THE EQUALITY CASE IN SOME RECENT CONVEXITY INEQUALITIES

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Abstract. In this paper, we investigate a functional equation related to some recently introduced and investigated convexity type inequalities.

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1. INTRODUCTION

In a recent paper [24] by Varošanec, a common generalization of convex and s-convex functions, Godunova-Levin functions, and $\mathcal{P}$-functions is introduced in the following way: Let $I$ be a nonvoid subinterval of $\mathbb{R}$ (the set of all real numbers), $h : [0, 1] \to \mathbb{R}$ and $f : I \to \mathbb{R}$ be real-valued functions satisfying the inequality

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)$$

for all $x,y \in I$ and $t \in [0,1]$. An even more general notion, the so-called $(T,h)$-convexity, can be found in Házy [11]: Let $X$ be a real or complex normed space, $D \subset X$ be a nonempty convex set, $\emptyset \neq T \subset [0,1]$, and $h : T \to \mathbb{R}$ be a function. A function $f : D \to \mathbb{R}$ is $(T,h)$-convex if (1.1) holds for all $x,y \in D$ and $t \in T$. It is clear that this generalizes the concepts of convexity ($h(t) = t$, $t \in [0,1]$, [24], [21]), the Breckner-convexity ($h(t) = t^s$, $t \in [0,1]$, for some $s \in \mathbb{R}$, [5], [6]), the Godunova-Levin functions ($h(t) = t^{-1}$, $t \in [0,1]$, [10]), the $\mathcal{P}$-functions ($h(t) = 1$, $t \in [0,1]$, [18]), and the $t$-convexity ($T = \{t, 1-t\}$, $h(t) = t$, $h(1-t) = 1-t$, where $0 < t < 1$ is a fixed number, Kuhn [14]). For further related results see Burai-Házy [1,2] and Burai-Házy-Juhász [3,4].

In this note, we focus on the functional equation related to these convexity properties and give the solutions of the following problem. Let $X$ be a real or complex topological vector space, $D \subset X$ be a nonempty open set, $T$ be a nonempty set,
and $\alpha, \beta, a, b : T \to \mathbb{R}$ be given functions. The problem is to find all the solutions $f : D \to \mathbb{R}$ of the functional equation

$$f(\alpha(t)x + \beta(t)y) = a(t)f(x) + b(t)f(y) \quad (x, y \in D, t \in T)$$

(1.2)

provided that $D$ is $(\alpha, \beta)$-convex, that is, $\alpha(t)x + \beta(t)y \in D$ whenever $x, y \in D$ and $t \in T$. To avoid the trivialities and the unimportant cases, we suppose that there exists an element $t_0 \in T$ such that

$$\alpha(t_0)\beta(t_0)a(t_0)b(t_0) \neq 0.$$  

(1.3)

We refer to the solutions of (1.2) as $(\alpha, \beta, a, b)$-affine functions and the solutions $f$ of the corresponding inequality

$$f(\alpha(t)x + \beta(t)y) \leq a(t)f(x) + b(t)f(y) \quad (x, y \in D, t \in T)$$

will be called $(\alpha, \beta, a, b)$-convex functions. Besides those convexity notions we listed above this is a generalization of $(t, q)$-convexity ($T = \{t\}, \alpha(t) = t, \beta(t) = 1 - t$, $a(t) = q, b(t) = 1 - q$, where $t, q \in [0, 1]$ are fixed numbers, Kuhn [15], Matkowski-Pycia [16]), and Orlicz $s$-convexity ($T = [0, 1], \alpha(t) = t^s, \beta(t) = (1 - t)^s$, $a(t) = t, b(t) = 1 - t$ for all $t \in T$ and for some $s \geq 1$, Orlicz [17], Hudzik-Maligranda [12]).

Our purpose is to describe the $(\alpha, \beta, a, b)$-affine functions. Throughout this paper $X$ denotes a real or complex topological vector space. A function $A : X \to \mathbb{R}$ is called additive if it satisfies the Cauchy functional equation

$$A(x + y) = A(x) + A(y) \quad (x, y \in X).$$

Given a subfield $S \subseteq \mathbb{R}$, a function $\varphi : S \to \mathbb{R}$ is said to be a field-homomorphism if $\varphi$ is additive and multiplicative on $S$, i.e.,

$$\varphi(s + t) = \varphi(s) + \varphi(t) \quad \text{and} \quad \varphi(st) = \varphi(s)\varphi(t) \quad (s, t \in S).$$

2. THE RESULTS

Our investigations are based on the following extension theorem which is an immediate consequence of Theorem 1 in Radó-Baker [19].

**Theorem 2.1.** Let $U$ be a nonempty, open, connected subset of $X \times X$ and define the following sets

- $U_0 := \{x + y \mid (x, y) \in U\}$,
- $U_1 := \{x \mid \exists y \in X : (x, y) \in U\}$, and
- $U_2 := \{y \mid \exists x \in X : (x, y) \in U\}$.

Suppose that the functions $f_i : U_i \to \mathbb{R}$, $(i = 0, 1, 2)$ satisfy the functional equation

$$f_0(x + y) = f_1(x) + f_2(y)$$
for all \((x, y) \in U\). Then there exist a unique additive function \(A : X \to \mathbb{R}\) and a unique pair \((c_1, c_2) \in \mathbb{R}^2\) such that

\[
\begin{align*}
f_0(x) &= A(x) + c_1 + c_2 \quad (x \in U_0), \\
f_1(x) &= A(x) + c_1 \quad (x \in U_1), \text{ and} \\
f_2(x) &= A(x) + c_2 \quad (x \in U_2).
\end{align*}
\]

An important consequence of the above theorem is the following result.

**Theorem 2.2.** Let \(\gamma, \delta, p, q \in \mathbb{R}\) and \(\emptyset \neq D \subset X\) be an open and connected set such that \(\gamma \delta pq \neq 0\) and \(\gamma x + \delta y \in D\) for all \(x, y \in D\). Then the function \(f : D \to \mathbb{R}\) satisfies the functional equation

\[
f(\gamma x + \delta y) = pf(x) + qf(y) \quad (x, y \in D) \tag{2.1}
\]

if, and only if, there exist an additive function \(A : X \to \mathbb{R}\) and a constant \(c \in \mathbb{R}\) such that

\[
\begin{align*}
A(\gamma x) &= pA(x) \quad (x \in X), \\
A(\delta x) &= qA(x) \quad (x \in X), \\
c(p + q - 1) &= 0, \quad \text{and} \\
f(x) &= A(x) + c \quad (x \in D). \tag{2.2}
\end{align*}
\]

**Proof.** Equation (2.1) implies that

\[
f(x + y) = pf\left(\frac{1}{\gamma}x\right) + qf\left(\frac{1}{\delta}y\right) \quad (x \in \gamma D, y \in \delta D).
\]

Applying Theorem 2.1 for the open and connected set \(U := (\gamma D) \times (\delta D)\) and the triplet of functions

\[
\begin{align*}
f_0(x) &:= f(x), \quad x \in \gamma D + \delta D \subset D, \\
f_1(x) &:= pf\left(\frac{1}{\gamma}x\right), \quad x \in \gamma D, \\
f_2(x) &:= qf\left(\frac{1}{\delta}x\right), \quad x \in \delta D,
\end{align*}
\]

we obtain that

\[
pf\left(\frac{1}{\gamma}x\right) = A_0(x) + c_0 \quad (x \in \gamma D)
\]

with some additive function \(A_0 : X \to \mathbb{R}\) and \(c_0 \in \mathbb{R}\). Thus

\[
f(x) = \frac{1}{p}A_0(\gamma x) + \frac{c_0}{p} \quad (x \in D),
\]

whence, with the definitions \(A(x) := \frac{1}{p}A_0(\gamma x), x \in X\) and \(c := \frac{c_0}{p}\),

\[
f(x) = A(x) + c \quad (x \in D)
\]

follows.
Obviously, \( A : X \to \mathbb{R} \) is additive. Replacing this form of \( f \) into (2.1), we find that
\[
A(\gamma x) - pA(x) + A(\delta y) - qA(y) - c(p + q - 1) = 0 \quad (x, y \in D).
\]
This shows that, for all fixed \( y \in D \), the polynomial function
\[
x \mapsto A(\gamma x) - pA(x) + A(\delta y) - qA(y) - c(p + q - 1) \quad (x \in X)
\]
vanishes on \( D \), therefore it vanishes everywhere on \( X \) (see Székelyhidi [23]). This implies the other equalities of (2.2), as well. The converse is straightforward.\( \square \)

In the result below we investigate homogeneity properties of additive functions. Given an additive function \( A : \mathbb{R} \to \mathbb{R} \), we introduce its set of homogeneity pairs \( H_A \) as follows:
\[
H_A := \{(s, t) \in \mathbb{R}^2 \mid A(sx) = tA(x) \text{ for all } x \in \mathbb{R}\}.
\]

**Theorem 2.3.** Let \( A : \mathbb{R} \to \mathbb{R} \) be a nonzero additive function. Then there exist a subfield \( S_A \subseteq \mathbb{R} \) (called the homogeneity field of \( A \)) and an injective field-homomorphism \( \varphi_A : S_A \to \mathbb{R} \) (called the homogeneity field-homomorphism of \( A \)) such that \( H_A \) is equal to the graph of \( \varphi_A \), i.e.,
\[
H_A = \{(s, \varphi_A(s)) \mid s \in S_A\}. \tag{2.3}
\]

Conversely, for every subfield \( S \subseteq \mathbb{R} \) and injective field-homomorphism \( \varphi : S \to \mathbb{R} \), there exists a nonzero additive function \( A : X \to \mathbb{R} \) such that \( S \subseteq S_A \) and \( \varphi_A|S = \varphi \).

**Proof.** Denote by \( S_A \) the domain of the relation \( H_A \). We show that, \( H_A \) is in fact a function. Assume that \((s, t_1), (s, t_2) \in H_A \). Then, for all \( x \in X \),
\[
(t_1 - t_2)A(x) = t_1A(x) - t_2A(x) = A(sx) - A(sx) = 0,
\]
which, by the nontriviality of \( A \), yields that \( t_1 = t_2 \) proving that the relation \( H_A \) is a function. This means that there exists a function \( \varphi_A : S_A \to \mathbb{R} \) such that (2.3) holds. It remains to show that \( S_A \) is a subfield of \( \mathbb{R} \) and \( \varphi_A \) is an injective field-homomorphism.

To prove the injectivity, let \((s_1, t), (s_2, t) \in H_A \). Then, for all \( x \in X \),
\[
A((s_1 - s_2)x) = A(s_1 x) - A(s_2 x) = tA(x) - tA(x) = 0,
\]
which, by the nontriviality of \( A \), yields that \( s_1 = s_2 \). By the injectivity, \( \varphi_A(s) \) is nonzero whenever \( s \) is different from zero.

Let \( s, t \in S \). Then, using (2.3), for all \( x \in X \), we get that
\[
A((s - t)x) = A(sx) - A(tx) = \varphi_A(s)A(x) - \varphi_A(t)A(x) = (\varphi_A(s) - \varphi_A(t))A(x).
\]
Hence, \((s - t, \varphi_A(s) - \varphi_A(t)) \in H_A \), which yields that \( s - t \in S \) and \( \varphi_A(s - t) = \varphi_A(s) - \varphi_A(t) \). Thus \( S \) is a group with respect to the addition and \( \varphi_A \) is additive.

Similarly, for all \( s \in S \), \( t \in S \setminus \{0\} \), and \( x \in X \), we obtain that
\[
\varphi_A(t)A\left(\frac{s}{t}x\right) = A(sx) = \varphi_A(s)A(x).
\]
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Hence \( \left( \frac{s}{t}, \varphi_A(s) \right) \in H_A \), which yields that \( \frac{s}{t} \in S \) and \( \varphi_A \left( \frac{s}{t} \right) = \frac{\varphi_A(s)}{\varphi_A(t)} \). This proves that \( S \) is a semigroup under the multiplication whose nonzero elements form a group and \( \varphi_A \) is also multiplicative.

To prove the reversed statement, let \( S \subseteq \mathbb{R} \) be a subfield and \( \varphi : S \to \mathbb{R} \) be an injective field-homomorphism. Consider \( X \) as a vector space over \( S \) and let \( \{ x_\gamma | \gamma \in \Gamma \} \) be a Hamel base of \( X \) over \( S \). In addition, let \( \{ a_\gamma | \gamma \in \Gamma \} \) be an arbitrary family of real numbers such that at least one of these elements is different from zero. Given an element \( x \in X \), it can uniquely be written in the form

\[
x = s_1x_{\gamma_1} + \ldots + s_mx_{\gamma_m},
\]

where \( m \in \mathbb{N} \cup \{0\} \), \( s_1, \ldots, s_m \in S \), and \( \gamma_1, \ldots, \gamma_m \) are pairwise distinct elements of the index set \( \Gamma \). Now define \( A(x) \) by

\[
A(x) := \varphi(s_1)a_{\gamma_1} + \ldots + \varphi(s_m)a_{\gamma_m}.
\]

Using the additivity of \( \varphi \), it is immediate to see that \( A \) is a nonzero additive function. It remains to show that, for all \( s \in S \), \( (s, \varphi(s)) \in H_A \), i.e.,

\[
A(sx) = \varphi(s)A(x) \quad (x \in X).
\]

If \( x \) is of the form (2.4), then \( sx = (ss_1)x_{\gamma_1} + \ldots + (ss_m)x_{\gamma_m} \) and hence, by the multiplicativity of \( \varphi \), we get

\[
A(sx) = \varphi(ss_1)a_{\gamma_1} + \ldots + \varphi(ss_m)a_{\gamma_m} = \varphi(s)(\varphi(s_1)a_{\gamma_1} + \ldots + \varphi(s_m)a_{\gamma_m}) = \varphi(s)A(x),
\]

which completes the proof of (2.5).

\[\square\]

**Remark 2.4.** The equality stated in (2.3) can be rewritten as the following identity:

\[
A(sx) = \varphi_A(s)A(x) \quad (s \in S_A, x \in X).
\]

The additive and multiplicative properties of \( \varphi_A \) imply that if \( s \in S \) is an algebraic number over a subfield of \( \mathbb{R} \) then \( \varphi_A(s) \) must be one of its algebraic conjugates. In particular, if \( s \) is a rational number then, \( \varphi_A(s) = s \). On the other hand, if \( s \in S \) is transcendental, then \( \varphi_A(s) \) can be any transcendental number. For an account of such results see the paper [8] by Z. Daróczy. Those real numbers \( s \) such that \( (s, s) \in H_A \) also form a subfield of \( \mathbb{R} \) (cf. Rätz [20]). This easily follows from the fact that they are characterized by the fixed point equation \( \varphi_A(s) = s \).

An easy consequence of Theorem 2.2 and Theorem 2.3 is the following result.

**Theorem 2.5.** Let \( T \) be a nonempty set, and \( \alpha, \beta, a, b : T \to \mathbb{R} \) be given functions satisfying property (1.3) for some \( t_0 \in T \). Let furthermore, \( \emptyset \neq D \subseteq X \) be an open connected and \( (\alpha, \beta) \)-convex set. Then \( f : D \to \mathbb{R} \) is a nonconstant \( (\alpha, \beta, a, b) \)-affine function if, and only if, there exist a nonzero additive function \( A : X \to \mathbb{R} \) and a
constant $c \in \mathbb{R}$ such that $\alpha(T) \cup \beta(T)$ is contained by the homogeneity field $S_A$ of $A$ and
\begin{align*}
  a(t) &= \varphi_A(\alpha(t)) & (t \in T), \\
  b(t) &= \varphi_A(\beta(t)) & (t \in T), \\
  c(a(t) + b(t) - 1) &= 0 & (t \in T), \quad \text{and} \\
  f(x) &= A(x) + c & (x \in D)
\end{align*}
where $\varphi_A : S_A \to \mathbb{R}$ is the homogeneity field-homomorphism of $A$.

\textbf{Proof.} Applying Theorem 2.2 with $\gamma := \alpha(t_0)$, $\delta := \beta(t_0)$, $p := \alpha(t_0)$, and $q := \beta(t_0)$, it follows that there exist an additive function $A : X \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that $f(x) = A(x) + c$ for all $x \in D$.

To see that the first three equations in (2.7) are valid, we substitute this form of $f$ into (1.2) and get that, for all $x, y \in D$ and $t \in T$,
\begin{equation}
  A(\alpha(t)x) - a(t)A(x) + A(\beta(t)y) - b(t)A(y) - c(a(t) + b(t) - 1) = 0. \tag{2.8}
\end{equation}
In other words, for all fixed $y \in D$ and $t \in T$, the polynomial function
\[ x \mapsto A(\alpha(t)x) - a(t)A(x) + A(\beta(t)y) - b(t)A(y) - c(a(t) + b(t) - 1) \quad (x \in X) \]
vanishes on the open set $D$, therefore it vanishes everywhere on $X$. (See Székelyhidi [23].) Analogously, for all fixed $x \in X$ and $t \in T$, the polynomial function
\[ y \mapsto A(\alpha(t)x) - a(t)A(x) + A(\beta(t)y) - b(t)A(y) - c(a(t) + b(t) - 1) \quad (y \in X) \]
vanishes on $D$, therefore it vanishes everywhere on $X$. Therefore, (2.8) holds for all $x, y \in X$ and $t \in T$.

Thus, with simple substitutions, for all $t \in T$ and $x \in X$, we obtain that
\begin{align*}
  A(\alpha(t)x) &= a(t)A(x), \\
  A(\beta(t)x) &= b(t)A(x), \\
  c(a(t) + b(t) - 1) &= 0.
\end{align*}
The first two equalities yield that $(\alpha(t), a(t))$ and $(\beta(t), b(t))$ belong to $H_A$ for all $t \in T$. Therefore, $\alpha(T) \cup \beta(T) \subseteq S_A$ and the first two equations in (2.7) are also satisfied. \hfill \Box

3. REMARKS AND EASY CONSEQUENCES OF THEOREM 2.5

\textbf{Remark 3.1.} Suppose that $\alpha, \beta, a, b : T \to \mathbb{R}$ are given functions, $\emptyset \neq D \subseteq X$ such that, for some $t \in T$,
\[ \alpha(t) + \beta(t) = a(t) + b(t) = 1, \ a(t) > 0, \ b(t) > 0, \ \text{and} \ \alpha(t)x + \beta(t)y, \ x + y \in D \]
whenever $x, y \in D$. Then every $(\alpha, \beta, a, b)$-convex function $f : D \to \mathbb{R}$ is Jensen convex, i.e.
\[ f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2} \quad (x, y \in D), \]
and every \((\alpha, \beta, a, b)\)-affine function \(f : D \to \mathbb{R}\) satisfies the Jensen equation

\[
f \left( \frac{x + y}{2} \right) = \frac{f(x) + f(y)}{2} \quad (x, y \in D).
\]

In Kuczma [13, p. 315], there is an extension theorem for the Jensen equation. There \(D\) is a subset of \(\mathbb{R}^n\) with nonempty interior. Our statements follow easily from the identity (see Daróczy-Páles [9], and also Matkowski-Pycia [16])

\[
\frac{x + y}{2} = \alpha(t) \left[ x + \beta(t) \frac{x + y}{2} \right] + \beta(t) \left[ \alpha(t) x + \beta(t) \frac{x + y}{2} \right] \quad (x, y \in D).
\]

Finally, we list some easy consequences of Theorem 2.5.

**Corollary 3.2.** If \(\alpha(T) \cup \beta(T)\) contains a set of positive Lebesgue measure then the additive function \(A\) in Theorem 2.5 is a linear functional on \(X\) and \(a = \alpha, b = \beta\).

**Proof.** In this case, by a well-known theorem of Steinhaus [22], the homogeneity field \(S_A\) must contain an interval of positive length. Therefore \(S_A = \mathbb{R}\). Thus, by the classical theorem of Darboux [7] and taking into consideration (1.3) to hold for some \(t_0 \in T\), we have that \(\varphi_A(t) = t\) for all \(t \in \mathbb{R}\). The remaining statements are obvious. \(\square\)

The following corollary is a trivial consequence of Corollary 3.2.

**Corollary 3.3.** Suppose that, for \(f : D \to \mathbb{R}\) and for all \(x, y \in D\), the equality holds in the defining inequality of Breckner-convexity or Orlicz-convexity. Then \(f\) must be the constant function except the case \(s = 1\).

**Proof.** Taking into consideration Remark 2.4 (see also Daróczy [8]), we have

**Corollary 3.4.** If \(t, q \in [0, 1]\) are fixed, \(T = \{t\}\), \(\alpha(t) = t, \beta(t) = 1 - t, a(t) = q, b(t) = 1 - q\) then there exists nonconstant \((\alpha, \beta, a, b)\)-affine function if, and only if, \(t\) and \(q\) are conjugate, \(i.e.,\) they are both transcendental or they are both algebraic and have the same minimal polynomial with rational coefficients.

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