Opuscula Mathematica • Vol. 31 • No. 2 • 2011

http://dx.doi.org/10.7494/OpMath.2011.31.2.279

POLYNOMIAL STABILITY OF EVOLUTION OPERATORS IN BANACH SPACES

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Abstract. The paper considers three concepts of polynomial stability for linear evolution operators which are defined in a general Banach space and whose norms can increase not faster than exponentially. Our approach is based on the extension of techniques for exponential stability to the case of polynomial stability. Some illustrating examples clarify the relations between the stability concepts considered in paper. The obtained results are generalizations of well-known theorems about the uniform and nonuniform exponential stability.

Keywords: evolution operator, polynomial stability, exponential stability.

Mathematics Subject Classification: 34D05, 34E05.

1. INTRODUCTION

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. Let $\triangle$ be the set defined by

$$\triangle = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}.$$

We recall that an operator-valued function $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is called an *evolution operator* on X if:
e_1) $\Phi(t,t) = I$ for every $t \geqslant 0$;
e_2) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$ for all $(t,s),(s,t_0) \in \Delta$.

In the examples considered in this paper we consider evolution operators on X defined by

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t,s)x = \frac{u(s)}{u(t)}x,$$

where $u : \mathbb{R}_+ \longrightarrow [1,\infty)$.

An evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ with the property
e_3) that there exists a nondecreasing function $\varphi : \mathbb{R}_+ \longrightarrow [1,\infty)$ such that

$$\|\Phi(t,s)\| \leq \varphi(t-s) \quad \text{for all} \quad (t,s) \in \Delta,$$

is called *with uniform growth.*

We recall three exponential stability concepts given by

Definition 1.1. The evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is called:

(i) *uniformly exponentially stable* (and denote u.e.s.) if there are $N \geq 1$ and $\alpha > 0$ such that

$$e^{\alpha t}\|\Phi(t,s)x\| \leq Ne^{\alpha s}\|x\|$$

for all $(t,s,x) \in \Delta \times X$;

(ii) *nonuniformly exponentially stable* (and denote e.s.) if there exist $\alpha > 0$ and a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1,\infty)$ such that

$$e^{\alpha t}\|\Phi(t,s)x\| \leq N(s)\|x\|$$

for all $(t,s,x) \in \Delta \times X$;

(iii) *exponentially stable in the Barreira-Valls sense* (and denote B.V.e.s.) if there are $N \geq 1$, $\beta \geq \alpha > 0$ such that

$$e^{\alpha t}\|\Phi(t,s)x\| \leq Ne^{\beta s}\|x\|$$

for all $(t,s,x) \in \Delta \times X$.

The concepts of uniform exponential stability and nonuniform exponential stability are well-known and the concept of exponentially stable in the Barreira-Valls sense has been considered in the works of L. Barreira and C. Valls, as for example [2] and [3].

Remark 1.2. It is obvious that

$$u.e.s. \Rightarrow B.V.e.s. \Rightarrow e.s.$$

The converse implications are not valid (see [8]).

A particular class of evolution operators is defined by

Definition 1.3. The evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is called *strongly measurable*, if for all $(s,x) \in \mathbb{R}_+ \times X$ the mapping defined by $t \longmapsto \|\Phi(t,s)x\|$ is measurable on $[s,\infty)$.

Characterizations for exponential stability properties of strongly measurable evolution operator with uniform growth are given in [1,6,7] for $u.e.s$, respectively in [4,8] and [10] for $e.s$, respectively in [2,3,8] and [9] for $B.V.e.s.$

Another class of evolution operators is introduced by

Definition 1.4. An evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is called $*$-*strongly measurable*, if for all $(t,x^*) \in \mathbb{R}_+ \times X^*$ the mapping defined by $s \longmapsto \|\Phi(t,s)^*x^*\|$ is measurable on $[0,t]$, where by X^* we denote the dual space of X.

Characterizations for exponential stability properties of $*$-strongly measurable evolution operators with uniform growth are known as "*Barbashin-type theorems*" and are given in [1, 5] for *u.e.s.*, respectively in [5, 8, 10] by *e.s.*, respectively in [8] and [9] for *B.V.e.s.*

In this paper we consider three concepts of polynomial stability and our main objectives are to extend the techniques from exponential stability theory to the cases of polynomial stabilities and to establish relations between these concepts.

2. NONUNIFORM POLYNOMIAL STABILITY

Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be an evolution operator on X.

Definition 2.1. The evolution operator Φ is called (*nonuniform*) *polynomially stable* (and denote p.s.) if there are $\alpha > 0$, $t_0 > 0$ and a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1, \infty)$ such that

$$t^{\alpha}\|\Phi(t,s)x\| \leq N(s)\|x\|$$

for all $(t, s, x) \in \Delta \times X$ with $s \geq t_0$.

Proposition 2.2. *If the evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is e.s. then Φ is p.s.*

Proof. If Φ is *e.s.* then there are $\alpha > 0$, $t_0 > 0$ and a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1, \infty)$ such that

$$t^{\alpha}\|\Phi(t,s)x\| \leq e^{\alpha t}\|\Phi(t,s)x\| \leq N(s)\|x\|$$

for all $t \geq s \geq t_0$ and all $x \in X$. This shows that Φ is *p.s.* □

The following example shows that the converse of Proposition 2.2 is not valid.

Example 2.3. (*Evolution operator which is polynomially stable and is not exponentially stable.*)

The evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(\mathbb{R}), \quad \Phi(t,s)x = \frac{s+1}{t+1}x$$

satisfies the inequality

$$t|\Phi(t,s)x| \leq s^2|x|$$

for all $(t, s, x) \in \Delta \times \mathbb{R}$ with $s \geq t_0 = 2$. Hence Φ is *p.s.* If we suppose that Φ is *e.s.* then there exist $\alpha > 0$ and a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1, \infty)$ such that

$$e^{\alpha t}(s+1) \leq (t+1)N(s)$$

for all $(t, s) \in \Delta$. For $t \longrightarrow \infty$, we obtain a contradiction.

Theorem 2.4. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a strongly measurable evolution operator with uniform growth. If there are $\gamma > 0$, $t_0 \geq 1$ and $M : \mathbb{R}_+ \longrightarrow [1, \infty)$ such that*

$$\int_s^\infty \tau^\gamma \|\Phi(\tau, s)x\| \, d\tau \leq M(s)\|x\|$$

for all $(s, x) \in \mathbb{R}_+ \times X$ with $s \geq t_0$, then Φ is polynomially stable

Proof. Let $x \in X$. If $t \geq s + 1$ and $s \geq t_0$, then using the monotony of the function

$$f : \mathbb{R}_+^* \longrightarrow \mathbb{R}_+^*, \; f(t) = \frac{e^t}{t},$$

where $\mathbb{R}_+^* = (0, \infty)$, we have that

$$\begin{aligned} t^\gamma \|\Phi(t, s)x\| &= \int_{t-1}^t t^\gamma \|\Phi(t, s)x\| \, d\tau \leq \\ &\leq \int_{t-1}^t t^\gamma \tau^\gamma \tau^{-\gamma} \|\Phi(t, \tau)\| \|\Phi(\tau, s)x\| \, d\tau \leq \\ &\leq \varphi(1) \int_{t-1}^t e^{\gamma(t-\tau)} \tau^\gamma \|\Phi(\tau, s)x\| \, d\tau \leq \\ &\leq \varphi(1)e^\gamma \int_s^t \tau^\gamma \|\Phi(\tau, s)x\| \, d\tau \leq M(s)\varphi(1)e^\gamma \|x\| \leq \\ &\leq M(s)\varphi(1)e^\gamma s^\gamma \|x\| = N(s) \, \|x\|. \end{aligned}$$

For $t \in [s, s + 1)$ we have

$$\begin{aligned} t^\gamma \|\Phi(t, s)x\| &= t^\gamma s^\gamma s^{-\gamma} \|\Phi(t, s)x\| \leq \\ &\leq \varphi(1)e^{\gamma(t-s)} s^\gamma \|x\| \leq \\ &\leq \varphi(1)e^\gamma s^\gamma \|x\| \leq N(s) \, \|x\|. \end{aligned}$$

So

$$t^\gamma \|\Phi(t, s)x\| \leq N(s)\|x\|$$

for all $(t, s, x) \in \Delta \times X$ with $s \geq t_0$, which shows that Φ is *p.s.* $\square$

Remark 2.5. In the case when the constant α given by Definition 2.1 satisfies the condition $\alpha > 1$ and the converse of Theorem 2.4 is valid (it is sufficient to consider $\gamma \in (0, \alpha - 1)$).

A particular case of polynomial stability is when the function N is constant. Thus we obtain a new stability concept studied in the next section.

3. UNIFORM POLYNOMIAL STABILITY

Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be an evolution operator on X.

Definition 3.1. The evolution operator Φ is called *uniformly polynomially stable* (and denote *u.p.s.*) if there are $\alpha > 0$, $t_0 > 0$ and $N \geq 1$ such that

$$t^\alpha \|\Phi(t, s)x\| \leq N s^\alpha \|x\|$$

for all $(t, s, x) \in \Delta \times X$ with $s \geq t_0$.

Remark 3.2. It is obvious that if Φ is *uniformly polynomially stable* then it is *polynomially stable.* The converse is not true (see Example 4.3).

Proposition 3.3. *If the evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is uniformly expotentially stable then it is uniformly polynomially stable*

Proof. If Φ is u.e.s. and $t \geq s \geq t_0 \geq 1$ then there are $N \geq 1$ and $\alpha > 0$ such that

$$t^\alpha \|\Phi(t, s)x\| \leq s^\alpha e^{\alpha(t-s)} \|\Phi(t, s)x\| \leq N s^\alpha \|x\|$$

and hence Φ is *u.p.s.* □

Example 3.4. (*Evolution operator which is uniformly polynomially stable and it is not uniformly exponentially stable.*)

The evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(\mathbb{R}), \quad \Phi(t, s)x = \frac{s^2 + 1}{t^2 + 1} x$$

has the property

$$t^2 |\Phi(t, s)x| \leq 2s^2 |x|$$

for all $(t, s, x) \in \Delta \times \mathbb{R}$ with $s \geq t_0 = 1$, which shows that Φ is *u.p.s.* If we suppose that Φ is *u.e.s.* then there are $N \geq 1$, $\alpha > 0$, such that

$$(s^2 + 1)e^{\alpha t} \leq N(t^2 + 1)e^{\alpha s}$$

for all $(t, s) \in \Delta$. For $s = 0$ and $t \to \infty$, we obtain a contradiction and hence Φ is not *u.e.s.*

From the proof of Theorem 2.4 it results in a sufficient condition for *uniformly polynomially stable* given by

Corollary 3.5. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a strongly measurable evolution operator with uniform growth. If there exist $M, t_0 \geq 1$ and $\gamma > 0$ such that*

$$\int_s^\infty \tau^\gamma \|\Phi(\tau, s)x\| \, d\tau \leq M \|x\|$$

for all $(s, x) \in \mathbb{R}_+ \times X$ with $s \geq t_0$, then Φ is uniformly polynomially stable.

Remark 3.6. As in Remark 2.5, the converse of Corollary 3.5 is valid in the case when the constant α given by Definition 3.1 is strictly greater than 1.

Another sufficient condition for *u.p.s.* is given by

Theorem 3.7. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a $*$-strongly measurable evolution operator with uniform growth. If there are $M, \gamma, t_0 \geq 1$ such that*

$$\int_s^t \left(\frac{t}{\tau}\right)^\gamma \|\Phi(t,\tau)^* x^*\| \, d\tau \leq M\|x^*\|$$

for all $(t,s,x^*) \in \Delta \times X^*$ with $s \geq t_0$ then Φ is uniformly polynomially stable.

Proof. Let $(x, x^*) \in X \times X^*$. If $t \geq s+1$ and $s \geq t_0 \geq 1$ then

$$\left(\frac{t}{s}\right)^\gamma |\langle x^*, \Phi(t,s)x\rangle| = \int_s^{s+1} \left(\frac{t}{\tau}\right)^\gamma \left(\frac{\tau}{s}\right)^\gamma |\langle \Phi(t,\tau)^* x^*, \Phi(\tau,s)x\rangle| \, d\tau \leq$$

$$\leq 2^\gamma \varphi(1)\|x\| \int_s^t \left(\frac{t}{\tau}\right)^\gamma \|\Phi(t,\tau)^* x^*\| \, d\tau \leq N\|x\|\|x^*\|,$$

where $N = 2^\gamma M \varphi(1)$. For $t \in [s, s+1)$, with $s \geq t_0$ we have

$$\left(\frac{t}{s}\right)^\gamma \|\Phi(t,s)x\| \leq 2^\gamma \varphi(1)\|x\| \leq N\|x\|.$$

Finally, it results that

$$t^\gamma \|\Phi(t,s)x\| \leq N s^\gamma \|x\|$$

for all $(t,s,x) \in \Delta \times X$ with $s \geq t_0$, which implies that Φ is *u.p.s.* □

4. POLYNOMIAL STABILITY IN THE BARREIRA-VALLS SENSE

Another particular case of polynomial stability is given by

Definition 4.1. The evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is called *polynomially stable in the Barreira-Valls sense* (and denote B.V.p.s.) if there are $N \geq 1$, $\alpha > 0$, $\beta \geq \alpha$ and $t_0 > 0$ such that:

$$t^\alpha \|\Phi(t,s)x\| \leq N s^\beta \|x\|$$

for all $(t,s,x) \in \Delta \times X$ with $s \geq t_0$.

Remark 4.2. It is obvious that

$$u.p.s. \Longrightarrow B.V.p.s. \Longrightarrow p.s.$$

The following two examples shows that the converse implications are not valid.

Example 4.3. (*Polynomial stable evolution operator which is not polynomially stable in the Barreira-Valls sense.*)

Consider the evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(\mathbb{R}), \quad \Phi(t,s)x = \frac{s^2 u(s)}{t^2 u(t)} x$$

where $u : \mathbb{R}_+ \longrightarrow [1, \infty)$ satisfies the conditions $u(n) = e^n$ and $u(n + \frac{1}{n}) = e^2$ for every $n \in \mathbb{N}^*$. Firstly, we have that

$$t|\Phi(t, s)x| \leq su(s)|x| = N(s)|x|$$

for all $(t, s, x) \in \Delta \times \mathbb{R}$ with $s \geq t_0 = 1$. this shows that Φ is *p.s.* If we suppose that Φ is *B.V.p.s.* then there are $N \geq 1$, $\alpha > 0$, $\beta \geq \alpha$ and $t_0 > 0$ such that:

$$t^{\alpha-2}u(s) \leq Nu(t)s^{\beta-2}$$

for all $(t, s) \in \Delta$ with $s \geq t_0$. Then for $s = n$ and $t = n + \frac{1}{n}$ we obtain

$$n^{\alpha-\beta}e^n(1 + \frac{1}{n^2})^{\alpha-2} \leq Ne^2$$

which for $n \longrightarrow \infty$ yields a contradiction.

Example 4.4. (*Evolution operator which is polynomially stable in the Barreira-Valls sense and it is not uniformly polynomially stable.*)

Consider the evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(\mathbb{R}), \quad \Phi(t, s)x = \frac{(s+1)^2}{(t+1)^2} \frac{(t+1)^{\cos \ln(t+1)}}{(s+1)^{\cos \ln(s+1)}} x$$

which satisfies

$$t|\Phi(t, s)x| \leq \frac{t(s+1)^3}{t+1}|x| \leq 8s^3|x|$$

for all $(t, s, x) \in \Delta \times \mathbb{R}$ with $s \geq t_0 = 1$. It results that Φ is *B.V.p.s.* Suppose that Φ is *u.p.s.* Then there are $N \geq 1$, $\alpha > 0$ and $t_0 > 0$ such that

$$t^\alpha(s+1)^2(t+1)^{\cos \ln(t+1)} \leq Ns^\alpha(t+1)^2(s+1)^{\cos \ln(s+1)}$$

for all $t \geq s \geq t_0$. From here for $t = -1 + \exp(2n\pi)$ and $s = -1 + \exp(2n\pi - \frac{\pi}{2})$ it results

$$[-1 + \exp(2n\pi)]^\alpha \exp(2n - 1)\pi \leq N[-1 + \exp(2n\pi - \frac{\pi}{2})]^\alpha,$$

which for $n \longrightarrow \infty$ yields a contradiction. Finally, we conclude that Φ is not *u.p.s.*

A sufficient condition for *B.V.p.s.* is given by

Corollary 4.5. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ by a strongly measurable evolution with uniform growth. If there exist $M \geq 1, \delta \geq \gamma > 0$ and $t_0 > 1$ such that*

$$\int_s^\infty \tau^\gamma \|\Phi(\tau, s)x\| \, d\tau \leq Ms^\delta \|x\|$$

for all $(s, x) \in \mathbb{R}_+ \times X$ with $s \geq t_0$, then Φ is polynomially stable in the sense of Barreira-Valls.

Proof. It results from the proof of Theorem 2.4 for $M(s) = Ms^\delta$. □

Remark 4.6. As in the case of polynomial stability the converse of the preceding corollary is valid in the case when the constant α given by Definition 3.1 is strictly greater than 1.

Example 4.7. (*Evolution operator which is polynomially stable in the sense of Barreira-Valls and it is not exponentially stable in the sense of Barreira-Valls.*)

The evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(\mathbb{R}), \quad \Phi(t,s)x = \frac{s^3+1}{t^3+1}x$$

verifies the inequality

$$t|\Phi(t,s)x| \leq 2s^3|x|$$

for all $t \geq s \geq t_0 = 1$ and all $x \in \mathbb{R}$, which shows that Φ is *B.V.p.s.* Suppose that Φ is *B.V.e.s.* Then it is *e.s.* and there exist $\alpha > 0$ and $N : \mathbb{R}_+ \longrightarrow [1,\infty)$ such that

$$e^{\alpha t}(s^3+1) \leq (t^3+1)N(s)$$

for all $t \geq s \geq 0$. From here, for s fixed and $t \longrightarrow \infty$ we obtain a contradiction.

Another sufficient condition for *B.V.p.s.* is given by

Theorem 4.8. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a $*$-strongly measurable evolution operator with uniform growth. If there are $M \geq 1$, $\delta \geq \gamma > 0$ and $t_0 \geq 1$ such that*

$$\int_s^t \left(\frac{t}{\tau}\right)^{\gamma} \|\Phi(t,\tau)^* x^*\| \, d\tau \leq Ms^{\delta}\|x^*\|$$

for all $(t,s,x^*) \in \Delta \times X^*$ with $s \geq t_0$, then Φ is polynomially stable in the sense of Barreira-Valls.

Proof. It is similar to the proof of Theorem 3.7. Let $(x,x^*) \in X \times X^*$. We observe that for $t \geq s+1 > s \geq t_0$ we have

$$\begin{aligned}\left(\frac{t}{s}\right)^{\gamma} |\langle x^*, \Phi(t,s)x\rangle| &= \int_s^{s+1} \left(\frac{t}{\tau}\right)^{\gamma}\left(\frac{\tau}{s}\right)^{\gamma} |\langle \Phi(t,\tau)^*x^*, \Phi(\tau,s)x\rangle| \, d\tau \leq \\ &\leq 2^{\gamma}\varphi(1)\|x\| \int_s^t \left(\frac{t}{\tau}\right)^{\gamma} \|\Phi(t,\tau)^*x^*\| \, d\tau \leq \\ &\leq 2^{\gamma}\varphi(1)M\|x\|\|x^*\| = N\|x\|\|x^*\|\end{aligned}$$

and hence

$$t^{\gamma}\|\Phi(t,s)x\| \leq Ns^{\delta+\gamma}\|x\|.$$

For $t \in [s,s+1)$, with $s \geq t_0$ we have

$$\begin{aligned}t^{\gamma}\|\Phi(t,s)x\| \leq t^{\gamma}\varphi(1)\|x\| &= \left(\frac{t}{s}\right)^{\gamma} \varphi(1)s^{\gamma}\|x\| \leq \\ &\leq 2^{\gamma}\varphi(1)s^{\gamma}\|x\| \leq Ns^{\gamma+\delta}\|x\|.\end{aligned}$$

Finally, we see that

$$t^{\gamma}\|\Phi(t,s)x\| \le N s^{\gamma+\delta}\|x\|$$

for all $(t,s,x) \in \Delta \times X$ with $s \ge t_0$, which implies that Φ is $B.V.p.s.$ $\square$

Remark 4.9. Theorem 4.8 can be considered as a variant of the classical Barbashin theorem ([1]) for polynomial stability in the Barreira-Valls sense.

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Received: June 29, 2010.
Revised: July 23, 2010.
Accepted: October 7, 2010.