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MONOTONE ITERATIVE TECHNIQUE FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS

Abstract. In this paper we develop Monotone Method using upper and lower solutions for fractional differential equations with periodic boundary conditions. Initially we develop a comparison result and prove that the solution of the linear fractional differential equation with periodic boundary condition exists and is unique. Using this we develop iterates which converge uniformly monotonically to minimal and maximal solutions of the nonlinear fractional differential equations with periodic boundary conditions in the weighted norm.

Keywords: Riemann-Liouville fractional derivative, monotone method, periodic boundary value problem.

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1. INTRODUCTION

Study of fractional differential equations with initial and boundary conditions has been represented as more appropriate models than its counterpart with integer derivatives. See [2,8] and [10] for more details. However, the existence of solutions by using upper and lower solutions or by monotone method, which is well established for integer derivative in [3] and [9], is not available in literature, except in the recent monograph [4] and [5–7] and [1].

In this paper we have developed a comparison result for the fractional differential equation with periodic boundary conditions. Using this comparison result and the property of Mittag-Leffler function we have proved that the solution to the linear fractional, nonhomogeneous periodic boundary value problem satisfies local Hölder continuity of order q. As an application of our comparison results, we prove the existence of minimal and maximal solutions of fractional differential equations with periodic boundary conditions, by combining the method of upper and lower solutions and monotone method.

2. COMPARISON THEOREM AND AUXILIARY RESULTS

In this section we develop some auxiliary results and comparison results relative to fractional differential equations with periodic boundary conditions. This will be useful to develop our main result. For that purpose we consider the Periodic Boundary Value Problem (PBVP):

$$\begin{aligned} D^q u(t) &= f(t, u(t)), \\ u(t)(t-a)^{1-q}|_{t=a} &= u(b) = x_0, \end{aligned} \tag{2.1}$$

or

$$\begin{aligned} D^q u(t) &= f(t, u(t)), \\ u(a) = u(t)(b-t)^{1-q}|_{t=b} &= y_0. \end{aligned} \tag{2.2}$$

Here $f \in C(J \times \mathbb{R}, \mathbb{R})$ where $J = [a, b]$.

In (2.1), $D^q u(t)$ is the Riemann-Liouville derivative (cf. [4]) for $t \in [a, b]$ having a singularity at $t = a$, and

$$u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_a^t \frac{u(s)ds}{(t-s)^q},$$

where as in (2.2),

$$u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_t^b \frac{u(s)ds}{(s-t)^q}.$$

In particular, if $q = 1$ then (2.1) and (2.2) will reduce to the first order PBVP as in [4].

We recall the following definitions.

Definition 2.1. A function $f : (a, b] \to \mathbb{R}$ is *Hölder continuous* if there are nonnegative real constants C, α such that $|f(x) - f(y)| \leq C|x - y|^\alpha$ for all $x, y \in (a, b]$.

Definition 2.2. Let $0 < q < 1$ and $p = 1 - q$. We denote by $C_{p_a}([a, b], \mathbb{R})$, the function space

$$C_{p_a}([a, b], \mathbb{R}) = \{u \in C((a, b], \mathbb{R}) \ / \ (t - a)^p u(t) \in C([a, b], \mathbb{R})\}.$$

Also, we denote by $C_{p_b}([a, b], \mathbb{R})$, the function space

$$C_{p_b}([a, b], \mathbb{R}) = \{u \in C([a, b), \mathbb{R}) \ / \ (b - t)^p u(t) \in C([a, b], \mathbb{R})\}.$$

Definition 2.3. For $u \in C_{p_a}$ we define the *weighted norm* as

$$|u(t)|_{C_p} = |(t - a)^{1-q} u(t)|.$$

Here and throughout this paper our results are all developed with this weighted norm.

Definition 2.4. Let $v(t)$ be C_{p_a} continuous. Furthermore if

$$D^q v(t) \le f(t, v(t)),$$

$$\Gamma(q) v(t)(t-a)^{1-q}|_{t=a} \le v(b),$$

then $v(t)$ is called a *lower solution* of (2.1). If the inequalities are reversed, then $v(t)$ is called an *upper solution*.

Note that any solution of (2.1) is C_{p_a} continuous and any solution of (2.2) is C_{p_b} continuous.

Next, we prove a comparison lemma relative to upper and lower solutions of (2.1). For that purpose we recall a known comparison theorem from [4].

Lemma 2.5. *Let $m \in C_{p_a}([a,b],\mathbb{R})$ be locally Hölder continuous with exponent $\alpha > q$ and for any $t_1 \in (a,b]$ we have that on (a,t_1), $m(t) \le 0$, $m(t_1) = 0$ and $m(t)(t-a)^{1-q}|_{t=a} \le 0$. Then $D^q m(t_1) \ge 0$.*

Proof. The proof follows from [4]. □

Similarly, we have the following.

Lemma 2.6. *Let $m \in C_{p_a}([a,b],\mathbb{R})$ be locally Hölder continuous with exponent $\alpha > q$ and for any $t_1 \in (a,b]$ we have that on $(t_1,b]$, $m(t) \le 0$, $m(t_1) = 0$ and $\Gamma(q) m(t)(t-a)^{1-q}|_{t=a} \le 0$. Then $D^q m(t_1) \le 0$.*

Remark 2.7. Here and throughout this paper we have assumed Hölder continuity of the function $m(t)$ or of some appropriate functions as needed. However, this is not really required. The C_{p_a} continuity of $m(t)$ on $[a,b]$ is enough for the conclusion of Lemma 2.5 and its applications throughout this paper to hold. This is precisely what we use in our main result on monotone method.

The following result is a comparison theorem which we will need for our main result.

Theorem 2.8. *Let $f \in C([a,b] \times \mathbb{R}, \mathbb{R})$, $v, w \in C_{p_a}([a,b],\mathbb{R})$ and Hölder continuous with exponent $\alpha > q$, with $0 < \alpha < 1$ and for $t \in (a,b)$,*

$$\begin{aligned} D^q v(t) &\le f(t, v(t)), \\ v_a = \Gamma(q) v(t)(t-a)^{1-q}|_{t=a} &\le v(b), \end{aligned} \tag{2.3}$$

$$\begin{aligned} D^q w(t) &\ge f(t, w(t)), \\ w_a = \Gamma(q) w(t)(t-a)^{1-q}|_{t=a} &\ge w(b). \end{aligned} \tag{2.4}$$

Suppose further that $f(t,x)$ is strictly decreasing in x for all $t \in [a,b]$, then

$$v(t) \le w(t) \textit{ for all } t \in (a,b]. \tag{2.5}$$

Proof. Assume that one of the inequalities is strict and let $m(t) = v(t) - w(t)$. If the conclusion of the theorem is not true, there exists $t_1 \in (a, b]$ such that $m(t_1) = 0$, $m(t) \leq 0$ on (a, t_1) and $m(t)(t-a)^{1-q}|_{t=a} \leq 0$. Or $t_1 = a$, in which case $m(t_1)(t_1 - a)^{1-q}|_{t_1=a} = 0$.

Consider the case when $t_1 \in (a, b]$, then $m(t_1) = 0$, $m(t) \leq 0$ on (a, t_1) and $m(t)(t-a)^{1-q}|_{t=a} \leq 0$. So from Lemma 2.5 we get that $D^q m(t_1) \geq 0$. Thus

$$f(t_1, v(t_1)) > D^q v(t_1) \geq D^q w(t_1) \geq f(t_1, w(t_1)) = f(t_1, v(t_1)),$$

which is a contradiction. Therefore $v(t) < w(t)$.

Suppose now that $t_1 = a$, then

$$v(b) > v_a = w_a > w(b) \geq v(b),$$

which is also a contradiction.

To prove the nonstrict inequality, let

$$v_\varepsilon(t) = v(t) - \varepsilon \left[(t-a)^{q-1} + \frac{(t-a)^q}{\Gamma(1+q)} \right],$$

then

$$\Gamma(q) v_\varepsilon(t)(t-a)^{1-q}|_{t=a} = \Gamma(q) v(t)(t-a)^{1-q}|_{t=a} - \varepsilon \Gamma(q),$$

therefore $v_\varepsilon(t)(t-a)^{1-q}|_{t=a} < v(t)(t-a)^{1-q}|_{t=a}$. Also $v_\varepsilon(t) < v(t)$, for each $\varepsilon > 0$.

Thus, it follows from (2.3) and the fact that $f(t, u(t))$ is strictly decreasing, that

$$D^q v_\varepsilon(t) = D^q v(t) - \varepsilon < f(t, v(t)) < f(t, v_\varepsilon(t)),$$

and from the result for strict inequalities we have that $v_\varepsilon(t) < w(t)$. Finally, by letting $\varepsilon \to 0$, (2.5) is true. □

As a special case, if $f(t, u) = -Lu$, with $L > 0$, then we have that

$$D^q v(t) \leq -Lv(t),$$

$$\Gamma(q) v(t)(t-a)^{1-q}|_{t=a} \leq v(b),$$

$$D^q w(t) \geq -Lw(t),$$

$$\Gamma(q) w(t)(t-a)^{1-q}|_{t=a} \geq w(b).$$

Therefore $v(t) \leq w(t)$.

This also yields the following result:

Corollary 2.9. *If L, M are positive constants, $m : [a, b] \to \mathbb{R}$ is Hölder continuous and satisfies*

$$D^q m(t) \leq -Lm(t), \text{ where } a \leq t \leq b,$$

$$\Gamma(q)m(t)(t-a)^{1-q}|_{t=a} \leq m(b),$$

then $m(t) \leq 0$ for $a \leq t \leq b$.
Similarly, if

$$D^q m(t) \geq -Mm(t), \text{ where } a \leq t \leq b,$$

$$\Gamma(q)m(t)(t-a)^{1-q}|_{t=a} \geq m(b),$$

then $m(t) \geq 0$ for $a \leq t \leq b$.

In order to prove that the solution of the fractional linear problem with periodic boundary conditions is well defined, we need to prove the following lemma. The next result is to establish a property of the Mittag-Leffler function.

Lemma 2.10. *If $\frac{(b-a)^{q-1}}{\Gamma(q)} < 1$ and $\lambda(b-a)^q \leq 1$, then $(b-a)^{q-1}E_{q,q}[-\lambda(b-a)^q] < 1$.*

Proof. By definition

$$(b-a)^{q-1}E_{q,q}[-\lambda(b-a)^q] = \frac{(b-a)^{q-1}}{\Gamma(q)} + \frac{-\lambda(b-a)^{2q-1}}{\Gamma(2q)} + \\ + \frac{(-\lambda)^2(b-a)^{3q-1}}{\Gamma(3q)} + \cdots + \frac{(-\lambda)^{n-1}(b-a)^{nq-1}}{\Gamma(nq)} + \cdots$$

Note that this is an alternating series. Initially we prove that it is absolutely convergent and, consequently, convergent to a number less than $\frac{(b-a)^{q-1}}{\Gamma(q)} < 1$.

To show that the series is absolutely convergent we will use the ratio test, i.e., we will evaluate

$$\lim_{n\to\infty} \frac{\lambda(b-a)^q \Gamma(nq)}{\Gamma((n+1)q)}.$$

We will do it by using the following definition of Γ,

$$\Gamma(x) = \frac{1}{x} \prod_{m=1}^{\infty} \frac{(1+\frac{1}{m})^x}{(1+\frac{x}{m})}.$$

Thus for each $n \geq 1$

$$\begin{aligned}
\frac{\Gamma((n+1)q)}{\Gamma(nq)} &= \frac{\frac{1}{(n+1)q}\prod_{m=1}^{\infty}\frac{(1+\frac{1}{m})^{(n+1)q}}{(1+\frac{(n+1)q}{m})}}{\frac{1}{nq}\prod_{m=1}^{\infty}\frac{(1+\frac{1}{m})^{nq}}{(1+\frac{nq}{m})}} = \frac{n}{n+1}\prod_{m=1}^{\infty}(1+\frac{1}{m})^q \frac{1+\frac{nq}{m}}{1+\frac{(n+1)q}{m}} = \\
&= \frac{n}{n+1}\prod_{m=1}^{\infty}(1+\frac{1}{m})^q \frac{m+nq}{m+(n+1)q} > \\
&> \frac{n}{n+1}\prod_{m=1}^{\infty}(1+\frac{1}{m})^q \left(\frac{n}{n+1}\right) = \frac{n^2}{(n+1)^2}\prod_{m=1}^{\infty}(1+\frac{1}{m})^q = \\
&= \frac{n^2}{(n+1)^2}\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\frac{(n+1)^2+1}{(n+1)^2}\prod_{m=(n+1)^2+1}^{\infty}(1+\frac{1}{m})^q = \\
&= \frac{n^2}{(n+1)^2}((n+1)^2+1)\prod_{m=(n+1)^2+1}^{\infty}(1+\frac{1}{m})^q > \\
&> n^2\prod_{m=(n+1)^2+1}^{\infty}(1+\frac{1}{m})^q > 1 \geq \lambda(b-a)^q.
\end{aligned}$$

Note that the last result implies that $\frac{\lambda(b-a)^q\Gamma(nq)}{\Gamma((n+1)q)} \to 0$ as $n \to \infty$. Thus, by the ratio test, the series is absolutely convergent. Furthermore the ratio test shows that the absolute values are decreasing for each $n \geq 1$, therefore the series converges to a number less than $\frac{(b-a)^{q-1}}{\Gamma(q)} < 1$ and

$$(b-a)^{q-1}E_{q,q}[-\lambda(b-a)^q] < 1.$$

□

Observe that in the above result we have obtained a set of conditions on the length $b-a$ of the interval. However it is feasible to obtain other conditions also.

The next lemma is related to the Mittag-Leffler function and shows that the solution of the linear PBVP exists.

Lemma 2.11. *Consider the linear Periodic Boundary Value Problem (PBVP),*

$$\begin{aligned} D^q x(t) &= -\lambda x(t) + f(t), \\ \Gamma(q)x(t-a)^{1-q}|_{t=a} &= x(b) = x_0, \end{aligned} \tag{2.6}$$

where $\lambda > 0$ *and* $f \in C_{p_a}([a,b],\mathbb{R})$. *Assume also that* $\frac{(b-a)^{q-1}}{\Gamma(q)} < 1$ *and* $\lambda(b-a)^q \leq 1$. *Then the solution to* (2.6) *exists and is unique.*

Proof. In order to compute the solution of the PBVP, we consider the solution of the Initial Value Problem (IVP)

$$D^q x(t) = -\lambda x(t) + f(t),$$

$$\Gamma(q)x(t-a)^{1-q}|_{t=a} = x_0.$$

The solution $x(t)$ to this problem can be obtained by induction, see [2] and [4]. It is given as follows,

$$x(t) = x_0(t-a)^{q-1}E_{q,q}[-\lambda(t-a)^q] + \int_a^t (t-s)^{q-1}E_{q,q}[-\lambda(t-s)^q]f(s)ds,$$

then we have that

$$x(b) = x_0(b-a)^{q-1}E_{q,q}[-\lambda(b-a)^q] + \int_a^b (b-s)^{q-1}E_{q,q}[-\lambda(b-s)^q]f(s)ds.$$

Since $x(b) = x_0$,

$$x_0 = x_0(b-a)^{q-1}E_{q,q}[-\lambda(b-a)^q] + \int_a^b (b-s)^{q-1}E_{q,q}[-\lambda(b-s)^q]f(s)ds = x_0.$$

Then,

$$x_0\left(1-(b-a)^{q-1}E_{q,q}[-\lambda(b-a)^q]\right) = \int_a^b (b-s)^{q-1}E_{q,q}[-\lambda(b-s)^q]f(s)ds,$$

therefore,

$$x_0 = \frac{1}{1-(b-a)^{q-1}E_{q,q}[-\lambda(b-a)^q]}\int_a^b (b-s)^{q-1}E_{q,q}[-\lambda(b-s)^q]f(s)ds. \tag{2.7}$$

Finally, the solution to the PBVP is,

$$x(t) = x_0(t-a)^{q-1}E_{q,q}[-\lambda(t-a)^q] + \int_a^t (t-s)^{q-1}E_{q,q}[-\lambda(t-s)^q]f(s)ds, \tag{2.8}$$

where x_0 is given above.

We will prove the uniqueness after Lemma 2.13. □

Remark 2.12. For the PBVP,

$$D^q x(t) = -\lambda x(t) + f(t),$$
$$x_0 = x(a) = \Gamma(q)x(t)(b-t)^{q-1}|_{t=b},$$

one can show that the following expression will give explicit solution which can be proved to be unique on $[a,b]$,

$$x(t) = x_0(b-t)^{q-1}E_{q,q}[-\lambda(b-t)^q] + \int_t^b (s-t)^{q-1}E_{q,q}[-\lambda(s-t)^q]f(s)ds.$$

The following lemma gives some properties of $x(t)$ for problem (2.6).

Lemma 2.13. *Let $x(t)$ be the solution of* (2.6)*, which is given by* (2.8)*, then $x(t) \in C_{p_a}([a,b],\mathbb{R})$ and it is locally Hölder continuous of order q on $(a,b]$.*

Proof. One can see that the solution given by (2.8) is a C_{p_a} continuous function. See [2] for details.

Now we will prove that this solution is locally Hölder continuous of order q. This means that we will prove that there exists a constant K such that for $t_1, t_2 \in (a,b]$ with $t_1 < t_2$,

$$\left|(t_2-a)^{1-q}x(t_2) - (t_1-a)^{1-q}x(t_1)\right| \leq K|t_2-t_1|^q.$$

For this purpose we consider the equivalent Volterra fractional integral Equation

$$x(t) = \frac{x_0(t-a)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}\big[-\lambda x(s) + f(s)\big]ds.$$

Thus,

$$\begin{aligned}\left|(t_2-a)^{1-q}x(t_2) - (t_1-a)^{1-q}x(t_1)\right| &= \\ &= \Bigg|\frac{1}{\Gamma(q)}\int_a^{t_2}(t_2-a)^{1-q}(t_2-s)^{q-1}\big[-\lambda x(s)+f(s)\big]ds - \\ &\quad - \frac{1}{\Gamma(q)}\int_a^{t_1}(t_1-a)^{1-q}(t_1-s)^{q-1}\big[-\lambda x(s)+f(s)\big]ds\Bigg|.\end{aligned}$$

Since $x(t)$ and $f(t)$ are C_{p_a} continuous functions, we will have that $(t-a)^{1-q}[-\lambda x(t)+f(t)]$ is bounded by, say, M_1. Then using this bound we can now write

$$\begin{aligned}\left|(t_2-a)^{1-q}x(t_2)-(t_1-a)^{1-q}x(t_1)\right| &\leq \\ \leq M_1\int_a^{t_1}\Big|&(t_2-a)^{1-q}(t_2-s)^{q-1}(s-a)^{q-1}- \\ &-(t_1-a)^{1-q}(t_1-s)^{q-1}(s-a)^{q-1}\Big|ds+ \\ +M_1\int_{t_1}^{t_2}&(t_2-a)^{1-q}(t_2-s)^{q-1}(s-a)^{q-1}ds.\end{aligned} \tag{2.9}$$

Since $s-a>0$ for s in (t_1,t_2), then $(s-a)^{q-1}$ is bounded by $(t_1-a)^{q-1}$. Thus we can now integrate the second term of (2.9)

$$\begin{aligned}M_1\int_{t_1}^{t_2}(t_2-s)^{q-1}(t_2-a)^{1-q}(s-a)^{q-1}ds &\leq \frac{M_1(t_2-a)^{1-q}}{(t_1-a)^{1-q}}\left[-\frac{(t_2-s)^q}{q}\right]_{t_1}^{t_2}= \\ &=\frac{M_1(t_2-a)^{1-q}(t_2-t_1)^q}{(t_1-a)^{1-q}}.\end{aligned}$$

In order to integrate the first term of (2.9), we observe that

$$\frac{(t_2-a)^{1-q}}{(t_2-s)^{1-q}}-\frac{(t_1-a)^{1-q}}{(t_1-s)^{1-q}}\leq 0.$$

Then,

$$\begin{aligned}&M_1\Big|\int_a^{t_1}[(t_2-a)^{1-q}(t_2-s)^{q-1}(s-a)^{q-1}ds- \\ &\quad-(t_1-a)^{1-q}(t_1-s)^{q-1}(s-a)^{q-1}]ds\Big|\leq \\ &\leq M_1\int_a^{t_1}\left[\frac{(t_1-a)^{1-q}}{(t_1-s)^{1-q}}-\frac{(t_2-a)^{1-q}}{(t_2-s)^{1-q}}\right](s-a)^{q-1}ds= \\ &=M_1\int_a^{t_1}\frac{(t_1-a)^{1-q}(s-a)^{q-1}}{(t_1-s)^{1-q}}ds-M_1\int_a^{t_1}\frac{(t_2-a)^{1-q}(s-a)^{q-1}}{(t_2-s)^{1-q}}ds= \\ &=M_1\int_a^{t_1}\frac{(t_1-a)^{1-q}(s-a)^{q-1}}{(t_1-s)^{1-q}}ds-M_1\int_a^{t_2}\frac{(t_2-a)^{1-q}(s-a)^{q-1}}{(t_2-s)^{1-q}}ds+ \\ &\quad+M_1\int_{t_1}^{t_2}\frac{(t_2-a)^{1-q}(s-a)^{q-1}}{(t_2-s)^{1-q}}ds.\end{aligned}$$

Now letting, $\sigma = s - a$ we obtain

$$= M_1(t_1-a)^{1-q}\int_0^{t_1-a}\frac{\sigma^{q-1}}{(t_1-a-\sigma)^{1-q}}d\sigma -$$

$$- M_1(t_2-a)^{1-q}\int_0^{t_2-a}\frac{\sigma^{q-1}}{(t_2-a-\sigma)^{1-q}}d\sigma +$$

$$+ M_1\int_{t_1}^{t_2}\frac{(t_2-a)^{1-q}(s-a)^{q-1}}{(t_2-s)^{1-q}}ds.$$

Then, letting $\sigma = (t_1-a)u_1$, $\sigma = (t_2-a)u_2$ and integrating the third term,

$$\leq M_1(t_1-a)^{1-q}(t_1-a)^{2q-1}\int_0^1 u_1^{q-1}(1-u_1)^{q-1}du_1 -$$

$$- M_1(t_2-a)^{1-q}(t_2-a)^{2q-1}\int_0^1 u_2^{q-1}(1-u_2)^{q-1}du_2 +$$

$$+ \frac{M_1(t_2-a)^{1-q}}{(t_1-a)^{1-q}}\left[-\frac{(t_2-s)^q}{q}\right]_{t_1}^{t_2} =$$

$$= M_1B(q,q)\big[(t_1-a)^q-(t_2-a)^q\big] + \frac{M_1(t_2-a)^{1-q}(t_2-t_1)^q}{(t_1-a)^{1-q}}$$

where B is the Beta function, and since $(t_1-a)^q-(t_2-a)^q < 0$, the last expression is less than

$$\frac{M_1(t_2-a)^{1-q}(t_2-t_1)^q}{(t_1-a)^{1-q}}.$$

Finally, we get that

$$\big|(t_2-a)^{1-q}x(t_2)-(t_1-a)^{1-q}x(t_1)\big| \leq K\big|t_2-t_1\big|^q,$$

for some constant K. Therefore $x(t)$ is locally Hölder continuous. □

Remark 2.14. The result of Lemma 2.13 is also valid if $f(t)$ is the sum of a C_{p_a} continuous function and a continuous function.

Remark 2.15. It is now easy to see that the solution of (2.6) given by (2.8) is unique by using Corollary 2.9 and Lemma 2.13.

For that purpose let $x_1(t)$ and $x_2(t)$ be any two solutions of (2.6). Then

$$D^q[x_1(t)-x_2(t)] = -\lambda[x_1(t)-x_2(t)],$$
$$\Gamma(q)\,[x_1(t)-x_2(t)]\,(t-a)^{1-q}|_{t=a} = x_1(b)-x_2(b).$$

Using Corollary 2.9, we have that $x_1(t) \leq x_2(t)$. Similarly, using (2.9) we have that $x_2(t) \leq x_1(t)$. Therefore $x_1(t) = x_2(t)$.

3. MONOTONE ITERATIVE TECHNIQUE

In this section, we develop the monotone method for the nonlinear fractional differential equation with periodic boundary conditions of the form (3.1) which is given below, by using upper and lower solutions of (3.1). Next we state our main result related to the corresponding nonlinear fractional differential equation with periodic boundary conditions. Consider the PBVP

$$\begin{aligned} D^q u(t) &= f(t, u(t)), \\ u(t-a)^{1-q}|_{t=a} &= u(b) = \frac{x_0}{\Gamma(q)}, \end{aligned} \tag{3.1}$$

where f is continuous.

Theorem 3.1. *Assume that:*

(i) $v_0, w_0 : [a,b] \to \mathbb{R}$ *are* C_{p_a} *continuous on* $[a,b]$ *such that*

$$D^q v_0 \le f(t, v_0(t)),$$

$$v_0(t)(t-a)^{1-q}|_{t=a} \le v_0(b),$$

$$D^q w_0 \ge f(t, w_0(t)),$$

$$w_0(t)(t-a)^{1-q}|_{t=a} \ge w_0(b),$$

$$v_0(t) \le w_0(t) \text{ for } t \in [a,b], \ \frac{(b-a)^{q-1}}{\Gamma(q)} < 1 \text{ and } \lambda(b-a)^q \le 1.$$

(ii) *there exists* $M > 0$ *such that* $f(t,u(t)) - f(t,\xi(t)) \ge -M(u(t)-\xi(t))$, *for* $t \in (a,b]$ *and* $v_0(t) \le \xi(t) \le u(t) \le w_0(t)$.

Then there exist monotone sequences $\{v_n\}, \{w_n\}$ *such that* $v_n \to v, w_n \to w$ *as* $n \to \infty$ *uniformly on* $[a,b]$ *and* v, w *are extremal solutions of the PBVP* (3.1) *in the weighted norm sense. That is if* $u(t)$ *is any solution of the periodic boundary value problem* (3.1) *such that* $v_0(t) \le u \le w_0(t)$ *, then* $v \le u \le w$.

Proof. From (3.1) we have the equivalent Volterra fractional integral equation given by,

$$u(t) = \frac{x_0(t-a)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s, u(s)) ds.$$

Now define the sequences

$$\begin{aligned} D^q v_n &= f(t, v_{n-1}(t)) - M(v_n - v_{n-1}), \\ v_n(t-a)^{1-q}|_{t=a} &= v_n(b) = \frac{v^{0,n}}{\Gamma(q)}, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} D^q w_n &= f(t, w_{n-1}(t)) - M(w_n - w_{n-1}), \\ w_n(t-a)^{1-q}|_{t=a} &= w_n(b) = \frac{w^{0,n}}{\Gamma(q)}. \end{aligned} \tag{3.3}$$

Then we have that

$$v_n(t) = \frac{v^{0,n}(t-a)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1}[f(s, v_{n-1}(s)) - M(v_n(s) - v_{n-1}(s))]ds,$$

and

$$w_n(t) = \frac{w^{0,n}(t-a)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1}[f(s, w_{n-1}(s)) - M(w_n(s) - w_{n-1}(s))]ds,$$

where $v^{0,n}, w^{0,n}$ can be computed like x_0 in (2.7) and v_n, w_n are unique by Lemma 2.11.

Apply Lemma 2.13 to (3.2) and (3.3) for $n = 1$ and it follows that v_1 and w_1 are C_{p_a} continuous. Proceeding inductively, we have that v_n and w_n are C_{p_a} continuous functions.

Also, it follows inductively from Lemma 2.13 that v_n and w_n are locally Hölder continuous.

Now define the mapping A by $v_1 = Av_0$, where v_1 is the unique solution of (3.2) and v_0 is a lower solution of (3.1). Also set $p(t) = v_0(t) - v_1(t)$, then from (i) and (3.2), we get that

$$D^q p = D^q v_0 - D^q v_1 \leq -M(v_0 - v_1) = -Mp,$$

$$p(t)(t-a)^{1-q}|_{t=a} = p(b), \quad a \leq t \leq b.$$

From Corollary 2.9 $p(t) \leq 0$ on $(a, b]$ and $p(t)(t-a)^{1-q}|_{t=a} \leq 0$, which means that $v_0 \leq Av_0$. Thus $v_0 \leq v_1$ on $(a, b]$ and $v_0(t)(t-a)^{1-q}|_{t=a} \leq v_1(t)(t-a)^{1-q}|_{t=a}$.

By a similar argument, if $w_1 = Aw_0$, where w_0 is an upper solution of (3.1), it can be shown that $w_0 \geq Aw_0 = w_1$ on $(a, b]$ and $w_0(t)(t-a)^{1-q}|_{t=a} \leq w_1(t)(t-a)^{1-q}|_{t=a}$.

Let η and μ be any two solutions such that $v_0 \leq \eta \leq \mu \leq w_0$. Assume that $u_1 = A\eta, u_2 = A\mu$.

Letting $p(t) = u_1(t) - u_2(t)$ and using assumption (ii), we have that

$$\begin{aligned} D^q p &= D^q u_1 - D^q u_2 = \\ &= f(t, \eta) - M(u_1 - \eta) - f(t, u) + M(u_2 - \mu) \leq \\ &\leq M(\mu - \eta) - M(u_1 - \eta) + M(u_2 - \mu) = \\ &= -M(u_1 - u_2) = -Mp \end{aligned}$$

with $p(t)(t-a)^{1-q}|_{t=a} = p(b)$. Hence $p(t) \leq 0$ and $u_1(t) \leq u_2(t)$ on $(a, b]$, $u_1(t)(t-a)^{1-q}|_{t=a} \leq u_2(t)(t-a)^{1-q}|_{t=a}$. This proves that A is monotone.

Define the sequences $\{v_n\}, \{w_n\}$ such that $v_n = Av_{n-1}, w_n = Aw_{n-1}$. Note that since $v_0 \leq Av_0 = v_1$ and $w_0 \geq Aw_0 = w_1$, by monotonicity of A we have that $v_0 \leq v_1 \leq w_1 \leq w_0$. Repeating the process we have that $v_2 \leq w_2$, and $v_0 \leq v_1 \leq v_2 \leq w_2 \leq w_1 \leq w_0$ Proceeding inductively it follows that $v_n \leq u \leq w_n$, then

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq w_n \leq \cdots w_1 \leq w_0,$$

on $(a, b]$, where $(t-a)^{1-q}v_0(t), (t-a)^{1-q}w_0(t)$ are bounded on $[a, b]$ because they are C_{p_a} continuous functions.

Also by $C_{p_a}-$continuity of $\{v_n\}, \{w_n\}$, we have that

$$\begin{aligned}(t-a)^{1-q}v_0(t) &\le (t-a)^{1-q}v_1(t) \le \cdots \le (t-a)^{1-q}v_n(t) \le \\ &\le (t-a)^{1-q}w_n(t) \le \cdots \le (t-a)^{1-q}w_1(t) \le (t-a)^{1-q}w_0(t),\end{aligned}$$

where

$$(t-a)^{1-q}v_n(t) = \frac{v^{0,n}}{\Gamma(q)} + \frac{(t-a)^{1-q}}{\Gamma(q)} \int_a^t (t-s)^{q-1}[f(s, v_{n-1}(s)) - M(v_n(s) - v_{n-1}(s))]ds,$$

and

$$(t-a)^{1-q}w_n(t) = \frac{w^{0,n}}{\Gamma(q)} + \frac{(t-a)^{1-q}}{\Gamma(q)} \int_a^t (t-s)^{q-1}[f(s, w_{n-1}(s)) - M(w_n(s) - w_{n-1}(s))]ds.$$

Now we show that $\{(t-a)^{1-q}v_n(t)\}$ and $\{(t-a)^{1-q}w_n(t)\}$ are uniformly bounded and equicontinuous.

First we show uniform boundedness. By hypothesis, both $(t-a)^{1-q}v_0(t)$, $(t-a)^{1-q}w_0(t)$ are bounded on $[a, b]$, then there exists $\bar{M}$ such that for any $t \in [a, b]$, $|(t-a)^{1-q}v_0(t)| \le \bar{M}$ and $|(t-a)^{1-q}w_0(t)| \le \bar{M}$. Since $v_0(t) \le v_n(t) \le w_0(t)$, it follows that

$$(t-a)^{1-q}v_0(t) \le (t-a)^{1-q}v_n(t) \le (t-a)^{1-q}w_0(t),$$

thus

$$0 \le (t-a)^{1-q}v_n(t) - (t-a)^{1-q}v_0(t) \le (t-a)^{1-q}(w_0(t) - v_0(t)),$$

and consequently, $\{(t-a)^{1-q}v_n(t)\}$ is uniformly bounded. By a similar argument $\{(t-a)^{1-q}w_n(t)\}$ is also uniformly bounded.

To prove that $\{(t-a)^{1-q}v_n(t)\}$ is equicontinuous, let $a \le t_1 \le t_2 \le b$. We consider two cases.

Case 1. $t_1 = a$.

Since $t_1 = a$,

$$v_n(t_1)(t_1-a)^{1-q}|_{t_1=a} = \frac{v^{0,n}}{\Gamma(q)},$$

and

$$\begin{aligned}(t_2-a)^{1-q}v_n(t_2) = \frac{v^{0,n}}{\Gamma(q)} + \frac{(t_2-a)^{1-q}}{\Gamma(q)} \int_a^{t_2} (t_2-s)^{q-1}\Big[f(s, v_{n-1}(s)) - \\ - M(v_n(s) - v_{n-1}(s))\Big]ds.\end{aligned}$$

Then,

$$|(t_2-a)^{1-q}v_n(t_2)-(t_1-a)^{1-q}v_n(t_1)| =$$
$$=\left|\frac{(t_2-a)^{1-q}}{\Gamma(q)}\int_a^{t_2}(t_2-s)^{q-1}(s-a)^{q-1}(s-a)^{1-q}[f(s,v_{n-1}(s))-M(v_n(s)-v_{n-1}(s))]ds\right|,$$

Since f is continuous on $[a,b]$ and $\{(t-a)^{1-q}v_n(t)\}$ are uniformly bounded, there exists $\bar{M}$ such that

$$=\left|\frac{(t_2-a)^{1-q}}{\Gamma(q)}\int_a^{t_2}(t_2-s)^{q-1}(s-a)^{q-1}(s-a)^{1-q}[f(s,v_{n-1}(s))-M(v_n(s)-v_{n-1}(s))]ds\right|\leq$$
$$\leq \bar{M}(t_2-a)^{1-q}\int_a^{t_2}(t_2-s)^{q-1}(s-a)^{q-1}ds.$$

Now, let $\sigma = s-a$, then the last expression becomes

$$\bar{M}(t_2-a)^{1-q}\int_0^{t_2-a}\frac{\sigma^{q-1}}{(t_2-a-\sigma)^{1-q}}d\sigma.$$

Next, let $\sigma=(t_2-a)u$, and the last expression gives us

$$\bar{M}(t_2-a)^{1-q}(t_2-a)^{q-1+q}\int_0^1 u^{q-1}(1-u)^{q-1}du = \bar{M}(t_2-a)^qB(q,q),$$

where B is the Beta function.

Therefore since $t_1=a$, for any $\varepsilon>0$ there exists $\delta>0$ such that for each n,

$$|(t_2-a)^{1-q}v_n(t_2)-(t_1-a)^{1-q}v_n(t_1)|<\varepsilon$$

provided that $|t_2-t_1|<\delta$.

Case 2. $t_1>a$.

In fact,

$$|(t_2-a)^{1-q}v_n(t_2)-(t_1-a)^{1-q}v_n(t_1)| =$$
$$=\left|\frac{1}{\Gamma(q)}\int_a^{t_2}(t_2-a)^{1-q}(t_2-s)^{q-1}\big[-M\big(v_n(s)-v_{n-1}(s)\big)+f(s,v_{n-1}(s))\big]ds-\right.$$
$$\left.-\frac{1}{\Gamma(q)}\int_a^{t_1}(t_1-a)^{1-q}(t_1-s)^{q-1}\big[-M\big(v_n(s)-v_{n-1}(s)\big)+f(s,v_{n-1}(s))\big]ds\right|.$$

Since $\{(t-a)^{1-q}v_n(t)\}$ are uniformly bounded and $f(t)$ is continuous on $[a,b]$, we will have that $(t-a)^{1-q}[f(t,v_{n-1}(t)) - M(v_n - v_{n-1})]$ is bounded by some constant M_1. Then using this bound we can now write

$$\begin{aligned}
&\big|(t_2-a)^{1-q}v_n(t_2) - (t_1-a)^{1-q}v_n(t_1)\big| \le \\
&\le M_1 \int_a^{t_1} \Big| (t_2-a)^{1-q}(t_2-s)^{q-1}(s-a)^{q-1} - \\
&\qquad - (t_1-a)^{1-q}(t_1-s)^{q-1}(s-a)^{q-1}\Big| ds + \\
&\quad + M_1 \int_{t_1}^{t_2} (t_2-a)^{1-q}(t_2-s)^{q-1}(s-a)^{q-1} ds.
\end{aligned}$$

Proceeding as in Lemma 2.13, we have that

$$\big|(t_2-a)^{1-q}v_n(t_2) - (t_1-a)^{1-q}v_n(t_1)\big| \le \bar{K}\big|t_2-t_1\big|^q,$$

for some constant $\bar{K}$.

Again, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each n,

$$|(t_2-a)^{1-q}v_n(t_2) - (t_1-a)^{1-q}v_n(t_1)| < \varepsilon,$$

provided that $|t_2 - t_1| < \delta$. This proves equicontinuity and by similar argument $\{(t-a)^{1-q}w_n(t)\}$ is equicontinuous.

This proves that $\{v_n(t)\}$ and $\{w_n(t)\}$ are equicontinuous and uniformly bounded on $[a,b]$ in the weighted norm. Hence by Arzela-Ascoli's theorem there exist subsequences $\{v_{n_k}(t)\}$ and $\{w_{n_k}(t)\}$ which converge to $v(t)$ and $w(t)$, respectively. Since the sequences are monotone, the entire sequences converge.

It is easy to observe that $v^{0,n}$ and $w^{0,n}$ converge to v^0 and w^0, respectively, where v^0 and w^0 are given by (2.7) where $f(t)$ is replaced by $f(t,v(t)) + Mv(t)$ and $f(t,w(t)) + Mw(t)$. This proves that $v(t)$ and $w(t)$ are solutions of the periodic boundary value problem (3.1).

It remains to show that $v(t)$ and $w(t)$ are extremal solutions of (3.1).

Assume that for some $k > 0$, $v_{k-1} \le u \le w_{k-1}$ on $[a,b]$ where u is a solution to (2.6) such that $v_0 \le u \le w_0$. Then setting $p = v_k - u$ we get from condition (ii) that

$$\begin{aligned}
D^q p = D^q v_k - D^q u &= f(t,v_{k-1}) - M(v_k - v_{k-1}) - f(t,u) \le \\
&\le M(u - v_{k-1}) - M(v_k - v_{k-1}) = -Mp
\end{aligned}$$

and $p(t)(t-a)^{1-q}|_{t=a} = p(b)$.

By Corollary 2.9, $p(t) \le 0$ on $a \le t \le b$, hence $v_k \le u$. By a similar argument $w_k \ge u$ on $[a,b]$.

Since $v_0 \le u \le w_0$, it follows by induction that $v_n \le u \le w_n$ on $[a,b]$, for all n. Hence $v \le u \le w$ on $[a,b]$, which shows that v and w are minimal and maximal solutions of (3.1), respectively. This completes the proof. □

Remark 3.2. We can develop the above result for the fractional differential equation with periodic boundary conditions (2.2) on similar lines.

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