma(q-m+1)}{\Gamma(2q-m+1)} \tau^q,$$

implying that $t^p D_t^{-q} f(t) \to \tau^p D_\tau^{-q} f(\tau)$ as $t \to \tau$, provided that $q \geq 1$. This proves that $D_t^{-q} f(t)$ is C_p continuous as long as $q \geq 1$ and $\tau > 0$.

Now we will consider the case where $q < 1$. First, in this case $m = 1$, implying that $p = 1 - q$. Next we will show that in this case

$$t^p (t-s)^{q-1} - \tau^p (\tau - s)^{q-1} \leq 0,$$

where $\tau > s$. To show this, consider the function

$$\phi(t) = t^p (t-s)^{q-1} = t^p (t-s)^{-p},$$

with $t > s \geq 0$. This function is nonincreasing in t on J, since

$$\frac{d}{dt}\phi(t) = pt^{p-1}(t-s)^{-p} - pt^p (t-s)^{-p-1} = -t^{p-1}(t-s)^{-p-1} ps \leq 0$$

for all $t \in J$. Thus implying that

$$\phi(t) - \phi(\tau) \leq 0.$$

From here, following the same process as before, we instead obtain

$$\left| t^p D_t^{-q} f(t) - \tau^p D_\tau^{-q} f(\tau) \right| \leq \frac{M}{\Gamma(q)} \frac{t^p}{\tau^p} \int_\tau^t (t-s)^{q-1} \, ds - \\ - \frac{M}{\Gamma(q)} \int_0^\tau [t^p (t-s)^{q-1} - \tau^p (\tau - s)^{q-1}] s^{q-m} \, ds.$$

This is analogous to the above result except for the negative second term. The remainder of the proof of this case follows exactly as before. Therefore $D_t^{-q} f(t)$ is C_p continuous for any $q > 0$, provided $\tau > 0$.

Finally, the case where $\tau = 0$ follows from the fact that

$$\left| t^p D_t^{-q} f(t) \right| \leq \frac{M\Gamma(q-m+1)}{\Gamma(2q-m+1)} t^q.$$

Thus proving that $D_t^{-q} f(t) \in C_p(J, \mathbb{R})$ for any value of q. □

From now on we choose q, such that $0 < q \leq 1$. Consider the following linear R-L fractional differential equation,

$$D_t^q x = \lambda x + f(t), \tag{2.1}$$

with initial condition

$$\Gamma(q)t^p x(t)\big|_{t=0} = x^0,$$

where λ and x^0 are constants and $f \in C_p(J, \mathbb{R})$.

Theorem 2.6. *The solution x of* (2.1) *exists, is unique, and is in $C_p(J, \mathbb{R})$.*

The proof that $x(t)$ exists on J and is unique can be found in [4], further the explicit solution of (2.1) is given as

$$x(t) = x^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s)\, ds. \tag{2.2}$$

That this x is in $C_p(J, \mathbb{R})$ follows from Lemma 2.5.

The next lemma is very similar to that found in [4], but we do not require the function in question to be locally Hölder continuous for our conclusion. Although the first part of the proof will follow on the same lines as that found in [4], we provide our modifications for clarity. The motivation for relaxing the assumptions comes from the requirements of the monotone method. For this iterative method we construct sequences from the solutions of linear R-L differential equations. As displayed previously the solution of equation (2.1) is given by equation (2.2), which can be rewritten as

$$x(t) = \frac{x^0}{\Gamma(q)} t^{q-1} + x^0 \sum_{k=1}^{\infty} \frac{\lambda^k t^{qk+q-1}}{\Gamma(qk+q)} + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s)\, ds.$$

In this form it can be seen that, in general, this function is not locally Hölder continuous of any order due to the term containing t^{q-1}. Therefore we must relax the requirements of the comparison result. Otherwise the resulting iterates of our constructed sequences in the monotone method cannot satisfy the requirements needed for convergence.

Lemma 2.7. *Let $m \in C_p(J, R)$ be such that for some $t_1 \in J$ we have $m(t_1) = 0$ and $m(t) \le 0$ for $t \in (0, t_1]$. Then $D_t^q m(t)\big|_{t=t_1} \ge 0$.*

Proof. Note that

$$D_t^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_0^t (t-s)^{-q} m(s)\, ds,$$

and let

$$H(t) = \int_0^t (t-s)^{-q} m(s)\, ds.$$

Now letting $h > 0$ be sufficiently small and following the same initial steps as in [4] we get

$$H(t_1) - H(t_1 - h) \ge \int_{t_1-h}^{t_1} (t_1 - s)^{-q} m(s)\, ds.$$

Noting that $t^p m(t)$ is continuous on $\bar{J}$ we may choose a $K_h > 0$ such that $|t_1 - s| < h$ implies $|t_1^p m(t_1) - s^p m(s)| < hK_h$, further implying that

$$m(s) > -hK_h s^{q-1}, \quad \text{for } s \in B_h(t_1).$$

This yields

$$\int_{t_1-h}^{t_1} (t_1 - s)^{-q} m(s)\, ds > -hK_h \int_{t_1-h}^{t_1} (t_1 - s)^{-q} s^{q-1}\, ds \geq -\frac{h^{2-q} K_h}{1-q}(t_1 - h)^{q-1}.$$

Thus we get,

$$\frac{H(t_1) - H(t_1 - h)}{h} > -\frac{h^{1-q} K_h}{1-q}(t_1 - h)^{q-1}, \quad \forall h > 0.$$

Which gives us that,

$$D_t^q m(t)\big|_{t=t_1} \geq 0,$$

this completes the proof. □

We will now analyze existence results for the scalar nonlinear R-L differential equation along with results pertaining to lower and upper solutions, that will aid in developing a monotone method for finite systems in the following section. For that purpose consider the R-L equation

$$D_t^q x = f(t, x), \quad \Gamma(q) t^p x(t)\big|_{t=0} = x^0, \tag{2.3}$$

where $f \in C(\bar{J} \times \mathbb{R}, \mathbb{R})$. Note that a solution $x \in C_p(J, \mathbb{R})$ of (2.3) also satisfies the equivalent R-L integral equation

$$x(t) = \frac{x^0}{\Gamma(q)} t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, x(s))\, ds. \tag{2.4}$$

Thus if $f \in C(\bar{J} \times \mathbb{R}, \mathbb{R})$ then (2.3) is equivalent to (2.4). See [2, 4] for details. Next we recall a comparison result, the special case of which is needed in our main result.

Theorem 2.8. *Let $f \in C(\bar{J} \times \mathbb{R}, \mathbb{R})$ and let $v, w \in C_p(J, \mathbb{R})$ be lower and upper solutions of* (2.3)*, i.e.*

$$D_t^q v \leq f(t, v), \quad \Gamma(q) t^p v(t)\big|_{t=0} = v^0 \leq x^0,$$

and

$$D_t^q w \geq f(t, w), \quad \Gamma(q) t^p w(t)\big|_{t=0} = w^0 \geq x^0.$$

If f satisfies the following Lipschitz condition

$$f(t, x) - f(t, y) \leq L(x - y), \quad \text{when } x \geq y,$$

then $v(t) \leq w(t)$ on J.

The proof follows as in [4] with appropriate modifications. For the corresponding case with R-L integral equations see [1]. In our main result, we need the special case of Theorem 2.8, which we state below as a corollary.

Corollary 2.9. *Let $f \in C_p(J,\mathbb{R})$ and let $v, w \in C_p(J,\mathbb{R})$ be lower and upper solutions of the linear equation* (2.1)*, i.e.*

$$D_t^q v \le \lambda v + f(t), \quad \Gamma(q)t^p v(t)\big|_{t=0} = v^0 \le x^0,$$

and

$$D_t^q w \ge \lambda w + f(t), \quad \Gamma(q)t^p w(t)\big|_{t=0} = w^0 \ge x^0.$$

Then $v(t) \le w(t)$ on J.

Note this follows since $F(t,x) = \lambda x + f(t)$ is Lipschitz with respect to x. Now, in the direction of proving an existence result relative to lower and upper solutions we state a Peano's type existence result for equation (2.3).

Theorem 2.10. *Suppose $f \in C(R_0,\mathbb{R})$ and $|f(t,x)| \le M$ on R_0, where*

$$R_0 = \{(t,x) \in \bar{J} \times C_p(J,\mathbb{R}) \ : \ |t^p x(t) - x^0| \le \eta\}.$$

Then the solution of (2.3) *exists on J.*

Although this result has been presented in [4], we require a modification of the set R_0 for our preceding results regarding existence by method of upper and lower solutions.

Proof. That the solution x exists on $(0,\alpha]$, where

$$\alpha = \min\left\{T, \frac{\eta\Gamma(q+1)}{M}\right\},$$

can be found in [4] with slight alterations. We have that since $x \in C_p((0,\alpha],\mathbb{R})$, and by the properties of C_p functions we have that x is defined at $t = \alpha$. Thus choose $x^\alpha \in \mathbb{R}$ such that $x(\alpha) = x^\alpha$, then the IVP

$$^cD_t^q x = f(t,x), \quad x(\alpha) = \alpha \tag{2.5}$$

exists on $[\alpha, \alpha+\alpha_1]$, where $^cD_t^q$ is the q-th Caputo derivative defined as

$$^cD_t^q x(t) = \frac{1}{\Gamma(p)} \int_\alpha^t (t-s)^{-q} x'(s)\, ds.$$

The Caputo differential equation covers the case where x is continuous on a compact interval and has initial condition as defined above. See [4] for more information. The existence of a solution to (2.5) where f is continuous and bounded on a rectangle follows in the same way as in the integer order case. From here, that the solution of (2.3) can be extended to J can be proved in the same way as in the integer order case. □

Now, if we know of the existence of lower and upper solutions v and w such that $v \leq w$, we can prove the existence of a solution in the set

$$\Omega = \{(t,y) \,:\, t^p v(t) \leq y \leq t^p w(t),\, t \in \bar{J}\}.$$

Note, for this result, we require f to be continuous on a compact set. The set Ω is defined in terms of the continuous extensions of v and w as a consequence of this requirement. We consider this result in the following theorem.

Theorem 2.11. *Let $v, w \in C_p(J, \mathbb{R})$ be lower and upper solutions of* (2.3) *such that $v(t) \leq w(t)$ on J and let $f \in C(\Omega, \mathbb{R})$, where Ω is defined as above. Then there exists a solution $x \in C_p(J, \mathbb{R})$ of* (2.3) *such that $v(t) \leq x(t) \leq w(t)$ on J.*

Proof. By the continuity of f on Ω there exists a function F such that $f(t,x) = F(t, t^p x)$. Now consider the function μ defined by

$$\mu(t,x) = \max\{t^p v(t), \min\{t^p x(t), t^p w(t)\}\},$$

and note that by the definition of μ we have,

$$t^p v(t) \leq \mu(t,x) \leq t^p w(t).$$

Therefore by Theorem 2.10 the R-L differential equation

$$D_t^q x = F(t, \mu(t,x)), \quad \Gamma(q) t^p x(t)\big|_{t=0} = x^0 \tag{2.6}$$

has a solution $x \in C_p(J, \mathbb{R})$.
Now we wish to show that $v(t) \leq x(t) \leq w(t)$ on J, where x is any solution of (2.6). To do so consider the functions v_ε and w_ε defined as follows

$$v_\varepsilon(t) = v(t) - \varepsilon\psi(t), \text{ and } w_\varepsilon(t) = w(t) + \varepsilon\psi(t),$$

where

$$\psi(t) = \frac{t^{q-1}}{\Gamma(q)} + \frac{t^q}{\Gamma(q+1)}.$$

We claim that $t^p v_\varepsilon(t) < t^p x(t) < t^p w_\varepsilon(t)$ on J. To prove this first note that

$$v_\varepsilon^0 = \Gamma(q) t^p (v(t) - \varepsilon\psi)\big|_{t=0} = v^0 - \varepsilon < v^0 \leq x^0.$$

Similarly $w_\varepsilon^0 > x^0$. Now suppose to the contrary that there exists a $t_1 > 0$ such that $t^p v_\varepsilon(t_1) = t^p x(t_1)$ and since $v_\varepsilon^0 < x^0$ we have that $t^p v_\varepsilon(t) \leq t^p x(t)$ on $[0, t_1]$. So by the continuity of v_ε and x we have that $v_\varepsilon(t) \leq x(t)$ on $(0, t_1]$. Thus letting $m(t) = v_\varepsilon(t) - x(t)$ we have by Lemma 2.7 that $D_t^q m(t)|_{t=t_1} \geq 0$. Thus we have that

$$\begin{aligned} F(t_1, t_1^p v(t_1)) = F(t_1, \mu(t_1, x(t_1))) \leq D_t^q v_\varepsilon(t)|_{t=t_1} = \\ = D_t^q v(t)|_{t=t_1} - \varepsilon D_t^q \psi(t)|_{t=t_1} \leq f(t_1, v(t_1)) - \varepsilon < F(t_1, t_1^p v(t_1)), \end{aligned}$$

which is a contradiction. Therefore $t^p v_\varepsilon(t) < t^p x(t)$ on $\bar{J}$. Similarly it can be shown that $t^p x(t) < t^p w_\varepsilon(t)$ on $\bar{J}$. Thus letting $\varepsilon \to 0$, and applying the continuity of v, x and w we get $v(t) \leq x(t) \leq w(t)$ on J. □

3. MONOTONE ITERATIVE TECHNIQUE

In this section we discuss the results for finite systems of R-L fractional differential equations of order q. For simplicity suppose that the subscript $i \in \{1, 2, 3, \dots, N\}$, and suppose that for any two vectors x and y, $x \leq y$ implies that $x_i \leq y_i$ for each i. We can extend the results of Theorems 2.10 and 2.11 to the nonlinear R-L fractional differential equation of the form

$$D_t^q x = f(t, x), \tag{3.1}$$

where $f \in C(\bar{J} \times \mathbb{R}^N, \mathbb{R}^N)$ with initial condition

$$\Gamma(q) t^p x(t)\big|_{t=0} = x^0.$$

Written in component form, equation (3.1) becomes

$$D_t^q x_i = f_i(t, x), \quad \Gamma(q) t^p x_i(t)\big|_{t=0} = x_i^0. \tag{3.2}$$

As in the integer order case, to develop a monotone iterative technique for the fractional system (3.1) we require the following generalizing concepts. For each fixed $i \in \{1, 2, 3, \dots N\}$, let r_i, s_i be two nonnegative integers such that $r_i + s_i = N - 1$ so that we can split the vector x into $x = (x_i, [x]_{r_i}, [x]_{s_i})$. Then system (3.1) can be written as

$$D_t^q x_i = f_i(t, x_i, [x]_{r_i}, [x]_{s_i}), \quad \Gamma(q) t^p x_i(t)\big|_{t=0} = x_i^0. \tag{3.3}$$

Definition 3.1. Let $v, w \in C_p(J, \mathbb{R}^N)$, v and w are coupled lower and upper quasisolutions of (3.3) if

$$D_t^q v_i \leq f_i(t, v_i, [v]_{r_i}, [w]_{s_i}), \quad \Gamma(q) t^p v_i(t)\big|_{t=0} = v_i^0 \leq x_i^0,$$

$$D_t^q w_i \geq f_i(t, w_i, [w]_{r_i}, [v]_{s_i}), \quad \Gamma(q) t^p w_i(t)\big|_{t=0} = w_i^0 \geq x_i^0.$$

On the other hand, v and w are coupled quasisolutions of (3.1) if

$$D_t^q v_i = f_i(t, v_i, [v]_{r_i}, [w]_{s_i}), \quad \Gamma(q) t^p v_i(t)\big|_{t=0} = x_i^0,$$

$$D_t^q w_i = f_i(t, w_i, [w]_{r_i}, [v]_{s_i}), \quad \Gamma(q) t^p w_i(t)\big|_{t=0} = x_i^0.$$

Further, one can define coupled extremal quasisolutions of (3.3) in the usual way.

Definition 3.2. A function $f \in C(\bar{J} \times \mathbb{R}^N, \mathbb{R}^N)$ possesses a mixed quasimonotone property if for each i, $f_i(t, x_i, [x]_{r_i}, [x]_{s_i})$ is monotone nondecreasing in $[x]_{r_i}$ and monotone nonincreasing in $[x]_{s_i}$.

A special case of the mixed quasimonotone property, specifically when either r_i or s_i is equal to zero, is defined below.

Definition 3.3. A function $f \in C(\bar{J} \times \mathbb{R}^N, \mathbb{R}^N)$ is quasimonotone nondecreasing (nonincreasing) if for each i, $x \leq y$ and $x_i = y_i$, then $f_i(t, x) \leq f_i(t, y)$ ($f_i(t, x) \geq f_i(t, y)$).

Next we state our main result. Using coupled lower and upper solutions relative to (3.3), we construct monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ such that $t^p v_n$ and $t^p w_n$ converge uniformly and monitonically to $t^p v$ and $t^p w$ respectively, where v and w are coupled minimal and maximal solutions of system (3.3).

Theorem 3.4. *Let $f \in C(J \times \mathbb{R}^N, \mathbb{R}^N)$ possess a mixed quasimonotone property and let v_0, w_0 be coupled lower and upper quasisolutions of system* (3.3) *such that $v_0 \le w_0$ on J. Suppose f also satisfies*

$$f_i(t, x_i, [x]_{r_i}, [x]_{s_i}) - f_i(t, y_i, [x]_{r_i}, [x]_{s_i}) \ge -M_i(x_i - y_i),$$

with $M_i \ge 0$, whenever $v_0^0 \le x^0 \le w_0^0$ and $v_{0i} \le y_i \le x_i \le w_{0i}$ on J. Then there exist monotone sequences $\{v_n\}, \{w_n\}$ such that

$$t^p v_n(t) \to t^p v(t),$$

and

$$t^p w_n(t) \to t^p w(t)$$

monotonically and uniformly on $\bar{J}$, where v and w are coupled minimal and maximal quasisolutions of (3.3) *provided $v_0^0 \le x^0 \le w_0^0$. Further if x is any solution of* (3.3) *such that $v_0 \le x \le w_0$ then $v \le x \le w$ on J.*

Proof. For any $\eta, \mu \in C_p(J, \mathbb{R}^N)$ such that $v_0 \le \eta, \mu \le w_0$ on J, define the function F as

$$F_i(t, x) = f_i(t, \eta_i, [\eta]_{r_i}, [\mu]_{s_i}) - M_i(x_i - \eta_i),$$

and consider the uncoupled linear fractional system

$$D_t^q x_i = F_i(t, x), \quad \Gamma(q) t^p x_i(t)\big|_{t=0} = x_i^0. \tag{3.4}$$

For each i (3.4) is a linear fractional differential equation of the form of (2.1). Therefore for each η, μ, there exists a unique solution $x(t) \in C_p(J, \mathbb{R}^N)$ of (3.4). So for each $\eta, \mu \in C_p(J, \mathbb{R}^N)$ with $v_0 \le \eta, \mu \le w_0$ on J, we can define the map A, where $A(\eta, \mu)$ is the unique solution of (3.4).

Note that A will define our sequences $\{v_n\}$, and $\{w_n\}$, in this direction, first note that $v_0 \le A(v_0, w_0)$ and $w_0 \ge A(w_0, v_0)$. To prove this let $v_1 = A(v_0, w_0)$, then note that

$$D_t^q v_{1i} = F_i(t, v_1),$$

and

$$D_t^q v_{0i} \le f_i(t, v_{0i}, [v_0]_{r_i}, [w_0]_{s_i}) - M_i(v_{0i} - v_{0i}) = F_i(t, v_0).$$

Therefore by applying Corollary 2.9 for each i, we have that $v_0 \le v_1 = A(v_0, w_0)$ on J. Further, letting $w_1 = A(w_0, v_0)$ and with a similar argument we have that $w_1 \le w_0$ on J. Now note that A possesses a mixed monotone property, to prove this, let $\eta, \xi, \mu \in C_p(J, \mathbb{R}^N)$ such that $v_0 \le \eta, \xi, \mu \le w_0$, and $\eta \le \xi$ on J. Now let $y = A(\eta, \mu)$ and $x = A(\xi, \mu)$. Then note that

$$D_t^q x_i = f_i(t, \xi_i, [\xi]_{r_i}, [\mu]_{s_i}) - M_i(x_i - \xi_i),$$

and by applying the mixed quasimonotone and Lipschitz properties of f,

$$D_t^q y_i = f_i(t, \eta_i, [\eta]_{r_i}, [\mu]_{s_i}) - M_i(y_i - \eta_i) \leq f_i(t, \xi_i, [\xi]_{r_i}, [\mu]_{s_i}) - M_i(y_i - \xi_i).$$

Thus by Corollary 2.9 we have $y \leq x$ on J. Proving that A is monotone nondecreasing in its first variable. By a similar argument we can prove that A is monotone nonincreasing in its second variable, proving that A possesses a mixed monotone property on J.

Now define the following sequences of C_p continuous functions

$$v_n = A(v_{n-1}, w_{n-1}), \text{ and } w_n = A(w_{n-1}, v_{n-1}).$$

Note that $\{v_n\}$ and $\{w_n\}$ are monotone nondecreasing and nonincreasing on J respectively. We will prove this by induction. Note we previously proved the basis step that $v_0 \leq v_1$ and $w_1 \leq w_0$ on J. So suppose that the hypothesis is true up to some $k \geq 1$, then by the mixed monotone property of A and applying the induction hypothesis we have

$$v_{k+1} = A(v_k, w_k) \geq A(v_{k-1}, w_{k-1}) = v_k,$$

on J. Similarly we can show that $w_{k+1} \leq w_k$ on J. Therefore by induction we have that $\{v_n\}$ and $\{w_n\}$ are monotone on J.

Now we wish to show that $v_n \leq w_n$ for all $n \geq 0$ on J. Note that $v_0 \leq w_0$ by definition, so suppose that the above hypothesis is true up to some $k \geq 1$, then by the mixed monotone property of A and applying the induction hypothesis we have

$$v_{k+1} \leq A(w_k, v_k) = w_{k+1}$$

on J, which by induction implies that $v_n \leq w_n$ on J for all $n \geq 1$.

Now following the same process as found in [7] modified slightly to include finite systems we get that $\{t^p v_n(t)\}$ and $\{t^p w_n(t)\}$ converge uniformly on $\bar{J}$. Therefore, we have that

$$t^p v_{ni}(t) = \frac{x_i^0}{\Gamma(q)} + \frac{t^p}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_i(t, v_{n-1i}, [v_{n-1}]_{r_i}, [w_{n-1}]_{s_i})\, ds -$$
$$- \frac{t^p}{\Gamma(q)} \int_0^t (t-s)^{q-1} M_i(v_{ni} - v_{n-1i})\, ds$$

converges uniformly and monotonically to

$$t^p v_i(t) = \frac{x_i^0}{\Gamma(q)} + \frac{t^p}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_i(t, v_i, [v]_{r_i}, [w]_{s_i})\, ds$$

on $\bar{J}$. Implying that

$$v_i(t) = \frac{x_i^0}{\Gamma(q)} t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f_i(t, v_i, [v]_{r_i}, [w]_{s_i})\, ds$$

on J, implying, along with a similar result for $\{t^p w(t)\}$, that

$$D_t^q v_i = f_i(t, v_i, [v]_{r_i}, [w]_{s_i}), \quad \Gamma(q) t^p v(t)|_{t=0} = x^0,$$

and

$$D_t^q w_i = f_i(t, w_i, [w]_{r_i}, [v]_{s_i}), \quad \Gamma(q) t^p w(t)|_{t=0} = x^0.$$

Therefore implying that v and w are coupled quasisolutions of (3.3) on J. Now we will show that v and w are extremal, to do this let x be a solution of (3.3) with $v_0^0 \leq x^0 \leq w_0^0$. We wish to show that $v_n \leq x \leq w_n$ on J. Suppose that x is a solution to (3.1), where $v_0 \leq x \leq w_0$. Letting this be our basis step, suppose our hypothesis is true up to some $k \geq 1$, then by the mixed monotone property of A and applying the induction hypothesis we have

$$v_{k+1} = A(v_k, w_k) \leq A(x, x) = x$$

on J, and by a symmetric argument we have that $x \leq w_{k+1}$ on J. Therefore, by induction, we have that $v_n \leq x \leq w_n$ on J implying that $t^p v \leq t^p x \leq t^p w$ on $\bar{J}$. By the continuity of v, x, and w this proves that $v \leq x \leq w$ on J. Thus v and w are coupled extremal solutions. □

Note that Theorem 3.4 is an extension of Theorem 1.4.1 in [3] to the R-L fractional differential system (3.3). Further, if $s_i = 0$, then v and w are coupled minimal and maximal solutions of (3.3) thus covering the case when f is quasimonotone nondecreasing. Also, if $r_i = 0$ then v and w are the most general coupled extremal solutions. In addition, if f satisfies uniqueness condition then we can prove that $v = w = x$ is the unique solution of (3.3).

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