GRAPH CHOOSABILITY
AND DOUBLE LIST COLORABILITY

Hamid-Reza Fanaï

Abstract. In this paper, we give a sufficient condition for graph choosability, based on Combinatorial Nullstellensatz and a specific property, called "double list colorability", which means that there is a list assignment for which there are exactly two admissible colorings.

Keywords: list coloring, choosability.

Mathematics Subject Classification: 05C15, 05C31.

1. INTRODUCTION

The list coloring problem in graph theory is an exciting research area. It was introduced by Vizing [6] and independently by Erdös, Rubin and Taylor [4]. In a list coloring problem, we have a graph G with a list of available colors at each vertex, and we are looking for a properly vertex coloring of G such that each vertex takes its color from its list. Choosability means that it is possible for G to be list colorable when only the size of lists is known. Alon and Tarsi in [2] defined a polynomial associated with a graph (Combinatorial Nullstellensatz) and used it to give sufficient conditions for choosability of a graph in terms of the existence of certain orientations on the edges of the graph. In [1], this powerful algebraic approach has been used to study some relations between choosability and unique list colorability. Here we use the same method for double list colorable graphs and under some conditions, we obtain a choosability result. Although, uniquely colorable graphs have been studied by many authors, see for example [3] and [7], the double colorable graphs have not been considered so much. However, the number of list-colorings was recently studied by Thomassen in [5] and seems to be an interesting problem. In this note, the graphs considered are finite, undirected and without loops. Let us recall some notation.

The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. We denote by n, the number of vertices of G which is called the order of G and by m, the number of edges of G which is called the size of G. For a graph G, a list assignment

L is a function that assigns to each vertex v of G a set L_v of colors. An L-coloring of G is a function that assigns to each vertex a color from its list such that two adjacent vertices in G receive different colors. If G admits an L-coloring, then G is called L-colorable. The graph G is said to be uniquely L-colorable, if there is exactly one L-coloring. Similarly, G is said to be double L-colorable, if there are exactly two L-colorings. For a function $f : V(G) \to \mathbb{N}$, we say that G is f-choosable if G is L-colorable for every list assignment L satisfying $|L_v| = f(v)$ for all $v \in V(G)$.

In [2], the graph polynomial $f_G(x_1, \ldots, x_n)$ of a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ is defined by

$$f_G(x_1, \ldots, x_n) = \prod \{(x_i - x_j) \mid i < j, v_i v_j \in E(G)\}.$$

Let D be an orientation of G. An oriented edge (v_i, v_j) of G is called decreasing if $i > j$. The orientation D is called even, if it has an even number of decreasing edges, otherwise, it is called odd. For non-negative integers $d_1, \ldots, d_n$, let $\mathsf{DE}(d_1, \ldots, d_n)$ and $\mathsf{DO}(d_1, \ldots, d_n)$ denote, respectively, the sets of all even and odd orientations of G, in which the outdegree of the vertex v_i is d_i for $1 \le i \le n$. We have the following lemma in [2].

Lemma 1.1. *In the above notation*

$$f_G(x_1, \ldots, x_n) = \sum_{d_1, \ldots, d_n \ge 0} (|\mathsf{DE}(d_1, \ldots, d_n)| - |\mathsf{DO}(d_1, \ldots, d_n)|) \prod_{i=1}^{n} x_i^{d_i}.$$

2. RESULTS

In this section, based on the algebraic technique developed by Alon and Tarsi in [2], and using the same idea of [1], a relation between choosability and double list colorability is obtained. First, we prove the following algebraic lemma which generalizes Lemma 1 in [1].

Lemma 2.1. *Let F be a field and let $P = P(x_1, \ldots, x_n)$ be a polynomial in n variables over F such that $\deg_{x_i}(P) \le d_i$ for $1 \le i \le n$. Furthermore, for $1 \le i \le n$, let S_i be a subset of F consisting of $d_i + 1$ elements and let $a_i, b_i \in S_i$ with $(a_1, \ldots, a_n) \ne (b_1, \ldots, b_n)$. Suppose that $P(a_1, \ldots, a_n) \ne 0$, $P(b_1, \ldots, b_n) \ne 0$ and $P(x_1, \ldots, x_n) = 0$ for every $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} S_i \setminus \{(a_1, \ldots, a_n), (b_1, \ldots, b_n)\}$. Then,*

$$P(x_1, \ldots, x_n) = c \prod_{j=1}^{n} \prod_{s \in S_j \setminus \{a_j\}} (x_j - s) + d \prod_{j=1}^{n} \prod_{s \in S_j \setminus \{b_j\}} (x_j - s)$$

for some constants $c, d \in F$.

Proof. Let $l_1 = \prod_{j=1}^{n} \prod_{s \in S_j \setminus \{a_j\}} (a_j - s)$, $l_2 = \prod_{j=1}^{n} \prod_{s \in S_j \setminus \{b_j\}} (b_j - s)$ and define the following polynomial,

$$Q(x_1, \ldots, x_n) = P(x_1, \ldots, x_n) - l_1^{-1} P(a_1, \ldots, a_n) \prod_{j=1}^{n} \prod_{s \in S_j \setminus \{a_j\}} (x_j - s) -$$
$$- l_2^{-1} P(b_1, \ldots, b_n) \prod_{j=1}^{n} \prod_{s \in S_j \setminus \{b_j\}} (x_j - s).$$

Clearly $\deg_{x_i}(Q) \leq d_i$, for any i, $1 \leq i \leq n$, and $Q(s_1, \ldots, s_n) = 0$ for each $(s_1, \ldots, s_n) \in \prod_{i=1}^{n} S_i$. Now the result follows from Lemma 2.1 of [2]. $\square$

We now use Lemma 2.1 to obtain a relation between choosability and double list colorability. Let G be a graph on a set $V = \{v_1, \ldots, v_n\}$ of $n \geq 1$ vertices. For $1 \leq i \leq n$, let L_{v_i} be a list of $d_i + 1$ colors, where $d_i \geq 0$ is a given integer. Suppose that G is double L-colorable. We would like to prove that G is f-choosable provided that $f(v_i) = d_i + 1$ for $1 \leq i \leq n$ and $d_1 + \ldots + d_n = m$, where m is the size of G. The latter condition was needed in [1] and we must have it as well. Let

$$f_G(x_1, \ldots, x_n) = \prod \{(x_i - x_j) \mid i < j, v_i v_j \in E(G)\}$$

be the graph polynomial of G. For $1 \leq i \leq n$, let $S_i = L_{v_i}$ and $S = \prod_{i=1}^{n} S_i$. Since G is double L-colorable, there are exactly two n-tuples $\underline{a} = (a_1, \ldots, a_n)$ and $\underline{b} = (b_1, \ldots, b_n)$ in S, such that

(1) $f_G(\underline{a}) \neq 0$, $f_G(\underline{b}) \neq 0$ and $f_G(x_1, \ldots, x_n) = 0, \forall (x_1, \ldots, x_n) \in S \setminus \{\underline{a}, \underline{b}\}$.

For an arbitrary color c, let V_c be the set of vertices which receive the color c in the L-coloring of the graph using the colors $a_1, \ldots, a_n$, and let V_c' be the set of vertices which receive the color c in the L-coloring of the graph using the colors $b_1, \ldots, b_n$. With these notations, we have the following result.

Proposition 2.2. *Let G be a graph on a set $V = \{v_1, \ldots, v_n\}$ of $n \geq 1$ vertices. For $1 \leq i \leq n$, let L_{v_i} be a list of $d_i + 1$ colors where $d_i \geq 0$ is a given integer. Suppose that G is double L-colorable and $d_1 + \ldots + d_n = m$, where m is the size of G. Suppose that for some color c, with the above notation, the following equation does not hold:*

$$\sum_{v \in V_c \setminus V_c'} \deg(v) - \sum_{w \in V_c' \setminus V_c} \deg(w) = \sum_{v \in V_c \setminus V_c'} (|L_v| - 2) - \sum_{w \in V_c' \setminus V_c} (|L_w| - 2). \quad (2.1)$$

Then the following statements hold: (i) $|\mathsf{DE}(d_1, \ldots, d_n)| \neq |\mathsf{DO}(d_1, \ldots, d_n)|$, *and* (ii) *$G$ is f-choosable provided that $f(v_i) = d_i + 1$ for $1 \leq i \leq n$.*

Proof. By a result of Alon and Tarsi in [2], as noted in [1], it is sufficient to show that $K \neq 0$, where

$$K = |\mathsf{DE}(d_1, \ldots, d_n)| - |\mathsf{DO}(d_1, \ldots, d_n)|.$$

We use the idea of [2] and proceed exactly as in [1]. The graph polynomial f_G is homogeneous and every monomial of f_G has degree m. We apply the same argument as in the proof of Theorem 2.1 in [2].

This argument, already used in [1], implies that there is a polynomial $\overline{f}_G(x_1, \ldots, x_n)$ satisfying the following conditions:

(2) $\overline{f}_G(x_1, \ldots, x_n) = f_G(x_1, \ldots, x_n)$ for every $(x_1, \ldots, x_n) \in S$.

(3) $\deg_{x_i}(\overline{f}_G) \le d_i$, for $1 \le i \le n$.

(4) The coefficient of $\prod_{i=1}^{n} x_i^{d_i}$ in $\overline{f}_G$ is equal to its coefficient in f_G.

By Lemma 1.1 and (4), it then follows that the coefficient of $\prod_{i=1}^{n} x_i^{d_i}$ in $\overline{f}_G$ is equal to K. Combining (1)–(3), and Lemma 2.1, we then conclude that $K = K_1 + K_2$, where

$$K_1 = \frac{\prod\{(a_i - a_j)|i < j, v_i v_j \in E(G)\}}{\prod_{j=1}^{n} \prod_{s \in S_j \setminus \{a_j\}} (a_j - s)}, K_2 = \frac{\prod\{(b_i - b_j)|i < j, v_i v_j \in E(G)\}}{\prod_{j=1}^{n} \prod_{s \in S_j \setminus \{b_j\}} (b_j - s)}.$$

If $K_1 + K_2 \ne 0$, then $K \ne 0$ and G would be f-choosable provided that $f(v_i) = d_i + 1$ for $1 \le i \le n$. Suppose that $K_1 + K_2 = 0$. From this equation, we obtain a contradiction with the assumption of the proposition concerning equation (2.1). Note that the equation $K_1 + K_2 = 0$ can be written as the following

$$\prod_{\substack{j=1 \\ a_j \ne b_j}}^{n} \prod_{s \in S_j \setminus \{a_j, b_j\}} \left(\frac{b_j - s}{a_j - s} \right) = (-1)^{k+1} \prod_{\substack{i < j \\ v_i v_j \in E(G)}} \left(\frac{b_j - b_i}{a_j - a_i} \right), \qquad (2.2)$$

where k is the number of indexes j for which $a_j \ne b_j$. Consider the above equation in terms of the indeterminate c for any color c, and compare the degree of c in the left and in the right side. Then we see that equation (2.1) must hold for every color c. Hence we have a contradiction. $\qquad \square$

Remark 2.3. We note that actually the equation $K_1 + K_2 = 0$ may be true. For example a triangle with the lists $\{1, 2\}, \{1, 2\}, \{2, 3\}$ is double L-colorable but is not 2-choosable.

Example 2.4. We adapt the example in Theorem 2 of [1] with a minor change. For any natural number $3 \le t$, consider the complete graph K_{2t-1}, with the vertex set $\{u_1, \ldots, u_t, v_1, \ldots, v_{t-1}\}$. For each $i, j, 1 \le i \le t, 1 \le j \le t - 1$, assign to u_i a list $L_{u_i} = \{1, \ldots, t\}$ and to v_j a list $L_{v_j} = \{1, \ldots, t + j\}$. By adding $t - 1$ independent new vertices $\{w_1, \ldots, w_{t-1}\}$ to the complete graph K_{2t-1} and joining the vertex $w_i, 1 \le i \le t - 1$, to all vertices $\{v_1, \ldots, v_{t-1}\}$ and $\{u_{i+1}, \ldots, u_t\}$, we get a graph G of order $3t - 2$ and size $\binom{2t-1}{2} + (t-1)^2 + 1 + 2 + \ldots + (t-1)$. For each $i, 1 \le i \le t - 2$, we put $L_{w_i} = \{t + 1, \ldots, 2t - 1\} \cup \{i\}$ and for w_{t-1} we put $L_{w_{t-1}} = L_{w_{t-2}}$ (this is the minor change with [1]). We can show that G is double L-colorable using the same argument as in [1]. In fact, all colors $\{1, \ldots, t\}$ appear in the vertices $\{u_1, \ldots, u_t\}$ and v_1 is adjacent to these vertices, so v_1 can only be colored by $t + 1$. In the same manner, v_2 can only be colored by $t + 2$ and similarly the color of v_i should be $t + i$, for $1 \le i \le t - 1$. Since w_i, for $1 \le i \le t - 1$, is adjacent to all vertices $\{v_1, \ldots, v_{t-1}\}$, its color is uniquely determined. Also w_1 is adjacent to the vertices $\{u_2, \ldots, u_t\}$ so the color of u_1 is 1. For any $j, 1 \le j \le t - 2$, the vertex u_j

can only be colored by j and for the vertices u_{t-1} and u_t we have two choices. Hence G is double L-colorable. We can check that the equation (2.1) does not hold for the color t. It is easy to verify that $\sum_{v \in V(G)}(|L_v| - 1)$ is equal to the size of G. Now by Proposition 2.2, G is f-choosable.

Now, with the above notation, suppose that the equation (2.1) holds for every color c. We are interested this time to the coefficient of the indeterminate c obtained by the equation (2.2). Fix a color c. Let V_c and V'_c be as above. We use indexing of the vertices in which the indexes of V'_c are greater than those of V_c. Suppose that there exist r edges between V_c and V'_c in the graph. It is not hard to see that just by comparing the coefficient of c in both sides of the equation (2.2), we must have that $k + r + 1$ is an even number. So if $k + r$ is an even number, we have choosability of the graph.

Proposition 2.5. *Let G be a graph on a set $V = \{v_1, \ldots, v_n\}$ of $n \geq 1$ vertices. For $1 \leq i \leq n$, let L_{v_i} be a list of $d_i + 1$ colors where $d_i \geq 0$ is a given integer. Suppose that G is double L-colorable and $d_1 + \ldots + d_n = m$, where m is the size of G. Let k be the number of vertices which receive different colors in the two L-colorings of G. Suppose that for some color c, in the above notation, there exist r edges between V_c and V'_c such that $k + r$ is an even number. Then the following statements hold: (i) $|\mathsf{DE}(d_1, \ldots, d_n)| \neq |\mathsf{DO}(d_1, \ldots, d_n)|$, and (ii) G is f-choosable provided that $f(v_i) = d_i + 1$ for $1 \leq i \leq n$.*

Example 2.6. Let G be the cycle of length $2n$. We assign to each vertex the same list $\{1, 2\}$ of colors. Then G is double L-colorable. In the above notation, the equation (2.1) holds for the colors $1, 2$. But the equation (2.2) does not hold. In fact we have $k = 2n$ and for each color 1 or 2, $r = 2n$. Hence the number $k + r$ is even and by Proposition 2.5, G is 2-choosable.

It is evident that in a similar way, we can study the relation between triple L-colorability and choosability of a graph. This leads us to study an equation in the form of the equation (2.2) with three terms involved. Determining the degree or the coefficient of an indeterminate color c in such equation seems to be more complicated.

REFERENCES

[1] S. Akbari, V.S. Mirrokni, B.S. Sadjad, *A relation between choosability and uniquely list colorability*, J. Combin. Theory Ser. B **96** (2006), 577–583.

[2] N. Alon, M. Tarsi, *Colorings and orientations of graphs*, Combinatorica **12** (1992), 125–134.

[3] C. Chao, Z. Chen, *On uniquely 3-colorable graphs*, Discrete Math. **112** (1993), 21–27.

[4] P. Erdös, A.L. Rubin, H. Taylor, *Choosability in graphs*, Congr. Numer. **26** (1979), 125–157.

[5] C. Thomassen, *The chromatic polynomial and list colorings*, J. Combin. Theory Ser. B **99** (2009), 474–479.

[6] V.G. Vizing, *Vertex colorings with given colors*, Metody Diskret. Anal. **29** (1976), 3–10 [in Russian].

[7] S. Xu, *The size of uniquely colorable graphs*, J. Combin. Theory Ser. B **50** (1990), 319–320.

Hamid-Reza Fanaï
fanai@sharif.edu

Sharif University of Technology
Department of Mathematical Sciences
P.O. Box 11155-9415
Tehran, Iran