AN ALGORITHM
FOR FINDING A COMMON SOLUTION FOR A SYSTEM
OF MIXED EQUILIBRIUM PROBLEM,
QUASI-VARIATIONAL INCLUSION PROBLEM
AND FIXED POINT PROBLEM
OF NONEXPANSIVE SEMIGROUP

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Abstract. In this paper, we introduce a hybrid iterative scheme for finding a common element of the set of solutions for a system of mixed equilibrium problems, the set of common fixed points for a nonexpansive semigroup and the set of solutions of the quasi-variational inclusion problem with multi-valued maximal monotone mappings and inverse-strongly monotone mappings in a Hilbert space. Under suitable conditions, some strong convergence theorems are proved. Our results extend some recent results in the literature.

Keywords: nonexpansive semigroup, mixed equilibrium problem, viscosity approximation method, quasi-variational inclusion problem, multi-valued maximal monotone mappings, α-inverse-strongly monotone mapping.

Mathematics Subject Classification: 47H09, 47H05.

1. INTRODUCTION

Throughout this paper we assume that H is a real Hilbert space and C is a nonempty closed convex subset of H.

In the sequel, we denote the set of fixed points of a mapping S by $F(S)$.

A bounded linear operator $A : H \to H$ is said to be *strongly positive*, if there exists a constant $\bar{\gamma}$ such that

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \ \ \forall \, x \in H. \tag{1.1}$$

Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a multi-valued mapping. The "so-called" *quasi-variational inclusion problem* (see, Chang [2,3]) is to find an $u \in H$ such that

$$\theta \in B(u) + M(u). \tag{1.2}$$

A number of problems arising in structural analysis, mechanics and economics can be studied in a framework of this kind of variational inclusions (see, for example [5]).

The set of solutions of quasi-variational inclusion (1.2) is denoted by **VI(H,B,M)**.

Special Case

If $M = \partial \delta_C$, where $\partial \delta_C$ is the subdifferential of δ_C, C is a nonempty closed convex subset of H and $\delta_C : H \to [0, \infty)$ is the indicator function of C, i.e.,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then the quasi-variational inclusion problem (1.2) is equivalent to find $u \in C$ such that

$$\langle B(u), v - u \rangle \ge 0, \quad \forall v \in C. \tag{1.3}$$

This problem is called the *Hartman-Stampacchia variational inequality problem* (see, for example [7]). The set of solutions of (1.3) is denoted by **VI(C, B)**.

Recall that a mapping $B : H \to H$ is called α*-inverse strongly monotone* (see [13]), if there exists an $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \ge \alpha \|Bx - By\|^2, \quad \forall x, y \in H.$$

A multi-valued mapping $M : H \to 2^H$ is called *monotone*, if for all $x, y \in H, u \in Mx$, and $v \in My$, implies that $\langle u-v, x-y \rangle \ge 0$. A multi-valued mapping $M : H \to 2^H$ is called *maximal monotone*, if it is monotone and if for any $(x, u) \in H \times H$

$$\langle u - v, \ x - y \rangle \ge 0, \quad \forall \, (y, v) \in Graph(M)$$

(the graph of mapping M) implies that $u \in Mx$.

Proposition 1.1 ([13]). *Let $B : H \to H$ be an α-inverse strongly monotone mapping, then:*

(a) *B is $\frac{1}{\alpha}$-Lipschitz continuous and a monotone mapping;*
(b) *If λ is any constant in $(0, 2\alpha]$, then the mapping $I - \lambda B$ is nonexpansive, where I is the identity mapping on H.*

Let C be a nonempty closed convex subset of H, $\Theta : C \times C \to R$ be an equilibrium bifunction (i.e., $\Theta(x, x) = 0$, $\forall x \in C$) and let $\varphi : C \to R$ be a real-valued function.

Recently, Ceng and Yao [1] introduced the following *mixed equilibrium problem* (MEP), i.e., to find $z \in C$ such that

$$MEP : \Theta(z, y) + \varphi(y) - \varphi(z) \ge 0, \ \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $MEP(\Theta, \varphi)$, i.e.,

$$MEP(\Theta) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) \geq 0, \ \forall y \in C\}.$$

In particular, if $\varphi = 0$, this problem reduces to the *equilibrium problem*, i.e., to find $z \in C$ such that

$$EP : \quad \Theta(z, y) \geq 0, \ \forall y \in C.$$

Denote the set of solution of EP by $EP(\Theta)$.

On the other hand, Li *et al.* [6] introduced a two step iterative procedure for the approximation of common fixed points of a nonexpansive semigroup $\{T(s) : 0 \leq s < \infty\}$ on a nonempty closed convex subset C in a Hilbert space.

Very recently, Saeidi [9] introduced a more general iterative algorithm for finding a common element of the set of solutions for a system of equilibrium problems and of the set of common fixed points for a finite family of nonexpansive mappings and a nonexpansive semigroup.

Recall that a family of mappings $\mathscr{T} = \{T(s) : 0 \leq s < \infty\} : C \to C$ is called *a nonexpansive semigroup*, if it satisfies the following conditions:

(a) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$ and $T(0) = I$;
(b) $\|T(s)x - T(s)y\| \leq \|x - y\|$, $\forall x, y \in C$.
(c) The mapping $T(\cdot)x$ is continuous, for each $x \in C$.

Motivated and inspired by Ceng and Yao [1], Li *et al.* [6] and Saeidi [9], the purpose of this paper is to introduce a hybrid iterative scheme for finding a common element of the set of solutions for a system of mixed equilibrium problems, the set of common fixed points for a nonexpansive semigroup and the set of solutions of the quasi-variational Inclusion problem with multi-valued maximal monotone mappings and inverse-strongly monotone mappings in Hilbert space. Under suitable conditions, some strong convergence theorems are proved. Our results extends the recent results in Zhang, Lee and Chan [13], Takahashi and Takahashi [11], Chang, Joseph Lee and Chan [4], Ceng and Yao [1], Li *et al.* [6] and Saeidi [9].

2. PRELIMINARIES

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and strong convergence of the sequence $\{x_n\}$ in H, respectively.

Definition 2.1. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \ \forall u \in H$$

is called the *resolvent operator associated with* M, where λ is any positive number and I is the identity mapping.

Proposition 2.2 ([13]). (a) *The resolvent operator $J_{M,\lambda}$ associated with M is single-valued and nonexpansive for all $\lambda > 0$, i.e.,*

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \ \ \forall x, y \in H, \ \forall \lambda > 0.$$

(b) *The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone, i.e.,*

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \leq \langle x - y, \ J_{M,\lambda}(x) - J_{M,\lambda}(y)\rangle, \ \ \forall x, y \in H.$$

Definition 2.3. A single-valued mapping $P : H \to H$ is said to be *hemi-continuous*, if for any $x, y \in H$, the mapping $t \mapsto P(x + ty)$ converges weakly to Px (as $t \to 0+$).

It is well-known that every continuous mapping must be hemi-continuous.

Lemma 2.4 ([8]). *Let E be a real Banach space, E^* be the dual space of E, $T : E \to 2^{E^*}$ be a maximal monotone mapping and $P : E \to E^*$ be a hemi-continuous bounded monotone mapping with $D(P) = E$ then the mapping $S = T + P : E \to 2^{E^*}$ is a maximal monotone mapping.*

For solving the equilibrium problem for bifunction $\Theta : C \times C \to R$, let us assume that Θ satisfies the following conditions:
(H_1) $\Theta(x, x) = 0$ for all $x \in C$.
(H_2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$.
(H_3) For each $y \in C$, $x \mapsto \Theta(x, y)$ is concave and upper semicontinuous.
(H_4) For each $x \in C$, $y \mapsto \Theta(x, y)$ is convex.
A map $\eta : C \times C \to H$ is called Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|\eta(x, y)\| \leq L\|x - y\|, \ \ \forall x, y \in C.$$

A differentiable function $K : C \to R$ on a convex set C is called:

(i) η-convex [1] if

$$K(y) - K(x) \geq \langle K'(x), \eta(y, x)\rangle, \ \ \forall x, y \in C,$$

where $K'(x))$ is the *Fréchet* derivative of K at x;

(ii) η-strongly convex[7] if there exists a constant $\mu > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y, x)\rangle \geq \left(\frac{\mu}{2}\right)\|x - y\|^2, \ \ \forall x, y \in C.$$

Let $\Theta : C \times C \to R$ be an equilibrium bifunction satisfying the conditions $(H_1) - (H_4)$. Let r be any given positive number. For a given point $x \in C$, consider the following *auxiliary problem for MEP* (for short, $MEP(x, r)$): to find $y \in C$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \geq 0, \ \ \forall z \in C,$$

where $\eta : C \times C \to H$ is a mapping and $K'(x)$ is the *Fréchet* derivative of a functional $K : C \to R$ at x. Let $V_r^{\Theta} : C \to C$ be the mapping such that for each $x \in C$, $V_r^{\Theta}(x)$ is the solution set of $MEP(x, r)$, i.e.,

$$V_r^{\Theta}(x) = \{y \in C : \Theta(y, z) + \varphi(z) - \varphi(y)+ $$
$$+ \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \geq 0, \ \ \forall z \in C\}, \ \ \forall x \in C. \tag{2.1}$$

Then the following conclusion holds:.

Proposition 2.5 ([1]). *Let C be a nonempty closed convex subset of H, $\varphi : C \to R$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to R$ be an equilibrium bifunction satisfying conditions* (H_1)–(H_4). *Assume that:*

(i) $\eta : C \times C \to H$ *is Lipschitz continuous with constant $L > 0$ such that:*
  (a) $\eta(x, y) + \eta(y, x) = 0, \ \forall x, y \in C$,
  (b) $\eta(\cdot, \cdot)$ *is affine in the first variable,*
  (c) *for each fixed $y \in C, x \mapsto \eta(y, x)$ is continuous from the weak topology to the weak topology;*
(ii) $K : C \to R$ *is η-strongly convex with constant $\mu > 0$ and its derivative K' is continuous from the weak topology to the strong topology;*
(iii) *for each $x \in C$, there exist a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$, the following holds:*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0.$$

Then the following holds:

(i) V_r^Θ *is single-valued;*
(ii) V_r^Θ *is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ such that $\mu \geq L\nu$;*
(iii) $F(V_r^\Theta) = MEP(\Theta)$;
(iv) $MEP(\Theta)$ *is closed and convex.*

Lemma 2.6 ([10]). *Let C be a nonempty bounded closed convex subset of H and let $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, then for any $h \geq 0$.*

$$\lim_{t \to \infty} \sup_{x \in C} \|\frac{1}{t} \int_0^t T(s)x\,ds - T(h)(\frac{1}{t} \int_0^t T(s)x\,ds)\| = 0.$$

Lemma 2.7 ([6]). *Let C be a nonempty bounded closed convex subset of H and let $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $\limsup_{s \to \infty} \limsup_{n \to \infty} \|T(s)x_n - x_n\| = 0$, then $z \in F(\Im)$.*

3. MAIN RESULTS

In order to prove the main result, we first give the following Lemma.

Lemma 3.1 ([13]). (a) $u \in H$ *is a solution of variational inclusion* (1.2) *if and only if $u = J_{M,\lambda}(u - \lambda Bu), \ \forall \lambda > 0, \ \ i.e.,$*

$$VI(H, B, M) = F(J_{M,\lambda}(I - \lambda B)), \ \forall \lambda > 0.$$

(b) *If $\lambda \in (0, 2\alpha]$, then $VI(H, B, M)$ is a closed convex subset in H.*

In the sequel, we assume that H, C, M, A, B, f, $\mathscr{T}$, $\mathscr{F}$, φ_i, $\eta_i, K_i(i = 1, 2, \cdots N)$ satisfy the following conditions:

(1) H is a real Hilbert space, $C \subset H$ is a nonempty closed convex subset;
(2) $A : H \to H$ is a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$, $f : H \to H$ is a contraction mapping with a contraction constant h $(0 < h < 1)$ and $0 < \gamma < \frac{\bar{\gamma}}{h}$, $B : C \to H$ is a α-inverse-strongly monotone mapping and $M : H \to 2^H$ is a multi-valued maximal monotone mapping;
(3) $\mathscr{T} = \{T(s) : 0 \le s < \infty\} : C \to C$ is a nonexpansive semigroup;
(4) $\mathscr{F} = \{\Theta_i : i = 1, 2, \cdots, N\} : C \times C \to R$ is a finite family of bifunctions satisfying conditions $(H_1) - (H_4)$ and $\varphi_i : C \to R(i = 1, 2, \cdots, N)$ is a finite family of lowersemi-continuous and convex functionals;
(5) $\eta_i : C \times C \to H$ is a finite family of Lipschitz continuous mappings with constant $L_i > 0 (i = 1, 2, \cdots, N)$ such that:
   (a) $\eta_i(x, y) + \eta_i(y, x) = 0$, $\forall x, y \in C$,
   (b) $\eta_i(\cdot, \cdot)$ is affine in the first variable,
   (c) for each fixed $y \in C, x \mapsto \eta_i(y, x)$ is continuous from the weak topology to the weak topology;
(6) $K_i : C \to R$ is a finite family of η_i-strongly convex with constant $\mu_i > 0$ and its derivative K_i' is not only continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu_i > 0$, $\mu_i \ge L_i \nu_i$.

In the sequel we always denote by $F(\mathscr{T})$ the set of fixed points of the nonexpansive semi-group $\mathscr{T}$, VI(H,B,M) the set of solutions to the variational inequality (1.2) and MEP($\mathscr{F}$) the set of solutions to the following *auxiliary problem for a system of mixed equilibrium problems*:

$$
\begin{cases}
\Theta_1(y_n^{(1)}, x) + \phi_1(x) - \phi_1(y_n^{(1)}) + \dfrac{1}{r_1}\langle K'(y_n^{(1)}) - K'(x_n), \, \eta_1(x, y_n^{(1)})\rangle \ge 0, \quad \forall x \in C, \\[2mm]
\Theta_2(y_n^{(2)}, x) + \phi_2(x) - \phi_2(y_n^{(2)}) + \dfrac{1}{r_2}\langle K'(y_n^{(2)}) - K'(y_n^{(1)}), \, \eta_2(x, y_n^{(2)})\rangle \ge 0, \quad \forall x \in C, \\[2mm]
\vdots \\[2mm]
\Theta_{N-1}(y_n^{(N-1)}, x) + \phi_{N-1}(x) - \phi_{N-1}(y_n^{(N-1)}) + \\[1mm]
\quad + \dfrac{1}{r_{N-1}}\langle K'(y_n^{(N-1)}) - K'(y_n^{(N-2)}), \, \eta_{N-1}(x, y_n^{(N-1)})\rangle \ge 0, \quad \forall x \in C, \\[2mm]
\Theta_N(y_n, x) + \phi_N(x) - \phi_N(y_n) + \\[1mm]
\quad + \dfrac{1}{r_N}\langle K'(y_n) - K'(y_n^{(N-1)}), \, \eta_N(x, y_n)\rangle \ge 0, \quad \forall x \in C,
\end{cases}
$$

where

$$
\begin{cases}
y_n^{(1)} = V_{r_1}^{\Theta_1} x_n, \\[2mm]
y_n^{(i)} = V_{r_i}^{\Theta_i} y_n^{(i-1)} = V_{r_i}^{\Theta_i} V_{r(i-1)}^{\Theta_{i-1}} y_n^{(i-2)} = V_{r_i}^{\Theta_i} \cdots V_{r_2}^{\Theta_2} y_n^{(1)} \\[2mm]
\quad = V_{r_i}^{\Theta_i} \cdots V_{r_2}^{\Theta_2} V_{r_1}^{\Theta_1} x_n, \quad i = 2, 3, \cdots, N-1, \\[2mm]
y_n = V_{r_N}^{\Theta_N} \cdots V_{r_2}^{\Theta_2} V_{r_1}^{\Theta_1} x_n,
\end{cases}
$$

and $V_{r_i}^{\Theta_i} : C \to C, \; i = 1, 2, \cdots, N$ is the mapping defined by (2.1)

In the sequel we denote by $\mathscr{V}^l = V_{r_l}^{\Theta_l} \cdots V_{r_2}^{\Theta_2} V_{r_1}^{\Theta_1}$ for $l \in \{1, 2, \cdots, N\}$ and $\mathscr{V}^0 = I$.

Theorem 3.2. *Let H, C, A, B, M, f, $\mathscr{T}$, $\mathscr{F}$, φ_i, $\eta_i, K_i (i = 1, 2, \cdots, N)$ be the same as above. Let $\{x_n\}$, $\{\rho_n\}$, $\{\xi_n\}$ and $\{y_n\}$ be the explicit iterative sequences generated by $x_1 \in H$ and*

$$
\begin{cases}
x_{n+1} = \alpha_n \gamma f\left(\dfrac{1}{t_n} \displaystyle\int_0^{t_n} T(s) x_n ds\right) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(s)\rho_n ds, \\[4pt]
\rho_n = J_{M,\lambda}(I - \lambda B)\xi_n, \\
\xi_n = J_{M,\lambda}(I - \lambda B)y_n, \\
y_n = V_{r_N}^{\Theta_N} \cdots V_{r_2}^{\Theta_2} V_{r_1}^{\Theta_1} x_n
\end{cases} \qquad \forall \, n \geq 1,
$$

(3.1)

where $r_i(i = 1, 2, \cdots, N)$ be a finite family of positive numbers, $\lambda \in (0, 2\alpha]$, $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ and $\{t_n\} \subset (0, \infty)$ is a sequence with $t_n \uparrow \infty$. If $\mathscr{G} := F(\mathscr{T}) \bigcap MEP(\mathscr{F}) \bigcap VI(H, B, M) \neq \emptyset$ and the following conditions are satisfied:

(i) *for each $x \in C$, there exist a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,*

$$
\Theta_i(y, z_x) + \varphi_i(z_x) - \varphi_i(y) + \frac{1}{r_i}\langle K_i'(y) - K_i'(x), \eta_i(z_x, y)\rangle < 0.
$$

(ii) *$\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$, then the sequence $\{x_n\}$ converges strongly to $x^* = P_{\mathscr{G}}(I - A + \gamma f)(x^*)$, provided that $V_{r_i}^{\Theta_i}$ is firmly nonexpansive where $P_{\mathscr{G}}$ is the metric projection of H onto $\mathscr{G}$.*

Proof. We observe that from conditions (ii), we can assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$.

Since A is a linear bounded self-adjoint operator on H, then

$$
\|A\| = \sup\{|\langle Au, u\rangle| : u \in H, \|u\| = 1\}.
$$

Since

$$
\langle((1 - \beta_n)I - \alpha_n A)u, u\rangle = 1 - \beta_n - \alpha_n\langle Au, u\rangle \geq 1 - \beta_n - \alpha_n\|A\| \geq 0,
$$

this implies that $(1 - \beta_n)I - \alpha_n A$ is positive. Hence we have

$$
\begin{aligned}
\|(1 - \beta_n)I - \alpha_n A\| &= \sup\{|\langle((1 - \beta_n)I - \alpha_n A)u, u\rangle| : u \in H, \|u\| = 1\} \\
&= \sup\{1 - \beta_n - \alpha_n\langle Au, \, u\rangle : u \in H, \|u\| = 1\} \leq \\
&\leq 1 - \beta_n - \alpha_n\bar{\gamma} < 1.
\end{aligned}
$$

Let $Q = P_{\mathscr{G}}$. Note that f is a contraction with coefficient $h \in (0, 1)$. Then, we have

$$\|Q(I - A + \gamma f)(x) - Q(I - A + \gamma f)(y)\| \leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \leq$$
$$\leq \|I - A\|\|x - y\| + \gamma\|f(x) - f(y)\| \leq$$
$$\leq (1 - \bar{\gamma})\|x - y\| + \gamma h\|x - y\| =$$
$$= (1 - (\bar{\gamma} - \gamma h))\|x - y\|,$$

for all $x, y \in H$. Therefore, $Q(I - A + \gamma f)$ is a contraction of H into itself, which implies that there exists a unique element $x^* \in H$ such that $x^* = Q(I - A + \gamma f)(x^*) = P_{\mathscr{G}}(I - A + \gamma f)(x^*)$.

Next, we divide the proof of Theorem 3.2 into 9 steps:

Step 1. First prove the sequences $\{x_n\}$, $\{\rho_n\}$, $\{\xi_n\}$ and $\{y_n\}$ are bounded.

(a) Pick $p \in \mathscr{G}$, since $y_n = \mathscr{V}^N x_n$ and $p = \mathscr{V}^N p$, we have

$$\|y_n - p\| = \|\mathscr{V}^N x_n - p\| \leq \|x_n - p\|. \tag{3.2}$$

(b) Since $p \in VI(H, B, M)$ and $\rho_n = J_{M,\lambda}(I - \lambda B)\xi_n$, we have $p = J_{M,\lambda}(I - \lambda B)p$, and so

$$\|\rho_n - p\| = \|J_{M,\lambda}(I - \lambda B)\xi_n - J_{M,\lambda}(I - \lambda B)p\| \leq$$
$$\leq \|(I - \lambda B)\xi_n - (I - \lambda B)p\| \leq \|\xi_n - p\| =$$
$$= \|J_{M,\lambda}(I - \lambda B)y_n - J_{M,\lambda}(I - \lambda B)p\| \leq$$
$$\leq \|y_n - p\| \leq \|x_n - p\|. \tag{3.3}$$

Letting $u_n = \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds$, $q_n = \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds$, we have

$$\|u_n - p\| = \|\frac{1}{t_n}\int\limits_0^{t_n} T(s)x_n ds - p\| \leq \frac{1}{t_n}\int\limits_0^{t_n} \|T(s)x_n - T(s)p\|ds \leq \|x_n - p\|. \tag{3.4}$$

Similarly, we have

$$\|q_n - p\| \leq \|\rho_n - p\|. \tag{3.5}$$

Form (3.1), (3.2), (3.3), (3.4)and (3.5), we have

$$\|x_{n+1} - p\| =$$
$$= \|\alpha_n\gamma f(u_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)q_n - p\| =$$
$$= \|\alpha_n\gamma(f(u_n) - f(p)) + \beta_n(x_n - p)+$$
$$+ ((1 - \beta_n)I - \alpha_n A)(q_n - p) + \alpha_n(\gamma f(p) - Ap)\| \leq$$
$$\leq \alpha_n\gamma h\|u_n - p\| + \beta_n\|x_n - p\| + ((1 - \beta_n) - \alpha_n\bar{\gamma})\|q_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \leq$$
$$\leq \alpha_n\gamma h\|x_n - p\| + \beta_n\|x_n - p\| + ((1 - \beta_n) - \alpha_n\bar{\gamma})\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \leq$$
$$\leq (1 - \alpha_n(\bar{\gamma} - \gamma h))\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \leq$$
$$\leq max\|x_n - p\|, \frac{1}{\bar{\gamma} - \gamma h}\|\gamma f(p) - Ap\|$$
$$\vdots$$
$$\leq max\|x_1 - p\|, \frac{1}{\bar{\gamma} - \gamma h}\|\gamma f(p) - Ap\|.$$

This implies that $\{x_n\}$ is a bounded sequence in H. Therefore $\{y_n\}$, $\{\rho_n\}$, $\{\xi_n\}$, $\{\gamma f(u_n)\}$ and $\{q_n\}$ are all bounded.

Step 2. Next we prove that

$$\|x_{n+1} - x_n\| \to 0 \ (n \to \infty). \tag{3.6}$$

In fact, let us define a sequence $\{z_n\}$ by

$$x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n \qquad \forall n \geq 1,$$

then we have

$$
\begin{aligned}
z_{n+1} - z_n &= \\
&= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \\
&= \frac{\alpha_{n+1}\gamma f(u_{n+1}) + ((1 - \beta_{n+1})I - \alpha_{n+1}A)q_{n+1}}{1 - \beta_{n+1}} - \\
&\quad - \frac{\alpha_n \gamma f(u_n) + ((1 - \beta_n)I - \alpha_n A)q_n}{1 - \beta_n} = \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}[\gamma f(u_{n+1}) - Aq_{n+1}] - \frac{\alpha_n}{1 - \beta_n}[\gamma f(u_n) - Aq_n] + q_{n+1} - q_n
\end{aligned}
$$

and so

$$
\begin{aligned}
\|z_{n+1} - z_n\| &- \|x_{n+1} - x_n\| \leq \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\|\gamma f(u_{n+1})\| + \|Aq_{n+1}\|) + \\
&\quad + \frac{\alpha_n}{1 - \beta_n}(\|\gamma f(u_n)\| + \|Aq_n\|) + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_{n+1}ds - \frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_n ds\| + \\
&\quad + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds\| - \|x_{n+1} - x_n\| \leq \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\|\gamma f(u_{n+1})\| + \|Aq_{n+1}\|) + \\
&\quad + \frac{\alpha_n}{1 - \beta_n}(\|\gamma f(u_n)\| + \|Aq_n\|) + \frac{1}{t_{n+1}}\int_0^{t_{n+1}} \|T(s)\rho_{n+1} - T(s)\rho_n\|ds + \\
&\quad + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds\| - \|x_{n+1} - x_n\|.
\end{aligned}
$$

$$\tag{3.7}$$

Since $\rho_n = J_{M,\lambda}(I - \lambda B)\xi_n$ and $y_{n+1} = \mathscr{V}^N(x_{n+1})$, $y_n = \mathscr{V}^N(x_n)$, from the nonexpansivity of $\mathscr{V}^N$. we have

$$
\begin{aligned}
\|\rho_{n+1} - \rho_n\| &= \|J_{M,\lambda}(I - \lambda B)\xi_{n+1} - J_{M,\lambda}(I - \lambda B)\xi_n\| \leq \\
&\leq \|\xi_{n+1} - \xi_n\| = \\
&= \|J_{M,\lambda}(I - \lambda B)y_{n+1} - J_{M,\lambda}(I - \lambda B)y_n\| \leq \\
&\leq \|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|.
\end{aligned}
\tag{3.8}
$$

Substituting (3.8) into (3.7), we get

$$
\begin{aligned}
&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\|\gamma f(u_{n+1})\| + \|Aq_{n+1}\|) + \\
&\quad + \frac{\alpha_n}{1 - \beta_n}(\|\gamma f(u_n)\| + \|Aq_n\|) + \frac{1}{t_{n+1}}\int_0^{t_{n+1}}\|T(s)\rho_{n+1} - T(s)\rho_n\|ds + \\
&\quad + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n}T(s)\rho_n ds\| - \|x_{n+1} - x_n\| \leq \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\|\gamma f(u_{n+1})\| + \|Aq_{n+1}\|) + \\
&\quad + \frac{\alpha_n}{1 - \beta_n}(\|\gamma f(u_n)\| + \|Aq_n\|) + \frac{1}{t_{n+1}}\int_0^{t_{n+1}}\|x_{n+1} - x_n\|ds + \\
&\quad + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n}T(s)\rho_n ds\| - \|x_{n+1} - x_n\| \leq \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\|\gamma f(u_{n+1})\| + \|Aq_{n+1}\|) + \frac{\alpha_n}{1 - \beta_n}(\|\gamma f(u_n)\| + \|Aq_n\|) + \\
&\quad + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n}T(s)\rho_n ds\|.
\end{aligned}
\tag{3.9}
$$

From conditions $t_n \subset (0, \infty)$ and $t_n \uparrow \infty$, we have

$$
\begin{aligned}
&\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}}T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n}T(s)\rho_n ds\| = \\
&= \|\frac{1}{t_{n+1}}(\int_0^{t_n}T(s)\rho_n ds + \int_{t_n}^{t_{n+1}}T(s)\rho_n ds) - \frac{1}{t_n}\int_0^{t_n}T(s)\rho_n ds\| \leq \\
&\leq \frac{1}{t_n t_{n+1}}\int_0^{t_n}\|(t_n - t_{n+1})T(s)\rho_n\|ds + \frac{1}{t_{n+1}}\int_{t_n}^{t_{n+1}}\|T(s)\rho_n\|ds = \\
&= \frac{t_{n+1} - t_n}{t_{n+1}}M + \frac{t_{n+1} - t_n}{t_{n+1}}M = 2M(1 - \frac{t_n}{t_{n+1}}) \to 0,
\end{aligned}
$$

where $M = \sup_{s \geq 0, n \geq 1} \|T(s)\rho_n\|$. From (3.9) and conditions $\lim_{n \to \infty} \alpha_n = 0$ and $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$ that

$$\limsup_{n \to \infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, we have

$$\lim_{n \to \infty} \|z_n - x_n\| = 0.$$

Consequently

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n)\|z_n - x_n\| = 0.$$

Step 3. Next we prove that

$$\lim_{n \to \infty} \|x_n - q_n\| = 0. \tag{3.10}$$

Since

$$\|x_n - q_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - q_n\| \leq$$
$$\leq \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(u_n) - Aq_n\| + \beta_n\|x_n - q_n\|,$$

simplifying it we have

$$\|x_n - q_n\| \leq \frac{1}{1 - \beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|\gamma f(u_n) - Aq_n\|.$$

Since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, and $\{\gamma f(u_n) - Aq_n\}$ is bounded, from the condition $\lim_{n \to \infty} \alpha_n = 0$ and $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$, we have $\|x_n - q_n\| \to 0$.
Step 4. Next we prove that

$$\|x_{n+1} - T(s)x_{n+1}\| \to 0 \ (n \to \infty). \tag{3.11}$$

Since $x_{n+1} = \alpha_n\gamma f(u_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)q_n$, then

$$\|x_{n+1} - q_n\| \leq \alpha_n\|\gamma f(u_n) - Aq_n\| + \beta_n\|x_n - q_n\|.$$

From condition $\lim_{n \to \infty} \alpha_n = 0$ and $\|x_n - q_n\| \to 0$, we have

$$\|x_{n+1} - q_n\| \to 0. \tag{3.12}$$

Let $K = \{w \in C : \|w - p\| \leq max\|x_1 - p\|, \frac{1}{\bar{\gamma} - \gamma h}\|\gamma f(p) - Ap\|$, then K is a nonempty bounded closed convex subset of C and $T(s)$-invariant. Since $\{x_n\} \subset K$ and K is bounded, there exists $r > 0$ such that $K \subset B_r$, it follows from Lemma 2.6 that

$$\lim_{n \to \infty} \|q_n - T(s)q_n\| \to 0. \tag{3.13}$$

From (3.12) and (3.13) , we have

$$\|x_{n+1} - T(s)x_{n+1}\| = \|x_{n+1} - q_n + q_n - T(s)q_n + T(s)q_n - T(s)x_{n+1}\| \leq$$
$$\leq \|x_{n+1} - q_n\| + \|q_n - T(s)q_n\| + \|T(s)q_n - T(s)x_{n+1}\| \leq$$
$$\leq \|x_{n+1} - q_n\| + \|q_n - T(s)q_n\| + \|q_n - x_{n+1}\| \to 0.$$

Step 5. Next we prove that

$$(i) \quad \lim_{n\to\infty} \|\mathscr{V}^{l+1}x_n - \mathscr{V}^l x_n\| = 0, \ \forall \ l \in \{0, 1, \cdots, N-1\};$$

$$(ii) \quad Especially, \ \lim_{n\to\infty} \|\mathscr{V}^N x_n - x_n\| = \lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.14}$$

In fact, for any given $p \in \mathscr{G}$ and $l \in \{0, 1, \cdots, N-1\}$, Since $V_{r_{l+1}}^{\Theta_{l+1}}$ is firmly nonexpansive, we have

$$\begin{aligned}
\|\mathscr{V}^{l+1}x_n - p\|^2 &= \|V_{r_{l+1}}^{\Theta_{l+1}}(\mathscr{V}^l x_n) - V_{r_{l+1}}^{\Theta_{l+1}} p\|^2 \leq \\
&\leq \langle V_{r_{l+1}}^{\Theta_{l+1}}(\mathscr{V}^l x_n) - p, \ \mathscr{V}^l x_n - p \rangle = \\
&= \langle \mathscr{V}^{l+1}x_n - p, \ \mathscr{V}^l x_n - p \rangle = \\
&= \frac{1}{2}(\|\mathscr{V}^{l+1}x_n - p\|^2 + \|\mathscr{V}^l x_n - p\|^2 - \|\mathscr{V}^l x_n - \mathscr{V}^{l+1}x_n\|^2).
\end{aligned}$$

It follows that

$$\|\mathscr{V}^{l+1}x_n - p\|^2 \leq \|x_n - p\|^2 - \|\mathscr{V}^l x_n - \mathscr{V}^{l+1}x_n\|^2. \tag{3.15}$$

From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \\
&= \|\alpha_n \gamma f(u_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)q_n - p\|^2 = \\
&= \|\alpha_n(\gamma f(u_n) - Ap) + \beta_n(x_n - q_n) + (I - \alpha_n A)(q_n - p)\|^2 \leq \\
&\leq \|(I - \alpha_n A)(q_n - p) + \beta_n(x_n - q_n)\|^2 + 2\alpha_n \langle \gamma f(u_n) - Ap, x_{n+1} - p \rangle \leq \\
&\leq [\|(I - \alpha_n A)(q_n - p)\| + \beta_n \|(x_n - q_n)\|]^2 + 2\alpha_n \langle \gamma f(u_n) - Ap, x_{n+1} - p \rangle \leq \\
&\leq [(1 - \alpha_n \bar{\gamma})\|\rho_n - p\| + \beta_n \|x_n - q_n\|]^2 + 2\alpha_n \langle \gamma f(u_n) - Ap, x_{n+1} - p \rangle = \\
&= (1 - \alpha_n \bar{\gamma})^2 \|\rho_n - p\|^2 + \beta_n^2 \|x_n - q_n\|^2 + 2(1 - \alpha_n \bar{\gamma})\beta_n \|\rho_n - p\| \cdot \|x_n - q_n\| + \\
&\quad + 2\alpha_n \|\gamma f(u_n) - Ap\| \cdot \|x_{n+1} - p\|.
\end{aligned} \tag{3.16}$$

Since

$$\|\rho_n - p\| \leq \|\xi_n - p\| \leq \|\mathscr{V}^N x_n - p\| \leq \|\mathscr{V}^{l+1}x_n - p\| \ \forall \ l \in \{0, 1, \cdots, N-1\}.$$

Substituting (3.15) into (3.16), it yields

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \\
&\leq (1 - \alpha_n \bar{\gamma})^2 \{\|x_n - p\|^2 - \|\mathscr{V}^l x_n - \mathscr{V}^{l+1}x_n\|^2\} + \beta_n^2 \|x_n - q_n\|^2 + \\
&\quad + 2(1 - \alpha_n \bar{\gamma}) \cdot \beta_n \|\rho_n - p\| \cdot \|x_n - q_n\| + 2\alpha_n \|\gamma f(u_n) - Ap\| \cdot \|x_{n+1} - p\| = \\
&= (1 - 2\alpha_n \bar{\gamma} + (\alpha_n \bar{\gamma})^2)\|x_n - p\|^2 - (1 - \alpha_n \bar{\gamma})^2 \|\mathscr{V}^l x_n - \mathscr{V}^{l+1}x_n\|^2 + \beta_n^2 \|x_n - q_n\|^2 + \\
&\quad + 2(1 - \alpha_n \bar{\gamma})\beta_n \|\rho_n - p\| \cdot \|x_n - q_n\| + 2\alpha_n \|\gamma f(u_n) - Ap\| \cdot \|x_{n+1} - p\|.
\end{aligned}$$

Simplifying it we have

$$
\begin{aligned}
(1 - \alpha_n \bar{\gamma})^2 \|\mathscr{V}^l x_n - \mathscr{V}^{l+1} x_n\|^2 \leq \\
\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \bar{\gamma}^2 \|x_n - p\|^2 + \\
+ \beta_n^2 \|x_n - q_n\|^2 + 2(1 - \alpha_n \bar{\gamma}) \beta_n \|\rho_n - p\| \cdot \|x_n - q_n\| + \\
+ 2\alpha_n \|\gamma f(u_n) - Ap\| \cdot \|x_{n+1} - p\|.
\end{aligned}
$$

Since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, $\|x_n - q_n\| \to 0$, it yields $\|\mathscr{V}^l x_n - \mathscr{V}^{l+1} x_n\| \to 0$.
Step 6. Now we prove that for any given $p \in \mathscr{G}$

$$
\lim_{n \to \infty} \|By_n - Bp\| = 0. \tag{3.17}
$$

In fact, it follows from (3.3) that

$$
\begin{aligned}
\|\rho_n - p\|^2 \leq \|\xi_n - p\|^2 &= \|J_{M,\lambda}(I - \lambda B)y_n - J_{M,\lambda}(I - \lambda B)p\|^2 \leq \\
&\leq \|(I - \lambda B)y_n - (I - \lambda B)p\|^2 = \\
&= \|y_n - p\|^2 - 2\lambda \langle y_n - p, By_n - Bp \rangle + \lambda^2 \|By_n - Bp\|^2 \leq \\
&\leq \|y_n - p\|^2 + \lambda(\lambda - 2\alpha)\|By_n - Bp\|^2 \leq \\
&\leq \|x_n - p\|^2 + \lambda(\lambda - 2\alpha)\|By_n - Bp\|^2.
\end{aligned} \tag{3.18}
$$

Substituting (3.18) into (3.16), we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \leq \\
\leq (1 - \alpha_n \bar{\gamma})^2 \{\|x_n - p\|^2 + \lambda(\lambda - 2\alpha)\|By_n - Bp\|^2\} + \beta_n^2 \|x_n - q_n\|^2 + \\
+ 2(1 - \alpha_n \bar{\gamma}) \beta_n \|\rho_n - p\| \cdot \|x_n - q_n\| + 2\alpha_n \|\gamma f(u_n) - Ap\| \cdot \|x_{n+1} - p\|.
\end{aligned}
$$

Simplifying it, we have

$$
\begin{aligned}
(1 - \alpha_n \bar{\gamma})^2 \lambda(2\alpha - \lambda)\|By_n - Bp\|^2 \leq \\
\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n \bar{\gamma}^2 \|x_n - p\|^2 + \beta_n^2 \|x_n - q_n\|^2 + \\
+ 2(1 - \alpha_n \bar{\gamma}) \beta_n \|\rho_n - p\| \cdot \|x_n - q_n\| + 2\alpha_n \|\gamma f(u_n) - Ap\| \cdot \|x_{n+1} - p\|.
\end{aligned}
$$

Since $\alpha_n \to 0$, $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$, $\|x_{n+1} - x_n\| \to 0$, $\|x_n - q_n\| \to 0$, and $\{\gamma f(u_n) - Ap\}, \{x_n\}$ are bounded, these imply that $\|By_n - Bp\| \to 0 \ (n \to \infty)$.
Step 7. Next we prove that

$$
\lim_{n \to \infty} \|y_n - \rho_n\| = 0. \tag{3.19}
$$

In fact, since
$$
\|y_n - \rho_n\| \leq \|y_n - \xi_n\| + \|\xi_n - \rho_n\|,
$$
for the purpose, it is sufficient to prove

$$
\|y_n - \xi_n\| \to 0 \ \ and \ \ \|\xi_n - \rho_n\| \to 0.
$$

(a) First we prove that $\|y_n - \xi_n\| \to 0$.
In fact, since

$$\|\xi_n - p\|^2 =$$
$$= \|J_{M,\lambda}(I - \lambda B)y_n - J_{M,\lambda}(I - \lambda B)p\|^2 \le$$
$$\le \langle y_n - \lambda By_n - (p - \lambda Bp), \xi_n - p \rangle =$$
$$= \frac{1}{2}\{\|y_n - \lambda By_n - (p - \lambda Bp)\|^2 + \|\xi_n - p\|^2 - \|y_n - \lambda By_n - (p - \lambda Bp) - (\xi_n - p)\|^2\} \le$$
$$\le \frac{1}{2}\{\|y_n - p\|^2 + \|\xi_n - p\|^2 - \|y_n - \xi_n - \lambda(By_n - Bp)\|^2\} \le$$
$$\le \frac{1}{2}\{\|y_n - p\|^2 + \|\xi_n - p\|^2 - \|y_n - \xi_n\|^2 + 2\lambda\langle y_n - \xi_n, By_n - Bp \rangle - \lambda^2\|By_n - Bp\|^2\}$$

we have

$$\|\xi_n - p\|^2 \le \|y_n - p\|^2 - \|y_n - \xi_n\|^2 + 2\lambda\langle y_n - \xi_n, By_n - Bp \rangle - \lambda^2\|By_n - Bp\|^2. \quad (3.20)$$

Substituting (3.20) into (3.16), it yields that

$$\|x_{n+1} - p\|^2 \le (1 - \alpha_n\bar{\gamma})^2\{\|y_n - p\|^2 - \|y_n - \xi_n\|^2 +$$
$$+ 2\lambda\langle y_n - \xi_n, By_n - Bp \rangle - \lambda^2\|By_n - Bp\|^2\} + \beta_n^2\|x_n - q_n\|^2 +$$
$$+ 2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p\| \cdot \|x_n - q_n\| + 2\alpha_n\|\gamma f(u_n) - Ap\| \cdot \|x_{n+1} - p\|.$$

Simplifying it we have

$$(1 - \alpha_n\bar{\gamma})^2\|y_n - \xi_n\|^2 \le$$
$$\le (\|x_n - x_{n+1}\|) \cdot (\|x_n - p\| + \|x_{n+1} - p\|) + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 +$$
$$+ 2(1 - \alpha_n\bar{\gamma}^2)\lambda\langle y_n - \xi_n, By_n - Bp \rangle - (1 - \alpha_n\bar{\gamma})^2\lambda^2\|By_n - Bp\|^2 + \beta_n^2\|x_n - q_n\|^2 +$$
$$+ 2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p\| \cdot \|x_n - q_n\| + 2\alpha_n\|\gamma f(u_n) - Ap\| \cdot \|x_{n+1} - p\|.$$

Since $\alpha_n \to 0$, $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, $\|x_n - q_n\| \to 0$, $\|By_n - Bp\| \to 0$ $(n \to \infty)$, $\|x_{n+1} - x_n\| \to 0$ and $\{\gamma f(u_n) - Ap\}, \{x_n\}, \{\rho_n\}$ are bounded, these imply that $\|y_n - \xi_n\| \to 0$ $(n \to \infty)$.
(b) Next we prove that
$$\lim_{n\to\infty}\|\xi_n - \rho_n\| = 0. \quad (3.21)$$

In fact, since $\|\xi_n - \rho_n\| = \|J_{M,\lambda}(I - \lambda B)y_n - J_{M,\lambda}(I - \lambda B)\xi_n\| \le \|y_n - \xi_n\| \to 0$, and so $\|y_n - \rho_n\| = \|y_n - \xi_n + \xi_n - \rho_n\| \le \|y_n - \xi_n\| + \|\xi_n - \rho_n\| \to 0$.
Step 8. Next we prove that

$$\limsup_{n\to\infty}\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \le 0. \quad (3.22)$$

(a) First, we prove that

$$\limsup_{n\to\infty}\langle\frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds - x^*, \gamma f(x^*) - Ax^*\rangle \le 0. \quad (3.23)$$

To see this, there exist a subsequence $\{\rho_{n_i}\}$ of $\{\rho_n\}$ such that

$$\limsup_{n\to\infty}\langle\frac{1}{t_n}\int_0^{t_n}T(s)\rho_n ds - x^*, \gamma f(x^*) - Ax^*\rangle =$$

$$= \limsup_{i\to\infty}\langle\frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)\rho_{n_i}ds - x^*, \gamma f(x^*) - Ax^*\rangle$$

we may also assume that $\rho_{n_i} \rightharpoonup w$, then $q_{n_i} = \frac{1}{t_{n_i}}\int_0^{t_{n_i}}T(s)\rho_{n_i}ds \rightharpoonup w$. Since $\|x_n - q_n\| \to 0$, we have $x_{n_i} \rightharpoonup w$.

Next, we prove that
$$w \in \mathscr{G}.$$

(1^0) We first prove that $w \in F(\mathscr{T})$. In fact, since $\{x_{n_i}\} \rightharpoonup w$. From Lemma 2.7 and Step 4, we obtain $w \in F(\mathscr{T})$.

(2^0) Now we prove that $w \in \cap_{l=1}^N MEP(\Theta_l, \varphi_l)$.

Since $x_{n_i} \rightharpoonup w$ and noting Step 5, without loss of generality, we may assume that $\mathscr{V}^l x_{n_i} \rightharpoonup w$, $\forall\, l \in \{0, 1, 2, \cdots, N-1\}$. Hence for any $x \in C$ and for any $l \in \{0, 1, 2, \cdots, N-1\}$, we have

$$\langle\frac{K_{l+1}'(\mathscr{V}^{l+1}x_{n_i}) - K_{l+1}'(\mathscr{V}^l x_{n_i})}{r_{l+1}}, \eta_{l+1}(x, \mathscr{V}^{l+1}x_{n_i})\rangle \geq$$

$$\geq -\Theta_{l+1}(\mathscr{V}^{l+1}x_{n_i}, x) - \varphi_{l+1}(x) + \varphi_{l+1}(\mathscr{V}^{l+1}x_{n_i}).$$

By the assumptions and by the condition (H_2) we know that the function φ_i and the mapping $x \mapsto (-\Theta_{l+1}(x, y))$ both are convex and lower semi-continuous, hence they are weakly lower semi-continuous. These together with $\frac{K_{l+1}'(\mathscr{V}^{l+1}x_{n_i}) - K_{l+1}'(\mathscr{V}^l x_{n_i})}{r_{l+1}} \to 0$ and $\mathscr{V}^{l+1}x_{n_i} \rightharpoonup w$, we have

$$0 = \liminf_{i\to\infty}\{\langle\frac{K_{l+1}'(\mathscr{V}^{l+1}x_{n_i}) - K_{l+1}'(\mathscr{V}^l x_{n_i})}{r_{l+1}}, \eta_{l+1}(x, \mathscr{V}^{l+1}x_{n_i})\rangle\} \geq$$

$$\geq \liminf_{i\to\infty}\{-\Theta_{l+1}(\mathscr{V}^{l+1}x_{n_i}, x) - \varphi_{l+1}(x) + \varphi_{l+1}(\mathscr{V}^{l+1}x_{n_i})\}.$$

i.e.,
$$\Theta_{l+1}(w, x) + \varphi_{l+1}(x) - \varphi_{l+1}(w) \geq 0$$

for all $x \in C$ and $l \in \{0, 1, \cdots, N-1\}$, hence $w \in \cap_{l=1}^N MEP(\Theta_l, \varphi_l)$.

(3^0) Now we prove that $w \in VI(H, B, M)$.

In fact, since B is α-inverse-strongly monotone, it follows from Proposition 1.1 that B is a $\frac{1}{\alpha}$-Lipschitz continuous monotone mapping and $D(B) = H$ (where $D(B)$ is the domain of B). It follows from Lemma 2.4 that $M + B$ is maximal monotone. Let $(\nu, g) \in Graph(M + B)$, i.e., $g - B\nu \in M(\nu)$. Since $x_{n_i} \rightharpoonup w$ and noting Step 5, without loss of generality, we may assume that $\mathscr{V}^l x_{n_i} \rightharpoonup w$, in particular, we have

$y_{n_i} = \mathscr{V}^N x_{n_i} \rightharpoonup w$. From $\|y_n - \rho_n\| \to 0$, we can prove that $\rho_{n_i} \rightharpoonup w$. Again since $\rho_{n_i} = J_{M,\lambda}(I - \lambda B)\xi_{n_i}$, we have

$$\xi_{n_i} - \lambda B\xi_{n_i} \in (I + \lambda M)\rho_{n_i}, \ i.e., \ \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i} - \lambda B\xi_{n_i}) \in M(\rho_{n_i}).$$

By virtue of the maximal monotonicity of M, we have

$$\langle \nu - \rho_{n_i}, g - B\nu - \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i} - \lambda B\xi_{n_i})\rangle \geq 0$$

and so

$$\langle \nu - \rho_{n_i}, g\rangle \geq \langle \nu - \rho_{n_i}, B\nu + \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i} - \lambda B\xi_{n_i})\rangle =$$

$$= \langle \nu - \rho_{n_i}, B\nu - B\rho_{n_i} + B\rho_{n_i} - B\xi_{n_i} + \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i})\rangle \geq$$

$$\geq 0 + \langle \nu - \rho_{n_i}, B\rho_{n_i} - B\xi_{n_i}\rangle + \langle \nu - \rho_{n_i}, \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i})\rangle.$$

Since $\|\xi_n - \rho_n\| \to 0$, $\|B\xi_n - B\rho_n\| \to 0$ and $\rho_{n_i} \rightharpoonup w$, we have

$$\lim_{n_i \to \infty} \langle \nu - \rho_{n_i}, g\rangle = \langle \nu - w, g\rangle \geq 0.$$

Since $M + B$ is maximal monotone, this implies that $\theta \in (M + B)(w)$, i.e., $w \in VI(H, B, M)$, and so $w \in \mathscr{G}$.

Since $x^* = P_{\mathscr{G}}(I - A + \gamma f)(x^*)$, we have

$$\limsup_{n \to \infty}\langle \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds - x^*, \gamma f(x^*) - Ax^*\rangle =$$

$$= \limsup_{i \to \infty}\langle \frac{1}{t_{n_i}}\int_0^{t_{n_i}} T(s)\rho_{n_i} ds - x^*, \gamma f(x^*) - Ax^*\rangle =$$

$$= \limsup_{i \to \infty}\langle q_{n_i} - x^*, \gamma f(x^*) - Ax^*\rangle =$$

$$= \langle w - x^*, \gamma f(x^*) - Ax^*\rangle \leq 0.$$

(b) Now we prove that

$$\limsup_{n \to \infty}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \leq 0.$$

From $\|x_{n+1} - q_n\| \to 0$ and (a), we have

$$\limsup_{n \to \infty}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle =$$

$$= \limsup_{n \to \infty}\langle \gamma f(x^*) - Ax^*, x_{n+1} - q_n + q_n - x^*\rangle \leq$$

$$\leq \limsup_{n \to \infty}\langle \gamma f(x^*) - Ax^*, x_{n+1} - q_n\rangle + \limsup_{n \to \infty}\langle \gamma f(x^*) - Ax^*, q_n - x^*\rangle \leq 0.$$

Step 9. Finally we prove that

$$x_n \to x^*.$$

Indeed, from (3.1) , we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \\
&= \|\alpha_n(\gamma f(u_n) - Ax^*) + \beta_n(x_n - x^*) + ((1 - \beta_n)I - \alpha_n A)(q_n - x^*)\|^2 \leq \\
&\leq \|\beta_n(x_n - x^*) + ((1 - \beta_n)I - \alpha_n A)(q_n - x^*)\|^2 + 2\alpha_n\langle\gamma f(u_n) - Ax^*, x_{n+1} - x^*\rangle \leq \\
&\leq [\|((1 - \beta_n)I - \alpha_n A)(q_n - x^*)\| + \beta_n\|x_n - x^*\|]^2 + \\
&\quad + 2\alpha_n\gamma\langle f(u_n) - f(x^*), x_{n+1} - x^*\rangle + 2\alpha_n\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \leq \\
&\leq [\|(1 - \beta_n - \alpha_n\bar{\gamma})\|\rho_n - x^*\| + \beta_n\|x_n - x^*\|]^2 + 2\alpha_n\gamma h\|x_n - x^*\| \cdot \|x_{n+1} - x^*\| + \\
&\quad + 2\alpha_n\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \leq \\
&\leq (1 - \alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + \alpha_n\gamma h\{\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\} + \\
&\quad + 2\alpha_n\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \\
&\leq \frac{(1 - \alpha_n\bar{\gamma})^2 + \alpha_n\gamma h}{1 - \alpha_n\gamma h}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma h}\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle = \\
&= [1 - \frac{2(\bar{\gamma} - \gamma h)\alpha_n}{1 - \alpha_n\gamma h}]\|x_n - x^*\|^2 + \frac{(\alpha_n\bar{\gamma})^2}{1 - \alpha_n\gamma h}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma h}\langle\gamma f(x^*) - \\
&\quad - Ax^*, x_{n+1} - x^*\rangle \leq \\
&\leq [1 - \frac{2(\bar{\gamma} - \gamma h)\alpha_n}{1 - \alpha_n\gamma h}]\|x_n - x^*\|^2 + \frac{2(\bar{\gamma} - \gamma h)\alpha_n}{1 - \alpha_n\gamma h}\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \gamma h)}\|x_n - x^*\|^2 + \\
&\quad + \frac{1}{\bar{\gamma} - \gamma h}\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle\} = \\
&= (1 - l_n)\|x_n - x^*\|^2 + \delta_n,
\end{aligned}
$$

where

$$
l_n = \frac{2(\bar{\gamma} - \gamma h)\alpha_n}{1 - \alpha_n\gamma h},
$$

and

$$
\delta_n = \frac{2(\bar{\gamma} - \gamma h)\alpha_n}{1 - \alpha_n\gamma h}\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \gamma h)}\|x_n - x^*\|^2 + \frac{1}{\bar{\gamma} - \gamma h}\langle\gamma f(x^*) - Ax^*, \; x_{n+1} - x^*\rangle\}.
$$

It is easy to see that $l_n \to 0$, $\sum_{n=1}^{\infty} l_n = \infty$ and $\limsup_{n\to\infty} \frac{\delta_n}{l_n} \leq 0$. Hence the sequence $\{x_n\}$ converges strongly to x^*.

This completes the proof of Theorem 3.2. $\square$

Corollary 3.3. *Let H, C, A, B, M, f, $\mathscr{T}$, $\mathscr{F}$, φ_i, $\eta_i, K_i(i = 1, 2, \cdots, N)$ be the same as Theorem 3.2. Let $\{x_n\}$, $\{\rho_n\}$, $\{\xi_n\}$ and $\{y_n\}$ be explicit iterative sequences generated by $x_1 \in H$ and*

$$
\begin{cases}
x_{n+1} = \alpha_n \gamma f(\dfrac{1}{t_n} \displaystyle\int_0^{t_n} T(s)x_n ds) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\dfrac{1}{t_n} \displaystyle\int_0^{t_n} T(s)\rho_n ds, \\
\rho_n = P_C(I - \lambda B)\xi_n, \\
\xi_n = P_C(I - \lambda B)y_n, \\
y_n = V_{r_N}^{\Theta_N} \ldots V_{r_2}^{\Theta_2} V_{r_1}^{\Theta_1} x_n,
\end{cases}
\qquad \forall n \geq 1,
$$

(3.24)

where $r_i(i = 1, 2, \cdots, N)$ are a finite family of positive numbers, $\lambda \in (0, 2\alpha]$, $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ and $\{t_n\} \subset (0, \infty)$ is a sequence with $t_n \uparrow \infty$. If $\mathscr{G} := F(\mathscr{T}) \bigcap MEP(\mathscr{F}) \bigcap VI(C, B) \neq \emptyset$ and the following conditions are satisfied:

(i) *for each $x \in C$, there exist a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,*

$$
\Theta_i(y, z_x) + \varphi_i(z_x) - \varphi_i(y) + \frac{1}{r_i}\langle K_i'(y) - K_i'(x), \eta_i(z_x, y)\rangle < 0,
$$

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then the sequence $\{x_n\}$ converges strongly to some point $x^* = P_{\mathscr{G}}(I - A + \gamma f)(x^*)$, provided that $V_{r_i}^{\Theta_i}$ is firmly nonexpansive.

Proof. Taking $M = \partial \delta_C : H \to 2^H$ in Theorem 3.2, where $\delta_C : H \to [0, \infty)$ is the indicator function of C, i.e.,

$$
\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}
$$

then the variational inclusion problem (1.2) is equivalent to variational inequality (1.3), i.e., to find $u \in C$ such that

$$
\langle B(u), v - u\rangle \geq 0, \forall v \in C.
$$

Again, since $M = \partial \delta_C$, the restriction of $J_{M,\lambda}$ on C is an identity mapping, i.e., $J_{M,\lambda}|_C = I$ and so we have

$$
P_C(I - \lambda B)k_n = J_{M,\lambda}(P_C(I - \lambda B)k_n); \quad P_C(I - \lambda B)y_n = J_{M,\lambda}(P_C(I - \lambda B)y_n).
$$

Hence the conclusion of Corollary 3.3 can be obtained form Theorem 3.2 immediately. $\quad\square$

REFERENCES

[1] L.C. Ceng, J.C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math. **214** (2008), 186–201.

[2] S. Chang, *Set-valued variational inclusions in Banach space*, J. Math. Anal. Appl. **248** (2000), 438–454.

[3] S. Chang, *Existence and approximation of solutions of Set-valued variational inclusions in Banach space*, Nonlinear Anal. **47** (2001) 1, 583–594.

[4] S. Chang, H.W. Joseph Lee, C. Chan, *A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization*, Nonlinear Anal. **70** (2009), 3307–3319.

[5] V.F. Demyanov, G.E. Stavroulakis, L.N. Polyakova, P.D. Panagiotopoulos, *Quasi-differentiability and Nonsmooth Modeling in Mechanics. Engineering and Economics*, Kluwer Academic, Dordrecht, 1996.

[6] S. Li, L. Li, Y. Su, *General iterative methods for a one-parameter nonexpansive semigroup in Hilbert space*, Nonlinear Anal., Theory Methods Appl. 70, No. 9 (A), 2065–2071 (2009), doi:10.1016/j.na.2008.04.007.

[7] J.L. Lions, G. Stampacchia, *Variational inequalities*, Comm. Pure. Appl. Math. **20** (1967), 493–517.

[8] D. Pascali, *Nolinear Mappings of Monotone Type*, Sijthoff and Noordhoff International Publishers, The Netherlands, 1978.

[9] S. Saeidi, *Iterative algorithms for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of families and semigroups of nonexpansive mappings*, Nonlinear Anal. (2008), doi:10.1016/j.na.2008.09.009.

[10] T. Shimizu, W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997), 71–83.

[11] S. Takahashi, W. Takahshi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert space*, J. Math. Anal. Appl. **331** (2007), 506–515.

[12] H.K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl. **116** (2003) 3, 659–678.

[13] S. Zhang, J.H. Joseph Lee, C.K. Chan, *Algorithms of common solutions for quasi-variational inclusion and fixed point problems*, Appl. Math. Mech. **29** (2008), 1–11.

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