DECOMPOSITION OF COMPLETE GRAPHS
INTO SMALL GRAPHS

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Abstract. In 1967, A. Rosa proved that if a bipartite graph $G$ with $n$ edges has an $\alpha$-labeling, then for any positive integer $p$ the complete graph $K_{2np+1}$ can be cyclically decomposed into copies of $G$. This has become a part of graph theory folklore since then. In this note we prove a generalization of this result. We show that every bipartite graph $H$ which decomposes $K_k$ and $K_m$ also decomposes $K_{km}$.

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Let $G$ be a graph with at most $n$ vertices. We say that the complete graph $K_n$ has a $G$-decomposition (or that it is $G$-decomposable) if there are subgraphs $G_0, G_1, G_2, \ldots, G_s$ of $K_n$, all isomorphic to $G$, such that each edge of $K_n$ belongs to exactly one $G_i$.

In 1967 A. Rosa [5] introduced some important types of vertex labelings. Graceful labeling (called $\beta$-valuation by AR) and rosy labeling (called $\rho$-valuation by AR) are useful tools for decompositions of complete graphs $K_{2n+1}$ into graphs with $n$ edges. A labeling of a graph $G$ with $n$ edges is an injection $\rho$ from $V(G)$, the vertex set of $G$, into a subset $S$ of the set $\{0, 1, 2, \ldots, 2n\}$ of elements of the additive group $\mathbb{Z}_{2n+1}$. The length of an edge $e = xy$ with endvertices $x$ and $y$ is defined as $\ell(e) = \min\{\rho(x) - \rho(y), \rho(y) - \rho(x)\}$. Notice that the subtraction is performed in $\mathbb{Z}_{2n+1}$ and hence $1 \leq \ell(e) \leq n$. If the set of all lengths of the $n$ edges is equal to $\{1, 2, \ldots, n\}$, then $\rho$ is a rosy labeling; if moreover $S \subseteq \{0, 1, \ldots, n\}$, then $\rho$ is a graceful labeling. A graceful labeling $\alpha$ is said to be an $\alpha$-labeling if there exists a number $\alpha_0$ with the property that for every edge $e$ in $G$ with endvertices $x$ and $y$ and with $\alpha(x) < \alpha(y)$ it holds that $\alpha(x) \leq \alpha_0 < \alpha(y)$. Obviously, $G$ must be bipartite to allow an $\alpha$-labeling.

For an exhaustive survey of graph labelings, see [3] by J. Gallian.

A. Rosa observed that if a graph $G$ with $n$ edges has a graceful or rosy labeling, then $K_{2n+1}$ can be cyclically decomposed into $2n+1$ copies of $G$. It is so because $K_{2n+1}$ has exactly $2n+1$ edges of length $i$ for every $i = 1, 2, \ldots, n$ and each copy of $G$ contains exactly one edge of each length. The cyclic decomposition is constructed
by taking a labeled copy of $G$, say $G_0$, and then adding a non-zero element $i \in \mathbb{Z}_{2n+1}$ to the label of each vertex of $G_0$ to obtain a copy $G_i$ for $i = 1, 2, \ldots, 2n$.

If $G$ with $n$ edges has an $\alpha$-labeling, then we can take $p$ copies of $G$, say $G_0, G_1, G_{p-1}$, and label them such that $G_0$ has the original labels induced by the $\alpha$-labeling, and for every $i = 1, 2, \ldots, p-1$ the vertices with lower labels (that is, with $\alpha(x) \leq \alpha_0$) will keep their labels, while the vertices with high labels will increase their labels by $in$. This way a copy $G_i$ contains edges of lengths $in + 1, in + 2, \ldots, (i+1)n$.

Therefore all $p$ copies together contain $np$ edges of lengths $1, 2, \ldots, np$. It follows that the graph consisting of these $p$ edge-disjoint copies of $G$ decompenses cyclically the complete graph $K_{2n+1}$ and consequently, $G$ itself decomposes $K_{2np+1}$.

We summarize these classical Rosa’s results in the following theorem.

**Theorem 1.** Let $G$ be a graph with $n$ edges. If $G$ allows a rosy labeling, then it decomposes $K_{2n+1}$, if $G$ allows an $\alpha$-labeling, then it decomposes $K_{2np+1}$ for every $p > 0$.

To guarantee a $G$-decomposition of $K_{2np+1}$, the condition of the existence of an $\alpha$-labeling can be relaxed. S. El-Zanati, C. Vanden Eynden, and N. Punnim [2] defined a $\rho^+$-labeling of a bipartite graph $G$ with bipartition $X, Y$ as a rosy labeling with the additional property that for every edge $xy \in E(G)$ with $x \in X, y \in Y$ it holds that $\rho^+(x) < \rho^+(y)$. Their theorem then follows by arguments similar to those for the $\alpha$-labeling.

**Theorem 2.** If a bipartite graph $G$ with $n$ edges has a $\rho^+$-labeling, then there exists a cyclic $G$-decomposition of $K_{2np+1}$ for any positive integer $p$.

In [1] M. Buratti and A. Pasotti proved a result on difference matrices, which is in [4] restated as follows.

**Theorem 3.** If a graph $G$ with $n$ edges and chromatic number $\chi(G)$ cyclically decomposes $K_k$ and $K_m$, where $k \equiv m \equiv 1 (\mod 2n)$ and $\chi(G)$ does not exceed the smallest prime factor of $m$, then there exists a cyclic $G$-decomposition of $K_{km}$.

Because a bipartite graph has $\chi(G) = 2$, the following corollary is easy to prove. It was stated in [1] in a more general form related to Theorem 3.

**Corollary 4.** If a bipartite graph $G$ with $n$ edges has a $\rho$-labeling, then there exists a cyclic $G$-decomposition of $K_{(2n+1)r}$ for any positive integer $r$.

Our goal is to show that if we restrict ourselves to bipartite graphs while assuming the existence of any $G$-decomposition rather than a cyclic one, we can still get a result similar to Theorem 3. First we prove a related useful result for decompositions of complete multipartite graphs. We recall that a composition $G[H]$ of graphs $G$ and $H$ (also called a lexicographic product) is a graph that arises from $G$ by replacing each vertex of $G$ by a copy of $H$ and each edge of $G$ by $K_{m,m}$, where $m$ is the order of $H$. In particular, if $H = K_m$, the graph consisting of $m$ isolated vertices, then we say that we blow up $G$ into $G[K_m]$. 
Observation 5. If a bipartite graph $G$ decomposes $K_k$, then $G$ also decomposes the complete $k$-partite graph $K_{m,m,...,m}$ for any $m \geq 2$.

Proof. Because $K_{m,m,...,m} = K_k[K_m]$, it is obvious that it can be decomposed into $G[K_m]$ by blowing up $K_k$ and concurrently every copy of $G$. Moreover, since $G$ is bipartite, $G[K_m]$ is also bipartite. Let $X,Y$ be the partite sets of $G$ and $X,Y$ the corresponding bipartition of $G[K_m]$. We need to decompose $G[K_m]$ into $m^2$ edge-disjoint copies of $G$.

We label the vertices of $K_k$ by the elements of $Z_k$ and the vertices of $K_{m,m,...,m} = K_k[K_m]$ by the elements $(i,j)$ of the group $Z_k \times Z_m$.

Now we construct $m^2$ copies of $G$, denoted by $G_{ij}$ for $i,j = 0,1,...,m-1$. If $xy$ is an edge in $G$ with $x \in X, y \in Y$ (we here identify vertices with their labels, so in fact $x$ and $y$ are the labels of these vertices), then in $G_{ij}$ there will be the edge $(x,i)(y,j)$. Therefore, every $G_{ij}$ contains an edge $(x,i)(y,j)$ if and only if $G$ contains the edge $xy$. On the other hand, for every complete bipartite graph $K_{x,y}$ in $G[K_m]$ corresponding to an edge $xy \in G$ we have each of its $m^2$ edges in precisely one of the graphs $G_{ij}$.

It should be clear that this way $G[K_m]$ is decomposed into $m^2$ copies of $G$. Because at the same time $G[K_m]$ decomposes $K_{m,m,...,m}$, it is obvious that $G$ decomposes $K_{m,m,...,m}$.

Now it is easy to observe that the following is true.

Theorem 6. If a bipartite graph $G$ decomposes $K_k$ and $K_{m,m,...,m}$, then $G$ also decomposes the complete graph $K_{km}$.

The following equivalent of Corollary 4 is now obvious.

Theorem 7. Let $G$ be a bipartite graph which decomposes $K_s$. Then $G$ decomposes also $K_{s^r}$ for any $r \geq 1$.

Proof. We prove the claim by induction on $r$. For $r = 1$ we get our assumption that $G$ decomposes $K_s$. Now assume that $G$ decomposes $K_{s^{r-1}}$. First decompose $K_s$ into $s$ copies of $K_{s^{r-1}}$ and a complete $s$-partite graph $B$ with partite sets of size $s^r - 1$. By induction hypothesis, each $K_{s^{r-1}}$ can be decomposed into $G$. The existence of a decomposition of the complete $s$-partite graph $B$ follows from our assumption that $G$ decomposes $K_s$ and from Observation 5, where we set $k = s$ and $m = s^{r-1}$. Therefore, $G$ decomposes $K_{s^r}$.

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