A RADIAL VERSION OF THE KONTOROVICH-LEBEDEV TRANSFORM IN THE UNIT BALL

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Abstract. In this paper we introduce a radial version of the Kontorovich-Lebedev transform in the unit ball. Mapping properties and an inversion formula are proved in $L_p$.

Keywords: Kontorovich-Lebedev transform, modified Bessel function, index transforms, Fourier integrals.

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1. INTRODUCTION

The Kontorovich-Lebedev transform (KL-transform) was introduced by the soviet mathematicians M.I. Kontorovich and N.N. Lebedev in 1938-1939 (see [4]) to solve certain boundary-value problems. The KL-transform arises naturally when one uses the method of separation of variables to solve boundary-value problems formulated in terms of cylindrical coordinate systems. It has been tabulated by Erdélyi et al., (see [3]) and Prudnikov et al., (see [11]). Its applications to the Dirichlet problem for a wedge were given by Lebedev in 1965 (see [5]), while Lowndes in 1959 (see [7]) applied a variant of it to a problem of diffraction of transient electromagnetic waves by a wedge. Some other applications can be found, for instance, in Skalskaya and Lebedev in 1974 (see [6]).

This transform was extended by Zemanian in 1975 (see [13]) to the distributional case, by Buggle in 1977 (see [1]) to some larger spaces of generalized functions. A possible extension to the multidimensional case of this index transform was investigated by the first author in his book (see [12]), where it was introduced the essentially multidimensional KL-transform.

The main goal of this work is to introduce a radial version of the KL-transform for the multidimensional case in the unit ball, prove its mapping properties and establish an inversion formula.
Formally, the one dimensional KL-transform is defined as

$$K_{i\tau}[f] = \int_{\mathbb{R}_+} K_{i\tau}(x) f(x) \, dx,$$  \hspace{1cm} (1.1)

where $K_{i\tau}$ denotes the modified Bessel function of pure imaginary index $i\tau$ (also called Macdonald’s function). The adjoint operator associated to (1.1) is

$$f(x) = \frac{2}{\pi x^2} \int_{\mathbb{R}_+} \tau \sinh(\pi\tau) K_{i\tau}(x) K_{i\tau}[f] \, d\tau.$$ \hspace{1cm} (1.2)

As we can see, in expression (1.2) the integration is realized with respect to the index $i\tau$ of the Macdonald’s function. This fact, for instance, carries extra difficulties in the deduction of norm estimates in certain function spaces. For more details about the one-dimensional KL-transform and other index transforms see [12].

The Macdonald’s function can be represented by the following Fourier integral (see [2])

$$K_{i\tau}(x) = \int_{\mathbb{R}_+} e^{-x \cosh u} \cos(\tau u) \, du, \quad x > 0 =$$ \hspace{1cm} (1.3)

$$= \frac{1}{2} \int_{\mathbb{R}} e^{-x \cosh u} e^{i\tau u} \, du, \quad x > 0.$$ \hspace{1cm} (1.4)

Making an extension of the previous integral equation to the strip $\delta \in [0, \frac{\pi}{2}]$ in the upper half-plane, we have, for $x > 0$, the following uniform estimate

$$|K_{i\tau}(x)| \leq \frac{1}{2} \int_{\mathbb{R}} e^{-x \cos \delta \cosh u} \, du =$$ \hspace{1cm} (1.5)

$$= e^{-\delta \tau} K_0(x \cos \delta), \quad x > 0$$

and in particular

$$|K_{i\tau}(x)| \leq K_0(x), \quad x > 0, \quad \tau \in \mathbb{R}.$$ \hspace{1cm} (1.6)

The modified Bessel function $K_\nu(x)$ function has the following asymptotic behavior (see [2] for more details) near the origin

$$K_\nu(x) = O \left(x^{-|\text{Re}(\nu)|}\right), \quad x \to 0, \quad \nu \neq 0,$$ \hspace{1cm} (1.7)

$$K_0(x) = O \left(\log x\right), \quad x \to 0^+.$$ \hspace{1cm} (1.8)

Using relation (2.16.52.8) in [11] we have the formulas

$$\int_{\mathbb{R}_+} \tau \sinh((\pi - \epsilon)\tau) K_{i\tau}(x) K_{i\tau}(y) \, d\tau =$$ \hspace{1cm} (1.9)

$$= \frac{\pi xy \sin \epsilon}{2} \frac{K_1((x^2 + y^2 - 2xy \cos \epsilon)^{\frac{1}{2}})}{(x^2 + y^2 - 2xy \cos \epsilon)^{\frac{1}{2}}}, \quad x, y > 0, \quad 0 < \epsilon \leq \pi.$$
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In the sequel we will appeal to the following definition of homogeneous functions:

**Definition 1.1** (c.f. [8]). Let $D \subseteq \mathbb{R}^n$. A function $f : D \to \mathbb{R}^n$ is said to be homogeneous of degree $\alpha$ in $D$ if and only if $f(\lambda x) = \lambda^\alpha f(x)$, for all $x \in D$, $\lambda > 0$ and $\lambda x \in D$. Here $\alpha \in \mathbb{R}$.

### 2. DEFINITION, BASIC PROPERTIES AND INVERSION

In this section we introduce the radial KL-transform. Given a function $f$ defined in $B_n^+$, the radial KL-transform of $f$ is given by

$$K_{i\tau}[f] = \int_{B_n^+} K_{i\tau}(|x|^2) \ f(x) \ dx,$$  \hspace{1cm} (2.1)

where $|x|^2 = x_1^2 + \cdots + x_n^2$, $dx = dx_1 \wedge \cdots \wedge dx_n$ and $B_n^+ = \{ x \in \mathbb{R}_n^+ : |x| \leq 1 \}$.

We remark that for the case of $n = 1$, the index transform (2.1) is a similar one used by Naylor in [9]. From (2.1) and definition of the Macdonald’s function (1.3), we conclude that the KL-transform of a function $f$ is an even function of real variable $\tau$ and, without loss of generality, we can consider only nonnegative variable $\tau$. From the asymptotic behavior of the Macdonald’s function given by (1.7), (1.8) and the Hölder inequality we observe that (2.1) is absolutely convergent for any function $f \in L_p(B_n^+)$. We have

**Lemma 2.1.** Let $f \in L_p(B_n^+)$, with $1 < p < +\infty$. Then the following uniform estimate by $\tau \geq 0$ for the KL-transform (2.1) holds

$$|K_{i\tau}[f]| \leq C_1 \ |f|_{L_p(B_n^+)},$$  \hspace{1cm} (2.2)

where $C$ is an absolute positive constant given by

$$C_1 = \left( \frac{(2\pi)^{2n-3}}{8q} \right)^{\frac{1}{4q}} \left( \frac{\pi}{4} \right)^{\frac{1}{2}} \Gamma \left( \frac{1}{2q} \right) \Gamma \left( \frac{1}{2} + \frac{1}{4q} \right),$$  \hspace{1cm} (2.3)

with $q = \frac{p}{p-1}$.
Proof. To establish (2.2) we appeal to (1.6) and the Hölder inequality in order to obtain

\[ |K_{i\tau}[f]| \leq \int_{B^n_+} K_0(|x|^2) \ |f(x)| \ dx = \]

\[ \leq \left( \int_{B^n_+} K_0^q(|x|^2) \ dx \right)^{\frac{1}{q}} \left( \int_{B^n_+} |f(x)|^p \ dx \right)^{\frac{1}{p}} = \]

\[ = \left( \int_{B^n_+} K_0^q(|x|^2) \ dx \right)^{\frac{1}{q}} \|f\|_{L^p(B^n_+)} \tag{2.4} \]

Further, using spherical coordinates, generalized Minkowski inequality and relation (2.5.46.6) in Prudnikov et al., [10], we get, in turn,

\[ \left( \int_{B^n_+} K_0^q(|x|^2) \ dx \right)^{\frac{1}{q}} \leq \int_{\mathbb{R}_+} \left( \int_{B^n_+} e^{-q|x|^2 \cosh u} \ dx \right)^{\frac{1}{q}} \ du = \]

\[ = \int_{\mathbb{R}_+} \left( \frac{2\pi)^{n-2}}{2} \int_0^{+\infty} e^{-q\rho^2 \cosh u} \rho^{n-1} d\rho \right)^{\frac{1}{q}} \ du \leq \]

\[ \leq \int_{\mathbb{R}_+} \left( \frac{2\pi)^{n-2}}{2} \int_0^{+\infty} e^{-q\rho^2 \cosh u} \rho^{n-1} d\rho \right)^{\frac{1}{q}} \ du = \]

\[ = \left( \frac{(2\pi)^{n-2}}{2} \sqrt{\frac{\pi}{q}} \right)^{\frac{1}{q}} \int_{\mathbb{R}_+} \frac{1}{(\cosh u)^{\frac{n-1}{2}}} \ du = \]

\[ = \left( \frac{2\pi)^{2n-3}}{8q} \right)^{\frac{1}{q}} \left( \frac{\pi}{4} \right)^{\frac{1}{2}} \left( \frac{1}{2} + \frac{1}{4q} \right) =: C_1. \tag{2.6} \]

The previous lemma shows that the KL-transform of a \( L^p \)-function is a continuous function on \( \tau \) in \( \mathbb{R}_+ \) in view of uniform convergence in (2.1). Moreover, we can deduce its differential properties. Precisely, performing the differentiation by \( \tau \) of arbitrary order \( k = 0, 1, \ldots \) under the integral representation (1.4) by Lebesgue’s theorem we find

\[ \frac{\partial^k}{\partial \tau^k} K_{i\tau}(|x|^2) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x|^2} \cosh u \ e^{i\tau u} \ (iu)^k \ du, \tag{2.5} \]

and

\[ \left| \frac{\partial^k}{\partial \tau^k} K_{i\tau}(|x|^2) \right| \leq \int_{\mathbb{R}_+} e^{-|x|^2} \cosh u \ u^k \ du. \tag{2.6} \]
Lemma 2.2. Under the conditions of Lemma 2.1 the KL-transform (2.1) is an infinitely differentiable function on the nonnegative real axis and for any \( k = 0, 1, \ldots \) we have the following estimate

\[
\left| \frac{\partial^k}{\partial \tau^k} K_{i\tau}[f] \right| \leq B_k \| f \|_{L_p(B^n_+)},
\]

where

\[
B_k = \left( \frac{(2\pi)^{n-1}}{4\sqrt{\pi q}} \right)^{\frac{1}{q}} \int_{\mathbb{R}^+} \frac{u^k}{(\cosh u)^{\frac{1}{q}}} \, du, \quad k = 0, 1, 2, \ldots.
\]

Proof. As in Lemma 2.1, making use of the Hölder inequality we derive

\[
\left| \frac{\partial^k}{\partial \tau^k} K_{i\tau}[f] \right| \leq \left( \int_{B^n_+} \left| \frac{\partial^k}{\partial \tau^k} K_{i\tau}(|x|^2) \right| \, dx \right)^{\frac{1}{q}} \| f \|_{L_p(B^n_+)}.
\]

Using estimate (2.6) it gives

\[
\left( \int_{B^n_+} \left| \frac{\partial^k}{\partial \tau^k} K_{i\tau}(|x|^2) \right| \, dx \right)^{\frac{1}{q}} \leq \int_{\mathbb{R}^+} u^k \left( \int_{B^n_+} e^{-q |x|^2 \cosh u} \, dx \right)^{\frac{1}{q}} \, du \leq \int_{\mathbb{R}^+} u^k \left( \frac{(2\pi)^{n-2}}{2} \sqrt{\frac{\pi}{q} \cosh u} \right)^{\frac{1}{q}} \, du = \left( \frac{(2\pi)^{n-1}}{4\sqrt{\pi q}} \right)^{\frac{1}{q}} \int_{\mathbb{R}^+} \frac{u^k}{(\cosh u)^{\frac{1}{q}}} \, du =: B_k.
\]

From the above properties of the KL-transform (2.1) one can discuss its belonging to \( L_r(\mathbb{R}^+) \) for some \( 1 < r < +\infty \), investigating only its behavior at infinity.

Lemma 2.3. The KL-transform (2.1) is a bounded map from any space \( L_p(B^n_+) \), with \( p \geq 1 \), into the space \( L_r(\mathbb{R}^+) \), where \( r \geq 1 \) and parameters \( p \) and \( r \) have no dependence.
Proof. Taking into account (1.5), with $\delta = \frac{\pi}{3}$, we obtain

$$|K_{i\tau}[f]| \leq e^{-\frac{\pi}{3}^{\frac{1}{4}} \int_{B_n^+} K_0 \left( \frac{|x|^2}{2} \right)|f(x)| dx \leq$$

$$\leq e^{-\frac{\pi}{3}^{\frac{1}{4}} \int_{B_n^+} (\int e^{-\frac{\mu^2 \cosh x}{2}} d\mu)^{\frac{1}{q}} dx \leq$$

$$\leq e^{-\frac{\pi}{3}^{\frac{1}{4}} \int_{\mathbb{R}^+} (\frac{2\pi)^{n-2}}{2} \int_0^{+\infty} e^{-\frac{\mu^2 \cosh u}{2}} \mu^{n-1} d\mu)^{\frac{1}{q}} du \|f\|_{L_p(B_n^+)} \leq$$

$$= e^{-\frac{\pi}{3}^{\frac{1}{4}} \int_{\mathbb{R}^+} (\frac{2\pi)^{n-2}}{2} \sqrt{\frac{2\pi}{q}} \int_{\mathbb{R}^+} \frac{1}{(\cosh u)^\frac{n}{q}} du \|f\|_{L_p(B_n^+)} =$$

$$= e^{-\frac{\pi}{3}^{\frac{1}{4}} \left( \frac{(2\pi)^{2n-3}}{4q} \right)^{\frac{1}{q}} \left( \frac{\pi}{4} \right)^{\frac{1}{q}} \left( \frac{\Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{1}{2} + \frac{1}{4} \right) \Gamma \left( \frac{1}{2} + \frac{1}{4} \right) \|f\|_{L_p(B_n^+)} =$$

$$= C_2 e^{-\frac{\pi}{3}^{\frac{1}{4}} \|f\|_{L_p(B_n^+)}}.$$
Our next aim is to obtain an inversion formula for the radial KL-transform (2.1). For this purpose we shall use the regularization operator of type

$$\left(I_{\epsilon} g\right)(x) = \frac{4|x|^{-n}(\sin \epsilon)^2}{(2\pi)^{n-1}} \int_{\mathbb{R}^+} \tau \ \sinh((\pi - \epsilon)\tau) \ K_{i\tau}(|x|^2) \ g(\tau) \ d\tau,$$

where $x \in B^n_+$ and $\epsilon \in [0, \pi]$.

**Theorem 2.5.** Let $p > 1$ and $n \in \mathbb{N}$. On functions $g(\tau) = K_{i\tau}[f]$ which are represented by (2.1) with density function $f \in L^p(B^n_+)$, operator (2.11) has the following representation

$$\left(I_{\epsilon} g\right)(x) = \frac{|x|^{-n+2}(\sin \epsilon)^2}{(2\pi)^{n-2}} \int_{B^n_+} K_1\left(\frac{|x|^4 + |y|^4 - 2|x|^2|y|^2 \cos \epsilon}{|y|^2 \cos \delta_2}\right) \ K_0\left(|y|^2 \cos \delta_2\right) \ |y|^2 \ f(y) \ dy,$$

where $K_1(z)$ is the Macdonald's function of index 1.

**Proof.** Substituting the value of $g(\tau)$ as the KL-transform (2.1) into (2.11), we change the order of integration by Fubini’s theorem taking into account the estimate (1.5)

$$|\left(I_{\epsilon} g\right)(x)| \leq \frac{4K_0(|x|^{2n} \cos \delta_1)(\sin \epsilon)^2}{|x|^n(2\pi)^{n-1}} \times \int_{\mathbb{R}^+} \tau \ \sinh((\pi - \epsilon)\tau) \ e^{-(\delta_1 + \delta_2)\tau} \int_{B^n_+} K_0(|y|^2 \cos \delta_2) \ |f(y)| \ dy \ d\tau,$$

where we choose $\delta_1, \delta_2$, such that $\delta_1 + \delta_2 + \epsilon > \pi$. Hence with (1.9) we get (2.12). \qed

An inversion formula of the KL-transform (2.1) is established by the following

**Theorem 2.6.** Let $p > 1$, $g(\tau) = K_{i\tau}[f]$ and $f \in L^p(B^n_+)$ be a radial function, i.e., $f(x) = h(|x|)$, where $h$ is a homogeneous of degree $2 - n$. Then

$$f(x) = \lim_{\epsilon \to 0} \frac{4|x|^{-n}(\sin \epsilon)^2}{(2\pi)^{n-1}} \int_{\mathbb{R}^+} \tau \ \sinh((\pi - \epsilon)\tau) \ K_{i\tau}(|x|^2) \ g(\tau) \ d\tau,$$

where the latter limit is with respect to $L^p$-norm in $L^p(B^n_+)$. 

\[\begin{align*}
\text{Proof.} \text{ Considering the integral (2.12) and the classical spherical coordinates multiplied by } |x| (sin \epsilon) \frac{1}{2}, \text{ we find} \\
\|(I_{L}g) - f\|_{L^{p}(B_{R}^{n})} = \\
\leq \left( \frac{\sin \epsilon}{2} \right)^{2} \int_{0}^{2\pi} \int_{0}^{\pi} \frac{R(|\cdot|, \rho, \epsilon) \rho^{3} h(|\cdot|) \rho \sin \phi \ d\rho \ d\phi \ d\theta \ldots \ d\theta_{n-2}}{[(\rho - \cot \epsilon)^{2} + 1]^2} \\
- \lim_{\epsilon \to 0^+} R(|\cdot|, \sqrt{\rho}, \epsilon) h(|\cdot|) \|_{L^{p}(B_{R}^{n})} \\
\leq \left( \frac{\sin \epsilon}{2} \right)^{2} \int_{0}^{2\pi} \int_{0}^{\pi} \frac{\rho}{[(\rho - \cot \epsilon)^{2} + 1]^2} \\
\left| R(|\cdot|, \sqrt{\rho}, \epsilon) h(|\cdot|) - \frac{1}{C_{\epsilon}(\cdot)} h(|\cdot|) \right| \|_{L^{p}(B_{R}^{n})} d\rho, \epsilon > 0, \\
(2.15) \end{align*}\]

where

\[R(|x|, \sqrt{\rho}, \epsilon) = |x|^{2} \sin \epsilon [(\rho - \cot \epsilon)^{2} + 1]^{\frac{1}{2}} K_{1}\left( |x|^{2} \sin \epsilon [(\rho - \cot \epsilon)^{2} + 1]^{\frac{1}{2}} \right), \quad \epsilon > 0,\]

and

\[C_{\epsilon}(x) = \sin \epsilon \int_{0}^{\rho} \frac{\rho}{[(\rho - \cot \epsilon)^{2} + 1]} \ d\rho = \]

\[= \cos \epsilon \left[ \arctan \left( \frac{\cos \epsilon}{\sin \epsilon} \right) - \arctan \left( \frac{|x|^{2} \cos \epsilon - 1}{|x|^{2} \sin \epsilon} \right) \right] + \]

\[+ \frac{\sin \epsilon}{2} \ln \left( \frac{(\cos \epsilon - |x|^{2})^{2} + (\sin \epsilon)^{2}}{|x|^{4}} \right), \quad \epsilon > 0.\]

For sufficiently small \(\epsilon > 0\) we have

\[0 < \pi - O(\epsilon) < C_{\epsilon}(x) < \pi + O(\epsilon).\]

Taking into account the relations (1.7) and (1.8), we have for \(R(|x|, \sqrt{\rho}, \epsilon)\) that

\[\lim_{\epsilon \to 0^+} R(|x|, \sqrt{\rho}, \epsilon) = 1,\]

and since \(x K_{1}(x) < 1\), for \(x > 0\), we conclude that \(R(|x|, \sqrt{\rho}, \epsilon)\) is bounded as a function of three variables. Further, since \(R(|x|, \sqrt{\rho}, \epsilon) < 1\) we obtain

\[\|(I_{L}g) - f\|_{L^{p}(B_{R}^{n})} \leq \frac{\sin \epsilon}{2} (C_{\epsilon} + 1) ||h||_{L^{p}(B_{R}^{n})} = O(\epsilon) \to 0, \quad \epsilon \to 0^+, \quad (2.16)\]

which leads to the equality (2.14). \(\square\)
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