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ON THE GLOBAL OFFENSIVE ALLIANCE NUMBER OF A TREE

Abstract. For a graph $G = (V, E)$, a set $S \subseteq V$ is a dominating set if every vertex in $V - S$ has at least a neighbor in S. A dominating set S is a global offensive alliance if for every vertex v in $V - S$, at least half of the vertices in its closed neighborhood are in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G and the global offensive alliance number $\gamma_o(G)$ is the minimum cardinality of a global offensive alliance of G. We first show that every tree of order at least three with ℓ leaves and s support vertices satisfies $\gamma_o(T) \geq (n - \ell + s + 1)/3$ and we characterize extremal trees attaining this lower bound. Then we give a constructive characterization of trees with equal domination and global offensive alliance numbers.

Keywords: global offensive alliance number, domination number, trees.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Let $G = (V, E)$ be a finite and simple graph of order n. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhoods* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of v, denoted by $\deg_G(v)$, is the size of its open neighborhood. A vertex of degree one is called a *pendent vertex* or a *leaf* and its neighbor is called a *support* vertex. If v is a support vertex, then L_v will denote the set of the leaves attached at v. We also denote the set of leaves of a graph G by $L(G)$, the set of support vertices by $S(G)$, and let $|L(G)| = \ell$, $|S(G)| = s$. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A *subdivided star* SS_q is obtained from a star $K_{1,q}$ by subdividing each edge by exactly one vertex.

For a graph $G = (V, E)$, a set S is a *dominating set* if every vertex in $V - S$ has at least a neighbor in S. A dominating set S is called a *global offensive alliance* if for every $v \in V - S$, $|N[v] \cap S| \geq |N[v] - S|$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and the *global offensive alliance number* $\gamma_o(G)$ is the minimum cardinality of a global offensive alliance of G. Clearly for every graph G,

$\gamma_o(G) \geq \gamma(G)$. Every graph has a global offensive alliance, since $S = V$ is such a set. We abbreviate global offensive alliance as *goa*. If S is a goa of G and $|S| = \gamma_o(G)$, then we say that S is a $\gamma_o(G)$-set. Alliances in graphs were introduced by Hedetniemi, Hedetniemi, and Kristiansen in [5]. For the study of offensive alliances we cite for example [1] and [2]. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [3, 4]. In this paper, we show that every tree of order at least three with ℓ leaves and s support vertices satisfies $\gamma_o(T) \geq (n - \ell + s + 1)/3$ and we characterize extremal trees attaining this lower bound. We also give a constructive characterization of trees with equal domination and global offensive alliance numbers.

2. LOWER BOUND

We begin with a couple of observations.

Observation 2.1. *If G is a connected graph of order at least three, then there is a $\gamma_o(G)$-set that contains all the support vertices.*

Observation 2.2. *Let T be a tree obtained from a nontrivial tree T' by attaching a star $K_{1,t}$ of center x with an edge xz at a support vertex z of T'. Then $\gamma_o(T) = \gamma_o(T') + 1$ and $\gamma(T) = \gamma(T') + 1$.*

Proof. By Observation 2.1 there is a $\gamma_o(T)$-set D that contains all the support vertices. Hence $x, z \in D$; so $D - \{x\}$ is a goa of T' and $\gamma_o(T') \leq \gamma_o(T) - 1$. Since every $\gamma_o(T')$-set can be extended to a goa of T by adding x, $\gamma_o(T) \leq \gamma_o(T') + 1$. It follows that $\gamma_o(T) = \gamma_o(T') + 1$. If D' is any $\gamma(T')$-set, then $D' \cup \{x\}$ is a dominating set of T, implying that $\gamma(T) \leq \gamma(T') + 1$. The equality comes by the fact that x, z belong to some $\gamma(T)$-set, and such a set minus x is a dominating set of T'. □

Let $\mathcal{F}$ be the family of trees of order at least three that can be obtained from r disjoint stars by first adding $r - 1$ edges so that they are incident only with centers of the stars and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 2.3. *Let T be a tree of order $n \geq 3$ with ℓ leaves and s support vertices. Then $\gamma_o(T) \geq (n - \ell + s + 1)/3$ with equality if and only if $T \in \mathcal{F}$.*

Proof. Let $T \in \mathcal{F}$. Then T contains $|S(T)| - 1$ vertices of degree two and the remaining vertices are leaves and support vertices. It follows that $n = \ell + 2s - 1$ and so $s = (n - \ell + s + 1)/3$. Now it is clear that every $\gamma_o(T)$-set contains at least $|S(T)|$ vertices and so $\gamma_o(T) \geq |S(T)|$. The equality follows from the fact that $S(T)$ is a global offensive alliance of T, implying that $\gamma_o(T) = |S(T)| = (n - \ell + s + 1)/3$.

To prove that if T is a tree of order $n \geq 3$, then $\gamma_o(T) \geq (n - \ell + s + 1)/3$ with equality only if $T \in \mathcal{F}$, we use an induction on the order n. If $\mathrm{diam}(T) = 2$, then T is a star with $\gamma_o(T) = 1 = (n - \ell + s + 1)/3$ and so $T \in \mathcal{F}$. If $\mathrm{diam}(T) = 3$, then $\gamma_o(T) = 2 > (n - \ell + s + 1)/3$. Assume that every tree T' of order n', $3 \leq n' < n$, with ℓ' leaves and s' support vertices satisfies $\gamma_o(T') \geq (n' - \ell' + s' + 1)/3$ with equality if and only if $T \in \mathcal{F}$. Let T be a tree of order n and diameter at least four having ℓ leaves and s support vertices.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T) \geqslant 4$. Let u be a support vertex at maximum distance from r, v be the parent of u, and w be the parent of v in the rooted tree. Note that $\deg_T(w) \geq 2$ and let D be a $\gamma_o(T)$-set that contains no leaves. Denote by T_x the subtree induced by a vertex x and its descendants in the rooted tree T. We distinguish between three cases.

Case 1. v is a support vertex. Let $T' = T - L_u \cup \{u\}$. Then $n' = n - 1 - |L_u| \geq 3$, $\ell' = \ell - |L_u|$ and $s' = s - 1$. By Observation 2.2, $\gamma_o(T) = \gamma_o(T') + 1$ and by induction on T' we obtain $\gamma_o(T) > (n - \ell + s + 1)/3$.

Case 2. $\deg_T(v) \geq 3$ and v is not a support vertex. Thus every child of v is a support vertex. Let k be the number of children of v and B the set of leaves in T_v. We first assume that $\deg_T(w) \geq 3$ and let $T' = T - T_v$. Then $n' = n - |B| - k - 1 \geq 3$, $\ell' = \ell - |B|$ and $s' = s - k$. Since D contains all children of v and does not contain v (else replace it by w), $D \cap V(T')$ is a goa of T'. It follows $\gamma_o(T') \leq \gamma_o(T) - k$ and by induction on T' we have

$$\gamma_o(T) \geq (n' - \ell' + s' + 1)/3 + k \geq (n - \ell + s + 1 + k - 1)/3$$

and therefore $\gamma_o(T) > (n - \ell + s + 1)/3$ since $k \geq 2$.

Now assume that $\deg_T(w) = 2$. Let $T' = T - (T_v - \{v\})$. Then $n' = n - |B| - k \geq 3$, $\ell' = \ell - |B| + 1$ and $s' = s - k + 1$. Clearly D contains all children of v and does not contain v (else replace it by w), and so D must contain w for otherwise w would have one neighbor in D and itself and v not in D. Thus $D \cap V(T')$ is goa of T' and hence $\gamma_o(T') \leq \gamma_o(T) - k$. By induction on T' we have

$$\gamma_o(T) \geq (n' - \ell' + s' + 1)/3 + k \geq (n - \ell + s + 1 + k)/3$$

and therefore $\gamma_o(T) > (n - \ell + s + 1)/3$.

Case 3. $\deg_T(v) = 2$. Then $u, w \in D$ and $v \notin D$. Assume that $\deg_T(w) = 2$ or $\deg_T(w) \geq 3$ and w is not a support vertex. Let $T' = T - L_u \cup \{u\}$. Then $D \cap V(T')$ is a goa of T' and so $\gamma_o(T') \leq \gamma_o(T) - 1$. Using the induction on T' and since $n' = n - 1 - |L_u| \geq 3$, $\ell' = \ell - |L_u| + 1$ and $s' = s$, we obtain

$$\gamma_o(T) \geq (n' - \ell' + s' + 1)/3 + 1 > (n - \ell + s + 1)/3.$$

We finally assume that $\deg_T(w) \geq 3$ and w is a support vertex. Let $T' = T - L_u \cup \{u, v\}$. Then $D \cap V(T')$ is a goa of T', $n' = n - 2 - |L_u| \geq 3, \ell' = \ell - |L_u|$ and $s' = s - 1$. Hence by induction on T', we have

$$\gamma_o(T) \geq \gamma_o(T') + 1 \geq (n' - \ell' + s' + 1)/3 + 1 \geq (n - \ell + s + 1)/3.$$

Further, if $\gamma_o(T) \geq (n - \ell + s + 1)/3$, then we have equality throughout this inequality chain. In particular, $\gamma_o(T') = (n' - \ell' + s' + 1)/3$. Thus by the inductive hypothesis on T', $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$. □

3. TREES T WITH $\gamma_o(T) = \gamma(T)$

Observation 3.1. *Let T be a tree obtained from a nontrivial tree T' by attaching a subdivided star SS_k, $k \geq 2$, of center x with an edge edge xy at a vertex y of T'. Then:*

(a) $\gamma_o(T') \leq \gamma_o(T) - k$, *with equality if y belongs to some $\gamma_o(T')$-set or a strict majority of its closed neighborhood belong to some $\gamma_o(T')$-set.*

(b) $\gamma(T) = \gamma(T') + k$.

Proof. (a) By Observation 2.1 there is a $\gamma_o(T)$-set S that contains all support vertices of the added subdivided star. Also we may assume that $x \notin S$ (else replace it by y). Thus $S \cap V(T')$ is a goa of T', and so $\gamma_o(T') \leq \gamma_o(T) - k$. Now if y belongs to some $\gamma_o(T')$-set or a strict majority of its closed neighborhood belong to some $\gamma_o(T')$-set, then such sets can be extended to a goa of T by adding the set of support vertices of SS_k. It follows that $\gamma_o(T) \leq \gamma_o(T') + k$ and the equality holds.

Item (b) is easy to show. □

In order to characterize trees with equal domination and global offensive alliance numbers we define the family $\mathcal{F}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ ($k \geq 1$) of trees, where $T_1 = P_2$, $T = T_k$, and, if $k \geq 2$, T_{i+1} is obtained recursively from T_i by one of the four operations defined below. Let one the vertices of T_1 be considered a support and the other a leaf.

— **Operation $\mathcal{O}_1$**: Attach a vertex by joining it to any support vertex of T_i.
— **Operation $\mathcal{O}_2$**: Attach a path $P_2 = xy$ by joining x to any support vertex z of T_i.
— **Operation $\mathcal{O}_3$**: Attach a subdivided star SS_k, $k \geq 2$, of center u by joining u to vertex v of T_i with the condition that if v does not belong to a $\gamma_o(T_i)$-set D, then a strict majority of $N_{T_i}[v]$ are in D.
— **Operation $\mathcal{O}_4$**: Attach a path $P_3 = xyz$ by joining x to any vertex of T_i that belongs to a $\gamma_o(T_i)$-set.

Lemma 3.2. *If $T \in \mathcal{F}$, then $\gamma_o(T) = \gamma(T)$.*

Proof. We use induction on the number of operations k performed to construct T. The property is true for $T_1 = P_2$. Suppose the property is true for all trees of $\mathcal{F}$ constructed with $k - 1 \geq 0$ operations. Let $T = T_k$ with $k \geq 2$, $T' = T_{k-1}$, and let D be a $\gamma_o(T)$-set that contains no leaf of T. We examine the following cases.

Clearly if T was obtained from T' by Operation $\mathcal{O}_1$, then $\gamma_o(T') = \gamma_o(T)$, $\gamma(T') = \gamma(T)$ and so $\gamma_o(T) = \gamma(T)$.

If T was obtained from T' by Operation $\mathcal{O}_2$, then by Observation 2.2, $\gamma_o(T) = \gamma_o(T') + 1$ and $\gamma(T) = \gamma(T') + 1$. Using the induction on T' it follows that $\gamma_o(T) = \gamma(T)$.

If T was obtained from T' by Operation $\mathcal{O}_3$, then by Observation 3.1 $\gamma_o(T) = \gamma_o(T') + k$ and $\gamma(T) = \gamma(T') + k$. By induction on T', we obtain $\gamma_o(T) = \gamma(T)$.

Finally assume that T was obtained from T' by Operation $\mathcal{O}_4$. Let $w \in V(T')$ be the neighbor of x. Then $y \in D$, and $x \notin D$ (else replace it by w). Thus $D \cap V(T')$ is a goa of T' and we have $\gamma_o(T') \leq \gamma_o(T) - 1$. Now since w belongs to a $\gamma_o(T')$-set, such a set can be extended to goa of T by adding y; so $\gamma_o(T) \leq \gamma_o(T') + 1$ and the equality follows. Also it can be seen easily that, $\gamma(T) = \gamma(T') + 1$. By induction on T', we obtain the desired result. □

Theorem 3.3. *Let T be a tree. Then $\gamma_o(T) = \gamma(T)$ if and only if $T = K_1$ or $T \in \mathcal{F}$.*

Proof. Clearly if $T = K_1$, then $\gamma_o(T) = \gamma(T)$. If $T \in \mathcal{F}$, then by Lemma 3.2, $\gamma_o(T) = \gamma(T)$. Now to prove the converse we use an induction on the order n of T. It is obvious that $\gamma_o(K_1) = \gamma(K_1)$. Let us assume that $n \geq 2$. If $n = 2$, then $T = P_2$ and T belongs to $\mathcal{F}$. If $n = 3$, then $T = P_3$ that belongs to $\mathcal{F}$ since it is obtained from P_2 by using Operation $\mathcal{O}_1$. Assume that every tree T' of order $n' \geq 2$ satisfying $\gamma_o(T') = \gamma(T')$ is in $\mathcal{F}$.

Let T be a tree of order $n > n'$ such that $\gamma_o(T) = \gamma(T)$. If T is a star $K_{1,t}$, then $\gamma_o(T) = \gamma(T)$ and $T \in \mathcal{F}$ because it is obtained from P_2 by using Operation $\mathcal{O}_1$. If T is a double star, then $\gamma_o(T) = \gamma(T)$ and $T \in \mathcal{F}$ because it is obtained from P_2 by using Operations $\mathcal{O}_2$ and $\mathcal{O}_1$. Thus we may assume that T has diameter at least four.

If any support vertex, say x, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to x. Then $\gamma_o(T') = \gamma_o(T)$, $\gamma(T') = \gamma(T)$ and so $\gamma_o(T') = \gamma(T')$. By induction on T', we have $T' \in \mathcal{F}$. Thus $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{O}_1$. Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T) \geqslant 4$. Let v be a support vertex at maximum distance from r, u be the parent of v, and w be the parent of u in the rooted tree. Let v' be the unique leaf adjacent to v. Note that $\deg_T(w) \geq 2$. Let D be a $\gamma_o(T)$-set that contains no leaves. We distinguish between three cases.

Case 1. u is a support vertex. Let $T' = T - \{v, v'\}$. Then by Observation 2.2, $\gamma_o(T) = \gamma_o(T') + 1$ and $\gamma(T) = \gamma(T') + 1$. Thus $\gamma_o(T') = \gamma(T')$ and by induction on T', we have $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$ and is obtained from T' by using Operation $\mathcal{O}_2$.

Case 2. u is not a support vertex but has at least one child besides v as a support vertex. Thus T_v is a subdivided star. Let $T' = T - T_v$. Then by Observation 3.1, $\gamma_o(T') \leq \gamma_o(T) - k$ and $\gamma(T) = \gamma(T') + k$, where k is the number of children of v. Assume now that $\gamma_o(T') < \gamma_o(T) - k$, then

$$\gamma_o(T') < \gamma_o(T) - k = \gamma(T) - k = (\gamma(T') + k) - k = \gamma(T')$$

and so $\gamma_o(T') < \gamma(T')$, a contradiction. Hence $\gamma_o(T') = \gamma_o(T) - k$ and $D' = D \cap V(T')$ is a $\gamma_o(T')$-set. It follows that $\gamma_o(T') = \gamma(T')$. Note that if $w \notin D'$, then since D is a $\gamma_o(T)$-set, $|N_{T'}[w] \cap D'| > |N_{T'}[w] - D'|$. Applying the inductive hypothesis T' belongs to $\mathcal{F}$, and so $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{O}_3$.

Case 3. u has degree two. Let $T' = T - \{v', v, u\}$. It can be seen that $\gamma(T') = \gamma(T) - 1$. Also $v \in D$, $u \notin D$ (else replace it by w), and so $w \in D$. Thus $D \cap V(T')$ is a goa of T' and $\gamma_o(T') \leq \gamma_o(T) - 1$. Now if $\gamma_o(T') < \gamma_o(T) - 1$, then

$$\gamma_o(T') < \gamma_o(T) - 1 = \gamma(T) - 1 = (\gamma(T') + 1) - 1 = \gamma(T')$$

and hence $\gamma_o(T') < \gamma(T')$, a contradiction. Therefore $\gamma_o(T') = \gamma_o(T) - 1$ and so $D \cap V(T')$ is a $\gamma_o(T')$-set containing w. It follows that $\gamma_o(T') = \gamma(T')$, and by the inductive hypothesis $T' \in \mathcal{F}$. Thus $T \in \mathcal{F}$ and is obtained from T' by using Operation $\mathcal{O}_4$. $\square$

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