STABILITY
OF THE POPOVICIU TYPE FUNCTIONAL EQUATIONS
ON GROUPS

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Abstract. We consider the stability problem for a class of functional equations related to the Popoviciu equation.

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1. INTRODUCTION

Dealing with some inequality for convex functions, T. Popoviciu [9] has introduced the functional equation
\[3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) = 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{x+z}{2}\right) + f\left(\frac{y+z}{2}\right)\right].\] (1.1)
The solution and stability of (1.1) have been investigated by J. Brzdęk [2], W. Smajdor [12] and T. Trif [10]. The analogous problems for the following generalization of (1.1)
\[m^2f\left(\frac{x+y+z}{m}\right)+f(x)+f(y)+f(z) = n^2\left[f\left(\frac{x+y}{n}\right)+f\left(\frac{x+z}{n}\right)+f\left(\frac{y+z}{n}\right)\right],\] (1.2)
where \(m\) and \(n\) are nonzero integers, have been studied in [7] (in the case where \(m = 3\) and \(n = 2\)) and [8] (in the case where \(m + 1 = 2n\)). Solutions of the particular case of (1.2), namely
\[f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(x+z) + f(y+z),\] (1.3)
have been considered by P. Kannappan [6]. Stability of (1.3) has been investigated by S.-M. Jung [5] and W. Fechner [4]. Solutions and stability of some further generalization of (1.1) have been investigated by T. Trif [11]. In [3] the general solution of the following functional equation
\[Mf\left(\frac{x+y+z}{m}\right)+f(x)+f(y)+f(z) = N\left[f\left(\frac{x+y}{n}\right)+f\left(\frac{x+z}{n}\right)+f\left(\frac{y+z}{n}\right)\right],\] (1.4)
where \( m, n, M, N \) are arbitrary positive integers, has been determined in the case where the unknown function \( f \) maps a commutative group uniquely divisible by \( m \) and \( n \) into a commutative group uniquely divisible by 2. Let us recall that a group \((X, +)\) is said to be \textit{uniquely divisible} by a given positive integer \( k \) provided, for every \( x \in X \), there exists a unique \( y \in X \) such that \( x = ky \); such an element will be denoted in the sequel by \( \frac{x}{k} \).

In the present paper we study the stability problem for (1.4) in a similar setting. Our work is inspired by the recent paper [1]. In the sequel we assume that \( m, n, M, N \) are positive integers, \((G, +)\) and \((H, +)\) are commutative groups, \((G, +)\) is uniquely divisible by \( m \) and \( n \), \((H, +)\) is uniquely divisible by 2 and \( d \) is a metric on \( H \) such that:

(i) \( d \) is invariant with respect to +, that is

\[
d(u + w, v + w) = d(u, v) \quad \text{for } u, v, w \in H;
\]

(ii) there exists a \( \xi \in (0, 1) \) such that

\[
d\left(\frac{u}{2}, \frac{v}{2}\right) \leq \xi d(u, v) \quad \text{for } u, v \in H;
\]

(iii) \((H, d)\) is a complete metric space.

2. RESULTS

We begin this section with the following simple observation.

**Remark 2.1.** Note that condition (1.5) implies the following two inequalities:

\[
d(u + w, v + r) \leq d(u, v) + d(w, r) \quad \text{for } u, v, w, r \in H;
\]

\[
d(-u, -v) = d(u, v) \quad \text{for } u, v \in H.
\]

In fact, for every \( u, v, w, r \in H \), we have

\[
d(u + w, v + r) = d(u - v, r - w) \leq d(u - v, 0) + d(0, r - w) = d(u, v) + d(w, r)
\]

and

\[
d(-u, -v) = d(-u + (u + v), -v + (u + v)) = d(v, u) = d(u, v).
\]

Note also that, in view of (2.1), by induction we get

\[
d(ku, kv) \leq kd(u, v) \quad \text{for } u, v \in H, \; k \in \mathbb{N}.
\]

In order to prove the stability result for (1.4) we need to recall that a function \( Q : G \to H \) is said to be \textit{quadratic} provided

\[
Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \quad \text{for } x, y \in G;
\]

and a function \( A : G \to H \) is said to be \textit{additive} provided

\[
A(x + y) = A(x) + A(y) \quad \text{for } x, y \in G.
\]
Theorem 2.2. Assume that a function \( f: G \rightarrow H \) satisfies inequality
\[
d\left(M f\left(\frac{x+y+z}{m}\right) + f(x) + f(y) + f(z), \left[f\left(\frac{x+y}{n}\right) + f\left(\frac{x+z}{n}\right) + f\left(\frac{y+z}{n}\right)\right]\right) \leq \delta(x,y,z)
\]
for \( x, y, z \in G \),
(2.4)
where \( \delta: G^3 \rightarrow [0, \infty) \) is an arbitrary function such that, for some \( 0 < \eta < \frac{1}{\xi} \), it holds
\[
\delta(2x, 2y, 2z) \leq \eta \delta(x, y, z) \quad \text{for} \quad x, y, z \in G.
\]
(2.5)
Then there exists a uniquely determined quadratic function \( Q: G \rightarrow H \) and an additive function \( A: G \rightarrow H \) such that
\[
(N - n^2)Q(x) = (M - m^2)Q(x) = 0 \quad \text{for} \quad x \in G,
\]
(2.6)
\[
(Mn + mn - 2mn)A(x) = 0 \quad \text{for} \quad x \in G
\]
(2.7)
and
\[
d(f(x), Q(x) + A(x) + f(0)) \leq \frac{\xi^4}{1 - \eta \xi} \left[\delta(x, x, 0) + \delta(-x, -x, 0) + \delta(2x, 0, 0) + \delta(-2x, 0, 0) + \delta(x, x, -x) + \delta(-x, -x, x) + \frac{\xi^3}{1 - \xi^2 \eta} \left(2 \delta(x, 0, -2x) + \delta(-2x, 0, 2x)\right) + \frac{\xi^3}{1 - \xi^2 \eta} (3 + 4 \xi) \delta(0, 0, 0)\right]
\]
for \( x \in G \).
(2.8)
Proof. Let \( f_e: G \rightarrow H \) and \( f_o: G \rightarrow H \) be given by
\[
f_e(x) := \frac{f(x) + f(-x)}{2} - f(0) \quad \text{for} \quad x \in G
\]
and
\[
f_o(x) := \frac{f(x) - f(-x)}{2} \quad \text{for} \quad x \in G,
\]
respectively. Then \( f_e \) is even, \( f_o \) is odd, \( f_o(0) = f_e(0) = 0 \) and
\[
f(x) = f_e(x) + f_o(x) + f(0) \quad \text{for} \quad x \in G.
\]
(2.9)
Applying (2.4) with \( x = y = z = 0 \), we obtain
\[
d((M + 3)f(0), 3Nf(0)) \leq \delta(0, 0, 0).
\]
(2.10)
Furthermore, by (2.4), we get
\begin{align*}
d\left( Mf\left( \frac{-(x+y+z)}{m} \right) + f(-x) + f(-y) + f(-z), \\
N\left[ f\left( \frac{-(x+y)}{n} \right) + f\left( \frac{-(x+z)}{n} \right) + f\left( \frac{-(y+z)}{n} \right) \right] \right) \leq \delta(-x,-y,-z)
\end{align*}
for \( x, y, z \in G \). Therefore, taking into account (1.6)–(2.1) and (2.10), from (2.4) and (2.11) we derive that
\begin{align*}
d\left( Mf_0\left( \frac{x+y+z}{m} \right) + f_0(x) + f_0(y) + f_0(z), \\
N\left[ f_0\left( \frac{x+y}{n} \right) + f_0\left( \frac{x+z}{n} \right) + f_0\left( \frac{y+z}{n} \right) \right] \right) \leq \xi \Delta(x,y,z) \quad \text{for} \quad x, y, z \in G
\end{align*}
and
\begin{align*}
d\left( Mf_e\left( \frac{x+y+z}{m} \right) + f_e(x) + f_e(y) + f_e(z), \\
N\left[ f_e\left( \frac{x+y}{n} \right) + f_e\left( \frac{x+z}{n} \right) + f_e\left( \frac{y+z}{n} \right) \right] \right) \leq \xi \left( \Delta(x,y,z) + \frac{1}{2} \Delta(0,0,0) \right)
\end{align*}
for \( x, y, z \in G \), where \( \Delta : G^3 \to [0, \infty) \) is given by
\begin{align*}
\Delta(x, y, z) = \delta(x, y, z) + \delta(-x,-y,-z) \quad \text{for} \quad x, y, z \in G.
\end{align*}
Obviously, by (2.5) and (2.14), we get
\begin{align*}
\Delta(2x, 2y, 2z) \leq \eta \Delta(x, y, z) \quad \text{for} \quad x, y, z \in G.
\end{align*}
Moreover, since \( f_0(0) = 0 \), taking in (2.12) \( z = 0 \) and then \( y = z = 0 \), we obtain
\begin{align*}
d\left( Mf_0\left( \frac{x+y}{m} \right) + f_0(x) + f_0(y), \\
N\left[ f_0\left( \frac{x+y}{n} \right) + f_0\left( \frac{x}{n} \right) + f_0\left( \frac{y}{n} \right) \right] \right) \leq \xi \Delta(x,y,0) \quad \text{for} \quad x, y \in G
\end{align*}
and
\begin{align*}
d\left( Mf_0\left( \frac{x}{m} \right) + f_0(x), 2Nf_0\left( \frac{x}{n} \right) \right) \leq \xi \Delta(x,0,0) \quad \text{for} \quad x \in G,
\end{align*}
respectively. Making use of (2.2), from (2.17) we derive that
\begin{align*}
d\left( -Mf_0\left( \frac{x+y}{m} \right) - f_0(x+y), -2Nf_0\left( \frac{x+y}{n} \right) \right) \leq \xi \Delta(x+y,0,0) \quad \text{for} \quad x, y \in G.
\end{align*}
Hence, taking into account (2.16), by (2.1), we get
\begin{align*}
d\left( f_0(x) + f_0(y) - f_0(x+y), N\left[ f_0\left( \frac{x}{n} \right) + f_0\left( \frac{y}{n} \right) - f_0\left( \frac{x+y}{n} \right) \right] \right) \leq \\
\leq \xi \left( \Delta(x,y,0) + \Delta(x+y,0,0) \right) \quad \text{for} \quad x, y \in G.
\end{align*}
On the other hand, as $f_0$ is odd, from (2.12) we derive that
\[
d \left( f_0(x) + f_0(y) - f_0(x + y), N \left[ f_0\left( \frac{x + y}{m} \right) - f_0\left( \frac{x}{n} \right) - f_0\left( \frac{y}{n} \right) \right] \right) = \\
= d\left( MF_0\left( \frac{x + y}{mn} \right) + f_0(x) + f_0(y) + f_0(-(x + y)), \right. \\
N \left[ f_0\left( \frac{x + y}{n} \right) + f_0\left( \frac{y - (x + y)}{n} \right) + f_0\left( \frac{x - (x + y)}{n} \right) \right] \right) \leq \\
\leq \xi \Delta(x, y, -(x + y)) \text{ for } x, y \in G.
\]

Therefore, using (2.1), we get
\[
d(2(f_0(x) + f_0(y) - f_0(x + y)), 0) \leq \xi[\Delta(x, y, 0) + \Delta(x + y, 0, 0) + \Delta(x, y, -(x + y))] \\
\text{for } x, y \in G, \text{ whence by (1.5) and (1.6), we obtain}
\]
\[
d(f_0(x + y), f_0(x) + f_0(y)) \leq \xi^2[\Delta(x, y, 0) + \Delta(x + y, 0, 0) + \Delta(x, y, -(x + y))] \\
\text{for } x, y \in G. \text{ Note also that in view of (2.15), the function } \chi_0 : G^2 \to [0, \infty) \text{ given by}
\]
\[
\chi_0(x, y) := \xi^2[\Delta(x, y, 0) + \Delta(x + y, 0, 0) + \Delta(x, y, -(x + y))] \text{ for } x, y \in G \quad (2.18)
\]
satisfies
\[
\chi_0(2x, 2y) \leq \eta \chi_0(x, y) \text{ for } x, y \in G. \quad (2.19)
\]

So, applying [1, Corollary 3.2], we conclude that there exists a unique additive function $A : G \to H$ such that
\[
d(f_0(x), A(x)) \leq \frac{\xi^2}{1 - \xi \eta} \chi_0(x, x) \text{ for } x \in G. \quad (2.20)
\]

Moreover, taking into account (2.3), from (2.20) we deduce that
\[
d(f_0(mx), A(mx)) \leq \frac{\xi^2}{1 - \xi \eta} \chi_0(mx, mx) \text{ for } x \in G, \quad (2.21)
\]
\[
d(Mf_0(nx), MA(nx)) \leq \frac{M \xi^2}{1 - \xi \eta} \chi_0(nx, nx) \text{ for } x \in G \quad (2.22)
\]
and
\[
d(2Nf_0(mx), 2NA(mx)) \leq \frac{2N \xi^2}{1 - \xi \eta} \chi_0(mx, mx) \text{ for } x \in G. \quad (2.23)
\]

Making use of (2.1), from (2.21) and (2.22) we derive that
\[
d(Mf_0(nx) + f_0(mx), MA(nx) + A(mx)) \leq \\
\leq \frac{\xi^2}{1 - \xi \eta} (\chi_0(mx, mx) + M \chi_0(nx, nx)) \text{ for } x \in G.
\]
On the other hand, by (2.17), for every \( x \in G \), we get
\[
d(Mf(x) + f_{\text{e}}(nx), 2Nf_{\text{e}}(mx)) \leq \xi (\Delta(mx, 0, 0)).
\]

Therefore, in view of (2.22), for every \( x \in G \), we obtain
\[
d((Mn + mn - 2Nm)A(x), 0) = d(MA(nx) + A(mnx), 2NA(mx)) \leq
\[
\leq \frac{\xi^2}{1 - \xi \eta} [\chi_0(mnx, mnx) + M\chi_0(nx, nx) + 2N\chi_0(mx, mx)] + \xi \Delta(mx, 0, 0).
\]

Thus, as \( A \) is additive, \( \Delta \) satisfies (2.15) and \( \chi_0 \) satisfies (2.19), using (1.5), we get
\[
d((Mn + mn - 2Nm)A(x), 0) = d(2^{-k}(Mn + mn - 2Nm)A(2^k x), 0) \leq
\]
\[
\leq (\xi \eta)^k \left[ \frac{\xi^2}{1 - \xi \eta} [\chi_0(mnx, mnx) + M\chi_0(nx, nx) + 2N\chi_0(mx, mx)]
\right.
\]
\[
+ \xi \Delta(mx, 0, 0) \] for \( x \in G, k \in \mathbb{N} \).

Since \( \eta < \frac{1}{2} \), this yields (2.7).

Next consider inequality (2.13). Since \( f_{\text{e}} \) is even and \( f_{\text{e}}(0) = 0 \), taking into account (2.13) \( z = -x \), we obtain
\[
d \left( Mf_{\text{e}} \left( \frac{y}{m} \right) + 2f_{\text{e}}(x) + f_{\text{e}}(y),
\right.
\]
\[
N \left[ f_{\text{e}} \left( \frac{x + y}{n} \right) + f_{\text{e}} \left( \frac{x - y}{n} \right) \right] \leq \xi \left[ \Delta(x, y, -x) + \frac{1}{2} \Delta(0, 0, 0) \right] \] (2.24)
for \( x, y \in G \). Applying (2.24), with \( y = 0 \) and next with \( x = 0 \), we get
\[
d \left( 2f_{\text{e}}(x), 2Nf_{\text{e}} \left( \frac{x}{n} \right) \right) \leq \xi \left( \Delta(x, 0, -x) + \frac{1}{2} \Delta(0, 0, 0) \right) \] (2.25)
for \( y \in G \) and
\[
d \left( Mf_{\text{e}} \left( \frac{y}{m} \right) + f_{\text{e}}(y), 2Nf_{\text{e}} \left( \frac{y}{n} \right) \right) \leq \xi \left( \Delta(0, y, 0) + \frac{1}{2} \Delta(0, 0, 0) \right) \]
for \( x \in G \), respectively. In view of (1.5), the last two inequalities imply that
\[
d \left( f_{\text{e}}(y), Mf_{\text{e}} \left( \frac{y}{m} \right) \right) \leq \xi \left( \Delta(y, 0, -y) + \Delta(0, y, 0) + \Delta(0, 0, 0) \right) \] (2.26)
for \( y \in G \). Note also that, by (1.6), from (2.25) we deduce that
\[
d \left( f_{\text{e}}(x), Nf_{\text{e}} \left( \frac{x}{n} \right) \right) \leq \xi^2 \left( \Delta(x, 0, -x) + \frac{1}{2} \Delta(0, 0, 0) \right) \] (2.27)
for \( x \in G \). Therefore, using (1.5) and (2.1), from (2.24), (2.26) and (2.27), we obtain
\[
d \left( f_{\text{e}}(x + y) + f_{\text{e}}(x - y), 2f_{\text{e}}(x) + 2f_{\text{e}}(y) \right) \leq
\leq \xi \left( \Delta(x, y, -x) + \Delta(y, 0, -y) + \Delta(0, y, 0) \right) +
\]
\[+ \xi^2 \left( \Delta(x + y, 0, -(x + y)) + \Delta(x - y, 0, -(x - y)) \right) +
\]
\[+ \left( \frac{3}{2} \xi + \xi^2 \right) \Delta(0, 0, 0) \] for \( x, y \in G \).
Furthermore, in view of (2.15), the function $\chi_1 : G^2 \rightarrow [0, \infty)$ given by
\[
\chi_1(x, y) := \xi(\Delta(x, y, -x) + \Delta(y, 0, -y) + \Delta(0, y, 0)) + \\
\xi^2(\Delta(x + y, 0, -(x + y)) + \Delta(x - y, 0, -(x - y))) + \\
\left(\frac{3}{2} \xi + \xi^2\right)\Delta(0, 0, 0)
\] (2.28)
for $x, y \in G$, satisfies
\[
\chi_1(2x, 2y) \leq \eta \chi_1(x, y)
\] for $x, y \in G$. (2.29)

Thus, as $f_e(0) = 0$, applying [1, Corollary 5.2], we obtain that there exists a unique quadratic function $Q : G \rightarrow H$ such that
\[
d(\bar{f}_e(x), Q(x)) \leq \frac{\xi^2}{1 - \xi^2 \eta} \chi_1(x, x)
\] for $x \in G$. (2.30)

Moreover, taking into account (2.3), from (2.26) we derive that
\[
d(MQ(x), MQ(x)) \leq d(Mf_e(x), MQ(x)) \leq \frac{\xi^2}{1 - \xi^2 \eta} [\chi_1(mx, mx) + M\chi_1(x, x)] + \\
+ \xi(M\chi_1(mx, 0, -mx) + \Delta(0, mx, 0) + \Delta(0, 0, 0))
\]
for $x \in G$. Since $Q$ is quadratic and the functions $\Delta$ and $\chi_1$ satisfy (2.15) and (2.29), respectively, making use of (1.5) and (1.6), from the latter inequality we obtain that
\[
d((M - m^2)Q(x), 0) = d(4^{-k}(M - m^2)Q(2^k x), 0) \leq \\
\leq (\xi^2 \eta)^k \left[\frac{\xi^2}{1 - \xi^2 \eta} (\chi_1(mx, mx) + M\chi_1(x, x)) + \\
+ \xi [\Delta(mx, 0, -mx) + \Delta(0, mx, 0) + \Delta(0, 0, 0)]\right]
\]
for every $x \in G$ and $k \in \mathbb{N}$. Since $\eta < \frac{1}{2} < \frac{1}{\xi}$, this means that $(M - m^2)Q(x) = 0$ for $x \in G$. In a similar way, using (2.27), we obtain that $(N - n^2)Q(x) = 0$ for $x \in G$. So, (2.6) holds.

Finally, (1.5), (2.1), (2.9), (2.20) and (2.30) imply that
\[
d(f(x), Q(x) + A(x) + f(0)) \leq \frac{\xi^2}{1 - \xi \eta} \chi_0(x, x) + \frac{\xi^2}{1 - \xi^2 \eta} \chi_1(x, x)
\]
for $x \in G$. Thus, taking into account (2.14), (2.18) and (2.28), after straightforward calculations, we get (2.8).

From Theorem 2.2 and [3, Theorem 1] we obtain the following stability result for (1.4).
Corollary 2.3. If \( f : G \rightarrow H \) satisfies (2.4) with \( \delta \) satisfying (2.5), then there exists a unique solution \( F : G \rightarrow H \) of (1.4) such that
\[
d(f(x), F(x)) \leq \frac{\xi^4}{1 - \xi \eta} [\delta(x, x, 0) + \delta(-x, -x, 0) + \delta(2x, 0, 0) + \delta(-2x, 0, 0) + \\
+ \delta(x, x, -2x) + \delta(-x, -x, 2x)] + \frac{\xi^4}{1 - \xi^2 \eta^2} [\delta(x, x) + \delta(-x, -x, x) + \\
+ \delta(x, 0, -x) + \delta(-x, 0, x) + \delta(0, x, 0) + \delta(0, -x, 0)] + \\
+ \frac{\xi^4}{1 - \xi^2 \eta^2} [\delta(2x, 0, -2x) + \delta(-2x, 0, 2x)] + \frac{\xi^4}{1 - \xi \eta^2} (3 + 4 \xi) \delta(0, 0, 0)
\]
for \( x \in G \).

In the case where \((H, \| \cdot \|)\) is a Banach space and \( \delta \in [0, \infty) \), conditions (1.5), (1.6) and (2.5) hold with \( \xi = \frac{1}{2} \) and \( \eta = 1 \). Therefore, from Theorem 2.2 we deduce the following result.

Corollary 2.4. Assume that \((H, \| \cdot \|)\) is a Banach space, \( \delta \in [0, \infty) \) and the function \( f : G \rightarrow H \) satisfies inequality
\[
\left\| Mf \left( \frac{x + y + z}{m} \right) + f(x) + f(y) + f(z) - \\
- N \left[ f \left( \frac{x + y}{n} \right) + f \left( \frac{x + z}{n} \right) + f \left( \frac{y + z}{n} \right) \right] \right\| \leq \delta \quad \text{for } x, y, z \in G.
\]
Then there exists a uniquely determined quadratic function \( Q : G \rightarrow H \) and an additive function \( A : G \rightarrow H \) such that (2.6), (2.7) hold and
\[
\| f(x) - Q(x) - A(x) - f(0) \| \leq \frac{11}{4} \delta \quad \text{for } x \in G.
\]

Finally note that if \((H, \| \cdot \|)\) is a Banach space and \( \delta : G^3 \rightarrow [0, \infty) \) is given by
\[
\delta(x, y, z) = \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \quad \text{for } x, y, z \in G,
\]
where \( \varepsilon \in [0, \infty) \) and \( p \in (0, 1) \) are fixed, then conditions (1.5), (1.6) and (2.5) hold with \( \xi = \frac{1}{2} \) and \( \eta = 2^p \). Therefore, applying Theorem 2.2, we obtain the following result.

Corollary 2.5. Assume that \((H, \| \cdot \|)\) is a Banach space, \( \varepsilon \in [0, \infty) \) and \( p \in (0, 1) \) are fixed and a function \( f : G \rightarrow H \) satisfies inequality
\[
\left\| Mf \left( \frac{x + y + z}{m} \right) + f(x) + f(y) + f(z) - \\
- N \left[ f \left( \frac{x + y}{n} \right) + f \left( \frac{x + z}{n} \right) + f \left( \frac{y + z}{n} \right) \right] \right\| \leq \\
\leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \quad \text{for } x, y, z \in G.
\]
Then there exists a uniquely determined quadratic function $Q : G \to H$ and an additive function $A : G \to H$ such that (2.6), (2.7) hold and

$$
\|f(x) - Q(x) - A(x) - f(0)\| \leq \left( \frac{2 + 2^p}{4 - 2^{p+1}} + \frac{6 + 2^p}{4 - 2^p} \right) \varepsilon \|x\|^p \quad \text{for} \quad x \in G.
$$

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