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ON SOME FAMILIES OF ARBITRARILY VERTEX DECOMPOSABLE SPIDERS

Abstract. A graph G of order n is called arbitrarily vertex decomposable if for each sequence $(n_1, \ldots, n_k)$ of positive integers such that $\sum_{i=1}^{k} n_i = n$, there exists a partition $(V_1, \ldots, V_k)$ of the vertex set of G such that for every $i \in \{1, \ldots, k\}$ the set V_i induces a connected subgraph of G on n_i vertices. A *spider* is a tree with one vertex of degree at least 3. We characterize two families of arbitrarily vertex decomposable spiders which are homeomorphic to stars with at most four hanging edges.

Keywords: arbitrarily vertex decomposable graph, trees.

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1. INTRODUCTION

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $|V(G)| = n$. A sequence $\tau = (n_1, \ldots, n_k)$ of positive integers is called *admissible for* G if $n_1 + \ldots + n_k = n$. We shall write $((n_1)^{s_1}, \ldots, (n_l)^{s_l})$ for the sequence $(\underbrace{n_1, \ldots, n_1}_{s_1}, \ldots, \underbrace{n_l, \ldots, n_l}_{s_l})$. If $\tau = (n_1, \ldots, n_k)$ is an admissible sequence for the graph G and there exists a partition $(V_1, \ldots, V_k)$ of the vertex set $V(G)$ such that for each $i \in \{1, \ldots, k\}$ the subgraph $G[V_i]$ induced by V_i is a connected graph on n_i vertices, then τ is called G-*realizable* or *realizable in* G and the sequence $(V_1, \ldots, V_k)$ is said to be a G-*realization of* τ or a *realization of* τ *in* G. Each set V_i will be called a τ-*part* of a realization of τ in G. A graph G is called *arbitrarily vertex decomposable* (*avd* for short) if each admissible sequence for G is realizable in G.

Arbitrarily vertex decomposable graphs have been investigated in several papers ([1–5] for example). The problem originated from some applications to computer networks ([1]).

The investigation of avd trees is motivated by the fact that a connected graph is avd if its spanning tree is avd.

In [4] the authors proved that every tree of maximum degree at least 7 is not avd and conjectured that every tree with maximum degree at least 5 is not avd. This conjecture was proved in [2]:

Theorem 1.1. *If tree T is arbitrarily vertex decomposable then $\Delta(T) \leq 4$. Moreover every vertex of degree four in T is adjacent to a leaf.*

Let $T = (V(T), E(T))$ be a tree. A vertex $v \in V(T)$ is called *primary* if $d(v) \geq 3$. A *leaf* is a vertex of degree one in T. Let the path P be a subgraph of T such that one of its end vertices is a leaf in T, the other one is a primary vertex in T and all internal vertices of P have degree two in T. We will call such a path an *arm* of T. Let v be a primary vertex of a tree T such that v is an end vertex of two arms A_1, A_2 of T. Let y_i be the other end vertex of A_i and $x_i \in V(A_i)$ the neighbour of v, $i = 1, 2$. Define $T(A_1, A_2)$ to be a tree with $V(T(A_1, A_2)) = V(T)$ and $E(T(A_1, A_2)) = E(T) - \{vx_2\} \cup \{y_1y_2\}$.

In [1] and, independently, in [5] the authors observed that:

Lemma 1.2. *Let T be an arbitrarily vertex decomposable tree and let A_1, A_2 be arms of T that share a primary vertex of T. Then the tree $T(A_1, A_2)$ is arbitrarily vertex decomposable, too.*

That gives a reason for the investigation of avd trees which are homeomorphic to a star $K_{1,q}$, where q is three or four. If $q = 2$ such a tree is a path which is avd.

A *spider* is a tree with one primary vertex. Such a tree has q arms A_i, $i = 1, \ldots, q$, where q is the degree of the primary vertex. Let a_i be the order of A_i, $i = 1, \ldots, q$. The structure of a spider is determined by the sequence of orders of its arms. Since the ordering of this sequence is not important, we will assume that $a_1 \leq a_2 \leq \ldots \leq a_q$ and we will denote the above defined spider by $S(a_1, \ldots, a_q)$.

The first result characterizing the avd spider was found in [1] and, independently, in [5].

We will denote by $\gcd(a, b)$ the greatest common divisor of two positive integers a and b.

Theorem 1.3. *The spider $S(2, b, c)$, $2 \leq b \leq c$ is arbitrarily vertex decomposable if and only if $\gcd(b, c) = 1$. Moreover, each admissible and non-realizable sequence in $S(2, b, c)$ is of the form $((d)^k)$, where $b \equiv c \equiv 0 (mod\ d)$ and $d \geq 2$.*

In [1] the authors characterized avd $S(2, 2, b, c)$ using avd $S(3, b, c)$:

Proposition 1.4. *The spider $S(2, 2, b, c)$, $2 \leq b \leq c$ is arbitrarily vertex decomposable if and only if the following conditions hold:*

1. *The spider $S(3, b, c)$ is arbitrarily vertex decomposable,*
2. *The numbers b, c are odd,*
3. *$b \not\equiv 2 (mod\ 3)$ or $c \not\equiv 2 (mod\ 3)$.*

In [3] the authors investigated two families of spiders: $S(2, 2, b, c)$ and $S(3, b, c)$.

Theorem 1.5. *The spider $S(2,2,b,c)$ of order n, $3 \leq b \leq c$, is arbitrarily vertex decomposable if and only if the following conditions hold:*

1. $\gcd(b,c) = 1$,
2. $\gcd(b+1,c) = 1$,
3. $\gcd(b,c+1) = 1$,
4. $\gcd(b+1,c+1) = 2$,
5. $n \neq \alpha b + \beta(b+1)$ *for* α, $\beta \in \mathbf{N}$.

Theorem 1.6. *The spider $S(3,b,c)$ of order n, $3 \leq b \leq c$, is arbitrarily vertex decomposable if and only if the following conditions hold:*

1. $\gcd(b,c) \leq 2$,
2. $\gcd(b+1,c) \leq 2$,
3. $\gcd(b,c+1) \leq 2$,
4. $\gcd(b+1,c+1) \leq 3$,
5. $n \neq \alpha b + \beta(b+1)$ *for* α, $\beta \in \mathbf{N}$.

The main result of this paper are Theorems 2.1 and 2.2 of Section 2 which give a complete characterization of avd spiders $S(2,3,b,c)$ and $S(4,b,c)$.To prove them we will also use the following results:

Proposition 1.7 ([1])**.** *The spider $S(a_1,a_2,a_3)$, $a_1 \leq a_2 \leq a_3$, is arbitrarily vertex decomposable if and only if every admissible sequence $((q)^{s_1},(q+1)^{s_2})$, $s_2 > 0$, $q \leq a_1+a_2-2$ and every admissible sequence $(m,(r)^{t_1},(r+1)^{t_2})$, $t_2 > 0$, $1 \leq m \leq r-1$, $r \leq a_1 - 3$, has a realization in $S(a_1,a_2,a_3)$.*

Proposition 1.8 ([2])**.** *The spider $S(2,a_1,a_2,a_3)$, $a_1 \leq a_2 \leq a_3$, is arbitrarily vertex decomposable if and only if the following conditions hold:*

1. *The spider $S(a_1,a_2,a_3)$, $a_1 \leq a_2 \leq a_3$, is arbitrarily vertex decomposable.*
2. *Every admissible sequence $((q)^{s_1},(q+1)^{s_2})$, $s_2 > 0$, $q \leq a_1+a_2-2$ and every admissible sequence $(m,(r)^{t_1},(r+1)^{t_2})$, $t_2 > 0$, $0 < m \leq r-1$, $r \leq a_1-3$, has a realisation in $S(2,a_1,a_2,a_3)$.*

Proposition 1.9 ([6])**.** *The graph G is arbitrarily vertex decomposable if and only if every admissible sequence $(n_1,\ldots,n_k)$ with $n_i \geq 2$ for each $i = 1,\ldots,k$, has a realization in G.*

Given an admissible sequence $\tau = (n_1,\ldots,n_k)$ for a graph G of order n, we will use the following convention to describe a realization $(V_1,\ldots,V_k)$ of τ in G. We choose an ordering $s = (v_1,\ldots,v_n)$ of the vertex set of G. Then we define the τ-parts according to the sequence s, that is $V_1 = \{v_1,\ldots,v_{n_1}\}$, $V_2 = \{v_{n_1+1},\ldots,v_{n_1+n_2}\}$ and so on.

2. ARBITRARILY VERTEX DECOMPOSABLE SPIDERS $S(2,3,b,c)$ AND $S(4,b,c)$

Theorem 2.1. *The spider $S(2,3,b,c)$ of order n, $3 \leq b \leq c$, is arbitrarily vertex decomposable if and only if the following conditions hold:*

(1) $\gcd(b,c) = 1$,

(2) $\max\{\gcd(b+1,c),\gcd(b,c+1),\gcd(b+1,c+1),\gcd(b+2,c),$
$\gcd(b,c+2)\} \le 2$,
(3) $\max\{\gcd(b+1,c+2),\gcd(b+2,c+1),\gcd(b+2,c+2)\} \le 3$,
(4) $n \ne \alpha b + \beta(b+1) + \gamma(b+2)$ *for* $\alpha,\ \beta,\ \gamma \in \mathbf{N}$,
(5) *If* $b = 2h$, $h \in \mathbf{N}$, $h \ge 3$ *then* $n \ne \alpha h + \beta(h+1)$ *for* $\alpha,\ \beta \in \mathbf{N}$.

Proof. Necessity. If $d_1 = \gcd(b,c) \ge 2$ or $d_2 = \max\{\gcd(b+1,c),\gcd(b,c+1)\} \ge 3$ or $d_3 = \max\{\gcd(b+1,c+1),\gcd(b+2,c),\gcd(b,c+2)\} \ge 3$ or $d_4 = \max\{\gcd(b+1,c+2),\gcd(b+2,c+1)\} \ge 4$ or $d_5 = \gcd(b+2,c+2) \ge 4$ then the following sequences $(2,(d_1)^{\frac{n-2}{d_1}})$ or $((d_2)^{\frac{n-1}{d_2}-1}, d_2+1)$ or $((d_3)^{\frac{n}{d_3}})$ or $(d_4-1,(d_4)^{\frac{n+1}{d_4}-1})$ or $((d_5-1)^2,(d_5)^{\frac{n+2}{d_5}-2})$, respectively, are admissible but not realizable. If $n = \alpha b + \beta(b+1) + \gamma(b+2)$, where $\alpha,\ \beta,\ \gamma \in \mathbf{N}$ then the sequence $((b)^\alpha,(b+1)^\beta,(b+2)^\gamma)$ is admissible and not realizable. If $n = \alpha h + \beta(h+1)$, where $h = \frac{b}{2} \in \mathbf{N}$, $h \ge 3$ then the sequence $((h)^\alpha,(h+1)^\beta)$ is admissible and not realizable.
Sufficiency. Let A_i, $i = 1,\ldots,4$ be arms of $S(2,3,b,c)$, $3 \le b \le c$, of orders 2, 3, b and c, respectively. Let v be a primary vertex of $S(2,3,b,c)$. Set $A_1 = \{v, v_1^2\}$, $A_2 = \{v, v_1^3, v_2^3\}$, $A_3 = \{v, v_1^b, \ldots, v_{b-1}^b\}$ and $A_4 = \{v, v_1^c, \ldots, v_{c-1}^c\}$, such that vv_1^2, vv_1^3, $v_1^3v_2^3$, vv_1^b, $v_i^bv_{i+1}^b$, vv_1^c, $v_j^cv_{j+1}^c$ are edges of $S(2,3,b,c)$, $i = 1,\ldots,b-2$, $j = 1,\ldots,c-2$. Let $\tau = (n_1,\ldots,n_k)$ be an admissible sequence for $S(2,3,b,c)$. We assume that $n_1 \le \ldots \le n_k$.

By Proposition 1.8, Proposition 1.9 and Theorem 1.6 we may assume that $\tau = ((n_1)^{k_1},(n_1+1)^{k_2})$, where $k_1,\ k_2 \in \mathbf{N}$ and $2 \le n_1 \le b+1$.

If $n_1 = 2$ then by Theorem 1.3 there is the realization $(V_2,\ldots,V_k)$ of the sequence $(n_2,\ldots,n_k)$ in $S(2,b,c)$ and hence $(\{v_1^3,v_2^3\},V_2,\ldots,V_k)$ is a realization of τ in $S(2,3,b,c)$. We may asume that $n_1 \ge 3$.

Since $\max\{\gcd(b+1,c+1),\gcd(b+2,c),\gcd(b,c+2)\} \le 2$, we have $\tau \ne ((3)^k)$ and hence especially $n_k \ge 4$. Since $n_k \le b+2$, by the condition (4), we obtain that $n_1 \le b-1$, $n_k \le b$. We define the sequence $(V_1,\ldots,V_k)$ of τ-parts according to $s^1 = (v_1^b, v_2^b, \ldots, v_{b-1}^b, v_{c-1}^c, \ldots, v_1^c, v, v_1^2, v_1^3, v_2^3)$. Suppose that the construction does not give a realization of τ in $S(2,3,b,c)$. It follows that there is i_0 such that v_{b-1}^b, $v_{c-1}^c \in V_{i_0}$. Since $n_k \le b$, $n_1 \le b-1$, we have $2 \le i_0 \le k-1$. If $|V_{i_0} \cap V(A_3)| \le n_k - 4$ then we modify the ordering of elements of τ, we obtain $\tau = (n_{i_0}, n_{i_0+1}, \ldots, n_k, n_1, \ldots, n_{i_0-1})$ and we define the sequence of τ-parts according to $s^2 = (v_{c-1}^c, v_{c-2}^c, \ldots, v_1^c, v, v_1^2 v_1^3, v_2^3, v_1^b, v_2^b, \ldots, v_{b-1}^b)$ and we obtain a realization of τ in $S(2,3,b,c)$. Hence we may assume that $|V_{i_0} \cap V(A_3)| \ge n_k - 3$.

We will use the following notation: $d = n_k - n_{i_0}$, $r = |V_{i_0} \cap V(A_3)| - (n_k - 4)$. It is easily seen that $d + r + |V_{i_0} \cap V(A_4)| = 4$. Since $|V_{i_0} \cap V(A_4)| \ge 1$, $d \le 1$, we obtain that $1 \le r \le 3$ or $1 \le r \le 2$ for $d = 0$ or $d = 1$, respectively. Observe that $b = \sum_{i=1}^{i_0-1} n_i + 1 + r + (n_k - 4) = \sum_{i=1}^{i_0-1} n_i + n_k + r - 3$ and $c = \sum_{i=i_0}^{k-1} n_i + 1 - r$.

Let us suppose that $n_{k-1} - n_1 \ge r$. We modify the ordering of elements of τ and we consider $\tau = (n_{k-1}, n_2, \ldots, n_{k-2}, n_1, n_k)$. We define the sequence of τ-parts according to s^1 and, since $0 \le |V_{i_0} \cap V(A_3)| - (n_{k-1} - n_1) \le n_k - 4$, either we obtain a realization of τ or v_{b-1}^b, $v_{c-1}^c \in V_{j_0}$, where $j_0 = i_o$ for $i_0 < k-1$ and $j_0 = 1$ for $i_0 = k-1$. In the second case we modify the ordering of elements of τ such that $\tau = (n_{i_0}, n_{i_0+1}, \ldots, n_1, n_k, n_{k-1}, n_2, \ldots, n_{i_0-1})$ if $i_0 < k-1$

or $\tau = (n_1, n_k, n_{k-1}, n_2, \dots, n_{k-2})$ if $i_0 = k-1$ and we define the sequence of τ-parts according to s^2. Since $|V_{j_0} \cap V(A_3)| \le n_k - 4$, we obtain a realization of τ. Hence we may assume that $n_{k-1} - n_1 < r$.

If $\tau = ((n_1)^k)$ then $b = i_0 n_1 + r - 3$, $c = (k-i_0)n_1 + 1 - r$ and hence $\max\{\gcd(b, c+2), \gcd(b+1, c+1), \gcd(b+2, c)\} \ge n_1 \ge 3$, contrary to (2). If $\tau = ((n_1)^{k-1}, n_1+1)$ then $d = 1$ and hence $r \in \{1, 2\}$. Since $b = i_0 n_1 + r - 2$, $c = (k - i_0)n_1 + 1 - r$, we obtain that $\max\{\gcd(b, c+1), \gcd(b+1, c)\} \ge n_1 \ge 3$, contrary to (2). Therefore we may assume that $n_{k-1} = n_1 + 1$.

Let us suppose that $\tau = (n_1, (n_1+1)^{k-1})$. Then $r \in \{2, 3\}$. Since $b = i_0(n_1+1) + r - 4$ and $c = (k-i_0)(n_1+1) + 1 - r$, we obtain that $\max\{\gcd(b+1, c+2), \gcd(b+2, c+1)\} \ge n_1 + 1 \ge 4$, contrary to (3). Hence we may assume that $n_2 = n_1$.

Let us suppose that $i_0 = 2$. Then $d = 1$, $r = 2$ and $b = 2n_1$, contrary to (5). We may assume that $i_0 \ge 3$, and hence $k \ge 4$.

If $i_0 = k-1$ then $b = \sum_{i=1}^{k-2} n_i + n_k + r - 3 \ge n_k + r$ and $c = n_k + 1 - r$, which contradicts the assumption $b \le c$. Hence we may assume that $i_0 \le k-2$ and hence $k \ge 5$.

Let us suppose that $(n_{k-1} + n_{k-2}) - (n_1 + n_2) \ge r$. We modify the ordering of elements of τ and we consider $\tau = (n_{k-1}, n_{k-2}, n_3, \dots, n_{k-3}, n_2, n_1, n_k)$. We define the sequence of τ-parts according to s^1. Combining condition $n_{k-1} - n_1 < r$ with the values of d and n_i, $i = 2, k-2, k-1$ we obtain that $0 \le |V_{i_0} \cap V(A_3)| - [(n_{k-1} + n_{k-2}) - (n_1 + n_2)] \le n_k - 4$. Then either we obtain a realization of τ or v_{b-1}^b, $v_{c-1}^c \in V_{j_0}$, where $j_0 = i_0$ for $i_0 < k-2$ and $j_0 = 2$ for $i_0 = k-2$. In the second case we modify the ordering of elements of τ such that $\tau = (n_{i_0}, n_{i_0+1}, \dots, n_{k-3}, n_2, n_1, n_k, n_{k-1}, n_{k-2}, n_3, \dots, n_{i_0-1})$ if $i_0 < k-2$ or $\tau = (n_2, n_1, n_k, n_{k-1}, n_{k-2}, n_3, \dots, n_{k-3})$ if $i_0 = k-2$ and we define the sequence of τ-parts according to s^2. Since $|V_{j_0} \cap V(A_3)| \le n_k - 4$, we obtain a realization of τ. Hence we may assume that $(n_{k-1} + n_{k-2}) - (n_1 + n_2) < r$.

It is not difficult to check that then we have two possibilities: either $\tau = ((n_1)^{k-2}, (n_1+1)^2)$, $r = 2$ or $n_1 = n_2$, $n_{k-2} = n_{k-1} = n_k = n_1 + 1$, $r = 3$.

If $\tau = ((n_1)^{k-2}, (n_1+1)^2)$ and $r = 2$ then $b = i_0 n_1$, $c = (k - i_0)n_1$ and hence $\gcd(b, c) \ge n_1 \ge 3$, contrary to (1). Hence $n_1 = n_2$, $n_{k-2} = n_{k-1} = n_k = n_1 + 1$ and $r = 3$. If $\tau = ((n_1)^2, (n_1+1)^{k-2})$ then $b = i_0(n_1+1) - 2$, $c = (k-i_0)(n_1+1) - 2$ and hence $\gcd(b+2, c+2) \ge n_1 + 1 \ge 4$, contrary to (3). Therefore we may assume that $k \ge 6$ and $n_3 = n_1$.

If $i_0 = 3$ then $d = 1$ and hence $r \le 2$, a contradiction. Hence $4 \le i_0$. If $i_0 = k-2$ then $4n_1 + 1 \le b \le c = 2n_1$, a contradiction. Hence $i_0 \le k-3$ and $k \ge 7$. We obtain that $n_1 = n_2 = n_3$, $n_{k-2} = n_{k-1} = n_k = n_1 + 1$, $r = 3$ and $4 \le i_0 \le k-3$. Then $d = 0$ and hence $n_{k-3} = n_1 + 1$. We modify the ordering of elements of τ and we consider $\tau = (n_{k-1}, n_{k-2}, n_{k-3}, n_4, \dots, n_{k-4}, n_3, n_2, n_1, n_k)$. We define the sequence of τ-parts according to s^1. Let us suppose that the construction does not give a realization of τ. Then we modify the ordering of elements of τ and we consider $\tau = (n_{i_0}, n_{i_0+1}, \dots, n_{k-4}, n_3, n_2, n_1, n_k, n_{k-1}, n_{k-2}, n_{k-3}, n_4, \dots, n_{i_0-1})$ if $i_0 < k-3$ or $\tau = (n_3, n_2, n_1, n_k, n_{k-1}, n_{k-2}, n_{k-3}, n_4, \dots, n_{k-4})$ if $i_0 = k-3$. We define the sequence of τ-parts according to s^2 and obtain a realization of τ. □

Theorem 2.2. *The spider $S(4,b,c)$ of order n, $4 \le b \le c$, is arbitrarily vertex decomposable if and only if the following conditions hold:*

(1) $\gcd(b,c) = 1$ *or* $\gcd(b,c) = 3$,
(2) $\max\{\gcd(b+1,c), \gcd(b,c+1), \gcd(b+1,c+1), \gcd(b+2,c), \gcd(b,c+2)\} \le 3$,
(3) $\max\{\gcd(b+1,c+2), \gcd(b+2,c+1), \gcd(b+2,c+2)\} \le 4$,
(4) $n \ne \alpha b + \beta(b+1) + \gamma(b+2)$ *for* α, β, $\gamma \in \mathbf{N}$,
(5) *If* $b = 2h$, $h \in \mathbf{N}$, $h \ge 4$ *then* $n \ne \alpha h + \beta(h+1)$ *for* α, $\beta \in \mathbf{N}$.

Proof. We will use the similar method to that in the proof of Theorem 2.1.

Necessity. If $d_1 = \gcd(b,c) \notin \{1,3\}$ or $d_2 = \max\{\gcd(b+1,c), \gcd(b,c+1)\} \ge 4$ or $d_3 = \max\{\gcd(b+1,c+1), \gcd(b+2,c), \gcd(b,c+2)\} \ge 4$ or $d_4 = \max\{\gcd(b+1,c+2), \gcd(b+2,c+1)\} \ge 5$ or $d_5 = \gcd(b+2,c+2) \ge 5$ then the following sequences $(2,(d_1)^{\frac{n-2}{d_1}})$ or $((d_2)^{\frac{n-1}{d_2}-1}, d_2+1)$ or $((d_3)^{\frac{n}{d_3}})$ or $(d_4-1,(d_4)^{\frac{n+1}{d_4}-1})$ or $((d_5-1)^2,(d_5)^{\frac{n+2}{d_5}-2})$, respectively, are admissible but not realizable. If $n = \alpha b + \beta(b+1) + \gamma(b+2)$, where α, β, $\gamma \in \mathbf{N}$ then the sequence $((b)^\alpha,(b+1)^\beta,(b+2)^\gamma)$ is admissible and not realizable. If $n = \alpha h + \beta(h+1)$, where $h = \frac{b}{2} \in \mathbf{N}$, $h \ge 4$ then the sequence $((h)^\alpha,(h+1)^\beta)$ is admissible and not realizable.

Sufficiency. Let A_i, $i = 1,2,3$ be arms of $S(4,b,c)$, $4 \le b \le c$, of orders 4, b and c, respectively. Let v be a primary vertex of $S(4,b,c)$. Set $A_1 = \{v, v_1^4, v_2^4, v_3^4\}$, $A_2 = \{v, v_1^b, \ldots, v_{b-1}^b\}$ and $A_3 = \{v, v_1^c, \ldots, v_{c-1}^c\}$, such that vv_1^4, $v_i^4 v_{i+1}^4$, vv_1^b, $v_j^b v_{j+1}^b$, vv_1^c, $v_l^c v_{l+1}^c$ are edges of $S(4,b,c)$, $i = 1,2$, $j = 1,\ldots,b-2$, $l = 1,\ldots,c-2$. Let $\tau = (n_1,\ldots,n_k)$ be an admissible sequence for $S(4,b,c)$. We assume that $n_1 \le \ldots \le n_k$.

If there is $i_0 \in \{1,\ldots,k\}$ such that $n_{i_0} = 3$ then we set $V_{i_0} = \{v_1^4, v_2^4, v_3^4\}$ and obtain a realization of τ in $S(4,b,c)$. Hence we may assume that $n_i \ne 3$ for $i \in \{1,\ldots,k\}$.

Let us suppose that $n_{i_0} = 2$ for any $i_0 \in \{1,\ldots,k\}$. Since $\tau \ne (2,(3)^{k-1})$, if we set $V_{i_0} = \{v_2^4, v_3^4\}$ then by Theorem 1.3 we obtain a realization of τ in $S(4,b,c)$. Hence we may assume that $n_i \ne 2$ for $i \in \{1,\ldots,k\}$. Then by Proposition 1.9 and Proposition 1.7 we have that $\tau = ((n_1)^{k_1},(n_1+1)^{k_2})$, where k_1, $k_2 \in \mathbf{N}$ and $4 \le n_1 \le b+2$. If $n_k = b+3$ then the sequence $(V_1,\ldots,V_k)$ such that $[V(A_1) \cup V(A_2)] \subset V_k$ and for $i = 1,\ldots,k-1$, $V_i \subset [V(A_3)\backslash\{v\}]$ is a realization of τ in $S(4,b,c)$. We may assume that $n_k \le b+2$. By the condition (4) we obtain that $n_1 \le b-1$, $n_k \le b$. We define the sequence $(V_1,\ldots,V_k)$ of τ-parts according to $s^1 = (v_1^b, v_2^b, \ldots, v_{b-1}^b, v_{c-1}^c, \ldots, v_1^c, v, v_1^4, v_2^4, v_3^4)$. Suppose that the construction does not give a realization of τ in $S(4,b,c)$. It follows that there is i_0 such that v_{b-1}^b, $v_{c-1}^c \in V_{i_0}$. Since $n_k \le b$ and $n_1 \le b-1$, we have $2 \le i_0 \le k-1$. Using similar arguments to that in the proof of Theorem 2.1 we may assume that $|V_{i_0} \cap V(A_2)| \ge n_k - 3$. We will use the following notation: $d = n_k - n_{i_0}$, $r = |V_{i_0} \cap V(A_2)| - (n_k - 4)$. It is easily seen that $d + r + |V_{i_0} \cap V(A_3)| = 4$. Since $|V_{i_0} \cap V(A_3)| \ge 1$, $d \le 1$, we obtain that $1 \le r \le 3$ or $1 \le r \le 2$ for $d = 0$ or $d = 1$, respectively. Observe that $b = \sum_{i=1}^{i_0-1} n_i + n_k + r - 3$, $c = \sum_{i=i_0}^{k-1} n_i + 1 - r$.

Using a similar method to that in the proof of Theorem 2.1 we obtain that if $n_{k-1} - n_1 \ge r$ then there is a realization of τ in $S(4,b,c)$. Hence we may assume that $n_{k-1} - n_1 < r$.

If $\tau = ((n_1)^k)$ then $\max\{\gcd(b+2,c), \gcd(b+1,c+1), \gcd(b,c+2)\} \geq n_1 \geq 4$, contrary to (2). If $\tau = ((n_1)^{k-1}, n_1+1)$ then $d = 1$ and hence $r \in \{1,2\}$ and $\max\{\gcd(b+1,c), \gcd(b,c+1)\} \geq n_1$, contrary to (2). If $\tau = (n_1, (n_1+1)^{k-1})$ then $r \in \{2,3\}$ and hence $\max\{\gcd(b+2,c+1), \gcd(b+1,c+2)\} \geq n_1+1 \geq 5$, contrary to (3). Hence we may assume that $k \geq 4$ and $n_1 = n_2$, $n_k = n_{k-1} = n_1+1$.

Using similar method to that in the proof of Theorem 2.1 we may assume that $k-2 \geq i_0 \geq 3$ and that $(n_{k-1}+n_{k-2}) - (n_1+n_2) < r$. Then we obtain that either $\tau = ((n_1)^{k-2}, (n_1+1)^2)$, $r = 2$ or $n_1 = n_2$, $n_{k-2} = n_{k-1} = n_k = n_1+1$, $r = 3$. In the first case $b = i_0 n_1$, $c = (k-i_0)n_1$ and $\gcd(b,c) \geq n_1 \geq 4$ contrary to (1). We may assume that $n_1 = n_2$, $n_{k-2} = n_{k-1} = n_k = n_1+1$ and $r = 3$.

If $\tau = ((n_1)^2, (n_1+1)^{k-2})$ then $b = i_0(n_1+1) - 2$, $c = (k-i_0)(n_1+1) - 2$ and $\gcd(b+2, c+2) \geq n_1+1 \geq 5$, contrary to (3). Hence we may assume that $k \geq 6$ and $n_3 = n_1$. Since $r = 3$, we obtain that $d = 0$ and hence $i_0 \geq 4$. If $i_0 = k-2$ then $4n_1+1 \leq b \leq c = 2n_1$, a contradiction. Hence $i_0 \leq k-3$ and $k \geq 7$.

Since $r = 3$, we have $n_{i_0} = n_k = n_1+1$ and especially $n_{k-3} = n_1+1$. Then, similarly to the proof of Theorem 2.1, we obtain a realization of τ in $S(4,b,c)$. □

Corollary 2.3. *The number of arbitrarily vertex decomposable spiders $S(2,3,b,c)$ and $S(4,b,c)$ is infinite.*

Proof. It is not difficult to check that for b and c such that $b \in \{60s+1, 60s+13, 60s+49, s \geq 0\}$, $c = b+3$ the assumptions (1)–(5) of Theorem 2.1 and assumptions (1)–(5) of Theorem 2.2 hold. □

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