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STRONG MAXIMUM PRINCIPLES FOR IMPLICIT PARABOLIC FUNCTIONAL-DIFFERENTIAL PROBLEMS TOGETHER WITH NONLOCAL INEQUALITIES WITH FUNCTIONALS

Abstract. The aim of the paper is to give strong maximum principles for implicit parabolic functional-differential problems together with nonlocal inequalities with functionals in relatively arbitrary $(n+1)$-dimensional time-space sets more general than the cylindrical domain.

Keywords: strong maximum principles, implicit parabolic problems, nonlocal inequalities.

Mathematics Subject Classification: 35B50, 35R45, 35K20.

1. INTRODUCTION

In this paper we consider implicit diagonal systems of nonlinear parabolic functional-differential inequalities of the form

$$\begin{aligned} &F_i(t,x,u(t,x),u_t^i(t,x),u_x^i(t,x),u_{xx}^i(t,x),u) \geq \\ &\geq F_i(t,x,v(t,x),v_t^i(t,x),v_x^i(t,x),v_{xx}^i(t,x),v) \quad (i=1,\dots,m) \end{aligned} \tag{1.1}$$

for $(t,x)=(t,x_1,\dots,x_n)\in D$, where $D\subset(t_0,t_0+T]\times\mathbb{R}^n$ is one of five relatively arbitrary sets more general than the cylindrical domain $(t_0,t_0+T]\times D_0\subset\mathbb{R}^{n+1}$. The symbol $w(=u$ or $v)$ denotes the mapping

$$w:\tilde{D}\ni(t,x)\to w(t,x)=(w^1(t,x),\dots,w^m(t,x))\in\mathbb{R}^m,$$

where $\tilde{D}$ is an arbitrary set contained in $(-\infty,t_0+T]\times\mathbb{R}^n$ and such that $\bar{D}\subset\tilde{D}$; $F_i(i=1,\dots,m)$ are functionals in w; $w_x^i(t,x)=grad_x w^i(t,x)$ $(i=1,\dots,m)$ and $w_{xx}^i(t,x)$ $(i=1,\dots,m)$ denote the matrices of second order derivatives of

$w^i(t,x)$ $(i = 1,\ldots,m)$ with respect to x. We give a theorem on strong maximum principles for problems with inequalities (1.1) and together with nonlocal inequalities with functionals.

The results obtained are a generalization of some results given by J. Chabrowski [5], R. Redheffer and W. Walter [7], J. Szarski [8, 9], P. Besala [1], W. Walter [10], N. Yoshida [11], the author [3, 4], and are based on those results.

Nonlinear functional-differential parabolic problems were investigated by S. Brzychczy in monograph [2].

Hyperbolic functional-differential inequalities were considered by Z. Kamont in monograph [6].

2. PRELIMINARIES

The notation, definitions and assumptions given in this section are applied throughout the paper.

We use the following notation: $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{N} = \{1,2,\ldots\}$, $x = (x_1,\ldots,x_n)$ $(n \in \mathbb{N})$.

For any vectors $z = (z_1,\ldots,z_m) \in \mathbb{R}^m, \tilde{z} = (\tilde{z}_1,\ldots,\tilde{z}_m) \in \mathbb{R}^m$, we write $z \leq \tilde{z}$ if $z_i \leq \tilde{z}_i$ $(i = 1,\ldots,m)$.

Let t_0 be a real number and let $0 < T < \infty$. A set $D \subset \{(t,x) : t > t_0, x \in \mathbb{R}^n\}$ (bounded or unbounded) is called a *set of type* (P) if:

(i) The projection of the interior of D on the t-axis is the interval $(t_0, t_0 + T)$.

(ii) For every $(\tilde{t},\tilde{x}) \in D$, there is a positive number r such that

$$\{(t,x) : (t-\tilde{t})^2 + \sum_{i=1}^{n}(x_i - \tilde{x}_i)^2 < r, t < \tilde{t}\} \subset D.$$

(iii) All the boundary points $(\tilde{t},\tilde{x})$ of D for which there is a positive number r such that

$$\{(t,x) : (t-\tilde{t})^2 + \sum_{i=1}^{n}(x_i - \tilde{x}_i)^2 < r, t \leq \tilde{t}\} \subset D$$

belong to D.

For any $t \in [t_0, t_0 + T]$, we define the following sets:

$$S_t = \begin{cases} int\{x \in \mathbb{R}^n : (t_0,x) \in \bar{D}\} & \text{for } t = t_0, \\ \{x \in \mathbb{R}^n : (t,x) \in D\} & \text{for } t \neq t_0 \end{cases}$$

and

$$\sigma_t = \begin{cases} int[\bar{D} \cap (\{t_0\} \times \mathbb{R}^n)] & \text{for } t = t_0, \\ D \cap (\{t\} \times \mathbb{R}^n) & \text{for } t \neq t_0. \end{cases}$$

Let $\tilde{D}$ be such a set that

$$\bar{D} \subset \tilde{D} \subset (-\infty, t_0 + T] \times \mathbb{R}^n$$

and let

$$\partial_p D := \tilde{D} \backslash D \text{ and } \Gamma := \partial_p D \backslash \sigma_{t_0}.$$

For an arbitrary fixed point $(\tilde{t}, \tilde{x}) \in D$, by $S^-(\tilde{t}, \tilde{x})$ we denote the set of points $(t, x) \in D$ that can be joined to $(\tilde{t}, \tilde{x})$ by a polygonal line contained in D along which the t-coordinate is weakly increasing from (t, x) to $(\tilde{t}, \tilde{x})$.

By $Z(\tilde{D}, \mathbb{R}^m)$ we denote the space of mappings

$$w : \tilde{D} \ni (t, x) \to w(t, x) = (w^1(t, x), \ldots, w^m(t, x)) \in \mathbb{R}^m$$

continuous in $\bar{D}$.

In the set of mappings bounded from above in $\tilde{D}$ and belonging to $Z(\tilde{D}, \mathbb{R}^m)$, we define the functional

$$[w]_t = \max_{i=1,\ldots,m} sup\{0, w^i(\tilde{t}, \tilde{x}) : (\tilde{t}, \tilde{x}) \in \tilde{D}, \tilde{t} \leq t\},\ t \leq t_0 + T.$$

By $Z^{1,2}(\tilde{D}, \mathbb{R}^m)$, we denote the space of all functions $w \in Z(\tilde{D}, \mathbb{R}^m)$ such that $w_t^i, w_x^i = grad_x w^i, w_{xx}^i = [w^i_{x_j x_k}]_{n \times n}$ $(i = 1, \ldots, m)$ are continuous in D. Moreover, by $M_{n \times x}(\mathbb{R})$, we denote the space of real square symmetric matrices $r = [r_{jk}]_{n \times n}$.

Let the mappings

$$F_i : D \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times M_{n \times n}(\mathbb{R}) \times Z(\tilde{D}, \mathbb{R}^m) \ni (t, x, z, p, q, r, w) \to$$
$$\to F_i(t, x, z, p, q, r, w) \in \mathbb{R}\ (i = 1, \ldots, m)$$

be given and let for an arbitrary function $w \in Z^{1,2}(\tilde{D}, \mathbb{R}^m)$,

$$F_i[t, x, w] := F_i(t, x, w(t, x), w_t^i(t, x), w_x^i(t, x), w_{xx}^i(t, x), w),$$
$$(t, x) \in D\ (i = 1, \ldots, m).$$

Assumption (W_+). *The mappings F_i $(i = 1, \ldots, m)$ are weakly increasing with respect to $z_1, \ldots, z_{i-1},\ z_{i+1}, \ldots, z_m$ $(i = 1, \ldots, m)$.*
Assumption (L). *There is a constant $L > 0$ such that*

$$F_i(t, x, z, p, q, r, w) - F_i(t, x, \tilde{z}, p, \tilde{q}, \tilde{r}, \tilde{w}) \leq$$
$$\leq L\big(\max_{k=1,\ldots,m} |z_k - \tilde{z}_k| + |x| \sum_{j=1}^{n} |q_j - \tilde{q}_j| + |x|^2 \sum_{j,k=1}^{n} |r_{jk} - \tilde{r}_{jk}| + [w - \tilde{w}]_t\big)$$
$$(i = 1, \ldots, m)$$

for all $(t, x) \in D,\ z = (z_1, \ldots, z_m) \in \mathbb{R}^m,\ \tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_m) \in \mathbb{R}^m,\ p \in \mathbb{R}, q = (q_1, \ldots, q_n) \in \mathbb{R}^n, \tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_n) \in \mathbb{R}^n,\ r = [r_{jk}] \in M_{n \times n}(\mathbb{R}), \tilde{r} = [\tilde{r}_{jk}] \in M_{n \times n}(\mathbb{R})$ and $w, \tilde{w} \in Z^{1,2}(\tilde{D}, \mathbb{R}^m)$, where $\sup_{(t,x) \in \tilde{D}} (w(t, x) - \tilde{w}(t, x)) < \infty$.

Assumption (M). *There are constants* $C_i > 0$ $(i = 1, 2)$ *such that*

$$F_i(t, x, z, p, q, r, w) - F_i(t, x, z, \tilde{p}, q, r, w) < C_1(\tilde{p} - p) \ (i = 1, \dots, m)$$

for all $(t, x) \in D, z \in \mathbb{R}^m, p > \tilde{p}, q \in \mathbb{R}^n, r \in M_{n\times n}(\mathbb{R}), w \in Z^{1,2}(\tilde{D}, \mathbb{R}^m)$
and

$$F_i(t, x, z, p, q, r, w) - F_i(t, x, z, \tilde{p}, q, r, w) < C_2(\tilde{p} - p) \ (i = 1, \dots, m)$$

for all $(t, x) \in D, z \in \mathbb{R}^m, p < \tilde{p}, q \in \mathbb{R}^n, r \in M_{n\times n}(\mathbb{R}), w \in Z^{1,2}(\tilde{D}, \mathbb{R}^m)$.

Each two mappings $u, v \in Z^{1,2}(\tilde{D}, \mathbb{R}^m)$ are said to be *solutions* of the system

$$F_i[t, x, u] \geq F_i[t, x, v] \ \ (i = 1, \dots, m) \tag{2.1}$$

in D if they satisfy (2.1) for all $(t, x) \in D$.

For every set $A \subset \tilde{D}$ and for each function $w \in Z^{1,2}(\tilde{D}, \mathbb{R}^m)$, we use the notation:

$$\max_{(t,x)\in A} w(t, x) := \left(\max_{(t,x)\in A} w^1(t, x), \dots, \max_{(t,x)\in A} w^m(t, x) \right).$$

Let $I = \mathbb{N}$ or I be a finite set of mutually different natural numbers.

Define the set

$$S = \bigcup_{i\in I} (\sigma_{T_{2i-1}} \cup \sigma_{T_{2i}}),$$

where, in the case of $I = \mathbb{N}$, the following conditions are satisfied:

(i) $t_0 < T_{2i-1} < T_{2i} \leq t_0 + T$ for $i \in I$ and
$T_{2i-1} \neq T_{2j-1},\ T_{2i} \neq T_{2j}$ for $i, j \in I,\ i \neq j$,

(ii) $T_0 := inf\{T_{2i-1} : i \in I\} > t_0$,

(iii) $S_t \supset S_{t_0}$ for every $t \in \bigcup_{i\in I} [T_{2i-1}, T_{2i}]$,

(iv) $S_t \supset S_{t_0}$ for every $t \in [T_0, t_0 + T]$,

and, in the case of I being a finite set of mutually different natural numbers, conditions (i),(iii) are satisfied.

An unbounded set D of type (P) is called a *set of type* $(P_{S\Gamma})$ if:

(a) $S \neq \phi$,

(b) $\Gamma \cap \bar{\sigma}_{t_0} \neq \phi$.

Let S_* denote a non-empty subset of S. We define the following set:

$$I_* = \{i \in I : (\sigma_{T_{2i-1}} \cup \sigma_{T_{2i}}) \subset S_*\}.$$

A bounded set D of type (P) satisfying condition (a) of the definition of a set of type $(P_{S\Gamma})$ is called a *set of type* (P_{SB}).

It is easy to see that if D is a set of type (P_{SB}) then D satisfies condition (b) of the definition of a set of type $(P_{S\Gamma})$. Moreover, it is obvious that if D_0 is a bounded subset [D_0 is an unbounded essential subset] of $\mathbb{R}^n$, then $D = (t_0, t_0 + T] \times D_0$ is a set of type (P_{SB}) [$(P_{S\Gamma})$, respectively].

Let

$$Z_i := S_{t_0} \times [T_{2i-1}, T_{2i}] \ \ (i \in I_*).$$

Assumption (N). *The functions* $g_i : S_{t_0} \times C(Z_i, \mathbb{R}) \to \mathbb{R}$ $(i \in I_*)$ *and* $h_i : S_{t_0} \to \mathbb{R}_-$ $(i \in I_*)$ *satisfy the following conditions:*

(N_1) $\quad g_i(x, \xi|_{Z_i}) \le \max_{t \in [T_{2i-1} T_{2i}]} \xi(t, x)$ *for* $x \in S_{t_0}, \xi \in C(\bar{D}, \mathbb{R})$ $(i \in I_*)$

and

(N_2) $\quad -1 \le \sum_{i \in I_*} h_i(x) \le 0$ *for* $x \in S_{t_0}$.

3. STRONG MAXIMUM PRINCIPLES

Theorem 3.1. *Assume that:*

(1) D *is a set of type* $(P_{S\Gamma})$ *or* (P_{SB}).
(2) *The functions* F_i $(i = 1, \ldots, m)$ *satisfy Assumptions* (W_+), (L), (M) *and the functions* g_i, h_i $(i \in I_*)$ *satisfy Assumption* (N).
(3) *The mapping* u *belongs to* $Z^{1,2}(\tilde{D}, \mathbb{R}^m)$ *and the maxima of* u *on* Γ *and* $\tilde{D}$ *are attained. Moreover,*

$$K = (K^1, \ldots, K^m) := \max_{(t,x) \in \Gamma} u(t, x) \tag{3.1}$$

and

$$M = (M^1, \ldots, M^m) := \max_{(t,x) \in \tilde{D}} u(t, x). \tag{3.2}$$

(4) *The mappings* F_i $(i = 1, \ldots, m)$ *are parabolic with respect to* u *in* D *and uniformly parabolic with respect to* M *in any compact subset of* D (*see* [4]). *Moreover,* u *and* $v = M$ *are solution of system* (2.1) *in* D.
(5) *The inequalities*

$$[u^j(t_0, x) - K^j] + \sum_{i \in I_*} h_i(x)[g_i(x, u^j|_{Z_i}) - K^j] \le 0 \tag{3.3}$$

$$\text{for} \quad x \in S_{t_0} \ \ (j = 1, \ldots, m)$$

are satisfied, where the series $\sum_{i \in I_*} h_i(x) g_i(x, u^j|_{Z_i})$ $(j = 1, \ldots, m)$ *are convergent for* $x \in S_{t_0}$ *if card* $I_* = \aleph_0$.

Then

$$\max_{(t,x) \in \tilde{D}} u(t, x) = \max_{(t,x) \in \Gamma} u(t, x). \tag{3.4}$$

Moreover, if there is a point $(\tilde{t}, \tilde{x}) \in D$ *such that* $u(\tilde{t}, \tilde{x}) = \max_{(t,x) \in \tilde{D}} u(t, x)$, *then*

$$u(t, x) = \max_{(t,x) \in \Gamma} u(t, x) \ \ \text{for} \ \ (t, x) \in S^-(\tilde{t}, \tilde{x}).$$

Proof. We shall prove Theorem 3.1 for a set of type $(P_{S\Gamma})$ only, since the proof of this theorem for a set of type (P_{SB}) is analogous.

Since each set of type $(P_{S\Gamma})$ is a set of type (P_{Γ}), from [4] it follows that in the case of $\sum_{i\in I_*} h_i(x) = 0$ for $x \in S_{t_0}$, Theorem 3.1 is a consequence of Theorem 4.1 of [4]. Therefore, we shall prove Theorem 3.1 only in the following case:

$$-1 \leq \sum_{i\in I_*} h_i(x) < 0 \;\text{ for }\; x \in S_{t_0}. \tag{3.5}$$

Assume that (3.5) holds and, since we shall argue by contradiction, suppose

$$M \neq K.$$

From (3.1) and (3.2) there follows that

$$K \leq M.$$

Consequently,

$$K < M. \tag{3.6}$$

Observe that by assumption (3.2)

$$\text{There is }\; (t^*, x^*) \in \tilde{D} \;\text{ such that }\; u(t^*, x^*) = M. \tag{3.7}$$

By (3.7), (3.1) and (3.6),

$$(t^*, x^*) \in \tilde{D}\backslash\Gamma = D \cup \sigma_{t_0}. \tag{3.8}$$

An argument analogous to the proof of Theorem 4.1 of [4] yields

$$(t^*, x^*) \notin D. \tag{3.9}$$

Conditions (3.8) and (3.9) give

$$(t^*, x^*) \in \sigma_{t_0}. \tag{3.10}$$

By the definition of sets I and I_*, we must consider the following cases:

(A) I_* is a finite set, i.e. (without loss of generality), there is a number $p \in \mathbb{N}$ such that $I_* = \{1, \ldots, p\}$.
(B) $card I_* = \aleph_0$.

First we shall consider case (A). And so, since $u \in C(\bar{D}, \mathbb{R}^m)$, it follows that for every $j \in \{1, \ldots, m\}$ and $i \in I_*$ there is $\tilde{T}_i^j \in [T_{2i-1}, T_{2i}]$ such that

$$u^j(\tilde{T}_i^j, x^*) = \max_{t\in[T_{2i-1}, T_{2i}]} u^j(t, x^*). \tag{3.11}$$

Consequently, by (3.3), Assumption (N_1), (3.11) and the inequality

$$u(t, x^*) < u(t_0, x^*) \;\text{ for }\; t \in \bigcup_{i=1}^{p} [T_{2i-1}, T_{2i}]$$

(being a consequence of (3.7), (3.10) and of (a)(i), (a)(iii) in the definition of a set type $(P_{S\Gamma})$), there is

$$\begin{aligned}
0 &\geq [u^i(t_0,x^*) - K^j] + \sum_{i=1}^{p} h_i(x^*)[g_i(x^*, u^j|_{Z_i}) - K^j] \geq \\
&\geq [u^j(t_0,x^*) - K^j] + \sum_{i=1}^{p} h_i(x^*)[u^j(x^*, \tilde{T}_i^j) - K^j] \geq \\
&\geq [u^j(t_0,x^*) - K^j] + \sum_{i=1}^{p} h_i(x^*)[u^j(t_0,x^*) - K^j] = \\
&= [u^j(t_0,x^*) - K^j] \cdot [1 + \sum_{i=1}^{p} h_i(x^*)] \qquad (j = 1,\ldots,m).
\end{aligned}$$

Hence

$$u(t_0,x^*) \leq K \ \text{ if } \ 1 + \sum_{i=1}^{p} h_i(x^*) > 0. \tag{3.12}$$

Then, from (3.6) and (3.10), we obtain contraction (3.12) with (3.7). Assume now that

$$\sum_{i=1}^{p} h_i(x^*) = -1. \tag{3.13}$$

Observe that for every $j \in \{1,\ldots,m\}$ there is a number $\ell_j \in \{1,\ldots,p\}$ such that

$$u^j(\tilde{T}_{l_j}^j, x^*) = \max_{i=1,\ldots,p} u^j(\tilde{T}_i^j, x^*). \tag{3.14}$$

Consequently, by (3.13), (3.14), (3.11), by Assumption (N_1), and by (3.3), we obtain

$$\begin{aligned}
u^i(t_0,x^*) - u^j(\tilde{T}_{l_j}^j, x^*) &= [u^j(t_0,x^*) - K^j] - [u^j(\tilde{T}_{l_j}^j, x^*) - K^j] = \\
&= [u^j(t_0,x^*) - K^j] + \sum_{i=1}^{p} h_i(x^*)[u^j(\tilde{T}_{l_j}^j, x^*) - K^j] \leq \\
&\leq [u^j(t_0,x^*) - K^j] + \sum_{i=1}^{p} h_i(x^*)[u^j(\tilde{T}_i^j, x^*) - K^j] \leq \\
&\leq [u^j(t_0,x^*) - K^j] + \sum_{i=1}^{p} h_i(x^*)[g_i(x^*, u^j|_{Z_i}) - K^j] \leq \\
&\leq 0 \qquad (j = 1,\ldots,m).
\end{aligned}$$

Hence

$$u^i(t_0,x^*) \leq u^j(\tilde{T}_{l_j}^j, x^*) \ (j = 1,\ldots,m) \ \text{ if } \ \sum_{i=1}^{p} h_i(x^*) = -1. \tag{3.15}$$

Since, by (a)(i) of the definition of a set of type $(P_{S\Gamma}), \tilde{T}^j_{l_j} > t_0$ $(j = 1, \ldots, m)$, from (3.10) we get that condition (3.15) contradicts (3.7). This completes the proof of (3.4) in case (A).

It remains to investigate case (B). Analogously as in the proof of (3.4) in case (A), by (3.3), Assumption (N_1), (3.11) and the inequality

$$u(t, x^*) < u(t_0, x^*) \text{ for } t \in \bigcup_{i \in I_*} [T_{2i-1}, T_{2i}]$$

(being a consequence of (3.7), (3.10), and of (a)(i), (a)(iii) of definition of a set of type $(P_{S\Gamma})$), there is

$$\begin{aligned} 0 &\geq [u^j(t_0, x^*) - K^j] + \sum_{i \in I_*} h_i(x^*)[g_i(x^*, u^j|_{Z_i}) - K^j] \geq \\ &\geq [u^j(t_0, x^*) - K^j] + \sum_{i \in I_*} h_i(x^*)[u^j(\tilde{T}^j_i, x^*) - K^j] \geq \\ &\geq [u^j(t_0, x^*) - K^j] + \sum_{i \in I_*} h_i(x^*)[u^j(t_0, x^*) - K^j] = \\ &= [u^j(t_0, x^*) - K^j] \cdot [1 + \sum_{i \in I_*} h_i(x^*)] \qquad (j = 1, \ldots, m). \end{aligned}$$

Hence

$$u(t_0, x^*) \leq K \text{ if } 1 + \sum_{i \in I_*} h_i(x^*) > 0. \tag{3.16}$$

Then, from (3.6) and (3.10), we obtain a contradiction of (3.16) with (3.7). Assume now that

$$\sum_{i \in I_*} h_i(x^*) = -1. \tag{3.17}$$

Let

$$\tilde{T}^j_* = \inf_{i \in I_*} \tilde{T}^j_i \ (j = 1, \ldots, m). \tag{3.18}$$

Since $u \in C(\bar{D}, \mathbb{R}^m)$ and since (by (3.10) and by (a)(iv), (a)(ii) of the definition of a set of type $(P_{S\Gamma})$) $x^* \in S_t$ for every $t \in [T_0, t_0 + T]$ if $I = \mathbb{N}$, from (3.18) it follows that for every $j \in \{1, \ldots, m\}$ there is a number $\hat{t}_j \in [\tilde{T}^j_*, t_0 + T]$ such that

$$u^j(\hat{t}_j, x^*) = \max_{t \in [\tilde{T}^j_*, t_0 + T]} u^j(t, x^*). \tag{3.19}$$

Consequently, by (3.17), (3.19), (3.11), by Assumption (N_1) and Assumption (5), there follows

$$\begin{aligned}u^j(t_0,x^*)-u^j(\hat{t}_j,x^*) &= [u^j(t_0,x^*)-K^j]-[u^j(\hat{t}_j,x^*)-K^j] = \\ &= [u^j(t_0,x^*)-K^j]+\sum_{i\in I_*}h_i(x^*)[u^j(\hat{t}_j,x^*)-K^j] \le \\ &\le [u^j(t_0,x^*)-K^j]+\sum_{i\in I_*}h_i(x^*)[u^j(\tilde{T}_i^j,x^*)-K^j] \le \\ &\le [u^j(t_0,x^*)-K^j]+\sum_{i\in I_*}h_i(x^*)[g_i(x^*,u^j|_{Z_i})-K^j] \le \\ &\le 0 \qquad (j=1,\ldots,m).\end{aligned}$$

Hence

$$u^j(t_0,x^*)\le u^j(\hat{t}_j,x^*) \ \ (j=1,\ldots,m) \ \text{ if } \sum_{i\in I_*}h_i(x^*)=-1. \tag{3.20}$$

Since, by (a)(ii) of the definition of a set of type $(P_{S\Gamma}), \hat{t}_j>t_0$ $(j=1,\ldots,m)$, from (3.10) we get that condition (3.20) contradicts condition (3.7). This completes the proof of equality (3.4).

The second part of Theorem 3.1 is a consequence of equality (3.4) and Lemma 3.1 from [4]. Therefore, the proof of Theorem 3.1 is complete. □

4. REMARKS

Remark 4.1. *It is easy to see that the functionals*

$$g_i: S_{t_0}\times C(Z_i,\mathbb{R})\to\mathbb{R}\ (i\in I_*)$$

given be the formulae

$$g_i(x,w^j|_{Z_i})=w^j(\cdot,x)|_{[T_{2i-1},T_{2i}]},\ x\in S_{t_0},\ w\in C(\bar{D},\mathbb{R}^m) \qquad (i\in I_*,\ j=1,\ldots,m)$$

or

$$g_i(x,w^j|_{Z_i})=\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}}w^j(\tau,x)d\tau,\ x\in S_{t_0},w\in C(\bar{D},\mathbb{R}^m)$$
$$(i\in I_*,\ j=1,\ldots,m)$$

satisfy Assumption (N_1).

Remark 4.2. *It is easy to see, from the proof of Theorem 3.1 hereof and proof of Theorem 4.1 in* [4]*, that if the functions $h_i (i\in I_*)$ from Assumption* (2) *of Theorem 3.1 satisfy the condition*

$$\Big[\sum_{i\in I_*}h_i(x)=0\Big]-1<\sum_{i\in I_*}h_i(x)<0 \ \ \textit{for}\ \ x\in S_{t_0}$$

then it is sufficient in that theorem to assume that [D *is an unbounded set of type* (P) *satisfying condition* (b) *of the definition of a set of type* $(P_{S\Gamma})$ *or* D *is a bounded set of type* (P)*, i.e., according to the terminology applied in* [4]*,* D *is a set of type* (P_Γ) *or* (P_B)*, respectively*] D *is an unbounded set of type* (P) *satisfying conditions* (a)(i), (a)(iii), *and* (b) *of the definition of a set of type* $(P_{S\Gamma})$ *or* D *is a bounded set of type* (P) *satisfying conditions* (a)(i) *and* (a)(iii) *of the definition of a set of type* $(P_{S\Gamma})$. *Moreover, if* I_* *is a finite set and*

$$-1 \leq \sum_{i \in I_*} h_i(x) < 0 \ \ \textit{for} \ \ x \in S_{t_0},$$

then it is sufficient in Theorem 3.1 to assume that D *is an unbounded set of type* (P) *satisfying condition* (a)(i), (a)(iii), *and* (b)*, or* D *is a bounded set of type* (P) *satisfying conditions* (a)(i) *and* (a)(iii).

REFERENCES

[1] P. Besala, *An extension of the strong maximum principle for parabolic equations,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **19** (1971), 1003–1006.

[2] S. Brzychczy, *Monotone iterative methods for nonlinear parabolic and elliptic differential-functional equations*, Dissertations Monographies 20, Wydawnictwa AGH, Cracow, 1995.

[3] L. Byszewski, *Differential and functional-differential problems together with nonlocal conditions,* Monograph 184, Politechnika Krakowska im. Tadeusza Kościuszki, Cracow, 1995 [in Polish].

[4] L. Byszewski, *Strong maximum principles for implicit parabolic functional-differential problems together with initial inequalities,* Annales Academiae Pedagogicae Cracoviensis, Studia Mathematica **23** (IV) (2004), 9–16.

[5] J. Chabrowski, *On the non-local problem with a functional for parabolic equation,* Funkcialaj Ekvacioj **27** (1984), 101–123.

[6] Z. Kamont, *Hyperbolic functional differential inequalities and applications,* Kluwer Academic Publishers, The Netherlands, 1999.

[7] R. Redheffer, W. Walter, *Das Maximumprinzip in unbeschränkten Gebieten für parabolische Ungleichungen mit Funktionalen,* Math. Ann. 226 (1977), 155–170.

[8] J. Szarski, *Differential Inequalities*, PWN, Warsaw, 1967.

[9] J. Szarski, *Strong maximum principle for non-linear parabolic differential-functional inequalities,* Ann. Polon. Math. **29** (1974), 207–217.

[10] W. Walter, *Differential and integral inequalities*, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[11] N. Yoshida, *Maximum principles for implicit parabolic equations,* Proc. Japan Acad. **49** (1973), 785–788.

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