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# INVARIANT MEASURES WHOSE SUPPORTS POSSESS THE STRONG OPEN SET PROPERTY

**Abstract.** Let $X$ be a complete metric space, and $S$ the union of a finite number of strict contractions on it. If $P$ is a probability distribution on the maps, and $K$ is the fractal determined by $S$, there is a unique Borel probability measure $\mu_P$ on $X$ which is invariant under the associated Markov operator, and its support is $K$. The Open Set Condition (OSC) requires that a non-empty, subinvariant, bounded open set $V \subset X$ exists whose images under the maps are disjoint; it is *strong* if $K \cap V \neq \emptyset$. In that case, the *core* of $V$, $\check{V} = \bigcap_{n=0}^{\infty} S^n(V)$ is non-empty and dense in $K$. Moreover, when $X$ is separable, $\check{V}$ has full $\mu_p$-measure for every $P$. We show that the strong condition holds for $V$ satisfying the OSC iff $\mu_P(\partial V) = 0$, and we prove a zero-one law for it. We characterize the complement of $\check{V}$ relative to $K$, and we establish that the values taken by invariant measures on cylinder sets defined by $K$, or by the closure of $V$, form multiplicative cascades.

**Keywords:** core, fractal, fractal measure, invariant measure, scaling function, scaling operator, strong open set condition, zero-one law.

**Mathematics Subject Classification:** Primary 28A80, Secondary 60F20.

## 1. INTRODUCTION

This paper is a companion to our article [2], which presents background and motivation. The difference is that there we used topological methods, while here we use measure-theoretic ones. We start with a resumé of the fundamental definitions and the principal results stated in [2].

Suppose that $X$ is a complete metric space and $w_i$, $i \in \{1, 2, \dots, N\}$, are strict contractions of $X$. Call the $w_i$ *scaling maps*, and define the *scaling operator* $S$ on $2^X$ as

$$S(E) = \bigcup_{i=1}^{N} w_i(E),$$

for $E \subset X$. Say $E$ is *subinvariant* under scaling if $S(E) \subseteq E$, *invariant* if equality holds.

These notions were introduced by Hutchinson in his fundamental paper [3]. There, he established the existence of a unique-empty bounded closed subset $K$ of $X$ that is invariant under scaling, and showed it to be compact. The set $K$ is termed the *fractal*, or *invariant set*, determined by the scaling maps.

He also introduced *fractal measures.* Each probability distribution $P = (p_1, p_2, \ldots, p_N)$ on the index set defines a *Markov operator*

$$M_P \mu = \sum_{i=1}^{N} p_i \cdot \mu \circ w_i^{-1},$$

acting on finite Borel measures $\mu$ and transforming them into probability measures on the Borel $\sigma$-algebra $\mathfrak{B}(X)$.

A probability measure $\mu_P$ on $\mathfrak{B}(X)$ for which $M_P \mu_P = \mu_P$ is called an *invariant* or *fractal measure.* Hutchinson [3], 733, proved the existence and uniqueness of such measures, assuming that $P$ is non-degenerate, and established that $K$ is their topological support.

The *Open Set Condition* (OSC) posits the existence of a non-empty subinvariant bounded open set $V$ whose images under the scaling maps are disjoint. If, in addition, $V$ meets $K$, it is called the *Strong Open Set Condition* (SOSC).

We establish, in Section 4, that the alternatives $K \cap V \neq \emptyset$ and its negation are reflected in the dichotomy $\mu_P(\partial V) = 0$ or 1, resp. In view of this, the SOSC can be formulated without any reference to $K$.

In [2] we studied the *core* $\check{V}$ of the set $V$ occurring in the OSC, defined by the formula

$$\check{V} = \bigcap_{n=0}^{\infty} S^n(V),$$

where $S^n$ denotes the $n$-th iterate of $S$. There, we showed by a category argument that $\check{V}$ is non-empty whenever the SOSC holds and the restriction of the scaling maps to $V$ are *open*: this includes the case of homeomorphisms. When it is non-empty, it is a dense subset of $K$. It is invariant under scaling. Conversely, $V \neq \emptyset$ implies the SOSC.

Our goal now is to remove the assumption that the maps be open, and show, by an appeal to measure theory, that the SOSC implies that $\check{V}$ is non-empty in general. This would happen if it were possible to show that it has positive $\mu_P$-measure for some $P$.

A difficulty that arises is that, without the openness assumption, there is no longer any reason to expect that the images of $V$ under the mappings $S^n$ will be Borel measurable. Indeed, as the space $X$ has not been assumed to be separable, the images of Borel sets under continuous maps may fail to be Borel, or even analytic, as pointed out by Sierpinski [8], 220. Consequently, there is no assurance that the sets $S^n(V)$ and the core are going to be $\mu_P$-measurable.

We are able to get around this obstacle by the following device. Consider $K$ with the induced metric as the ambient space, in place of $X$, and replace $V$ by $U = K \cap V$. Then $U$ is open in the topology of $K$ and if the OSC or the SOSC holds for $V$, it also holds for $U$.

However, continuous images of $U$ belong to the Borel $\sigma$-algebra $\mathfrak{B}(K)$, for $U$ is an $F_\sigma$ subset of a compact space, thus it is $\sigma$-compact. Consequently, any continuous image of $U$ is likewise $\sigma$-compact, hence Borel, in the topology of $K$.

## 2. INVARIANT MEASURES AND THE CORE

Define the core $\check{U}$ of $U$ in the same way that $\check{V}$ was defined for $V$:

$$\check{U} = \bigcap_{n=0}^{\infty} S^n(U).$$

The properties established in [2] for $\check{V}$ hold for $\check{U}$. In particular, $\check{U}$ is invariant under scaling, i.e., $S(\check{U}) = \check{U}$.

What is new is that $\check{U} \in \mathfrak{B}(K)$. This puts us in a position to show, in Corollary 2.6, that when $V$ and, therefore, $U$ satisfy the SOSC, the core of $U$ is non-empty, because it has positive $\mu_p$-measure. Since $\check{U} \subset \check{V}$, it follows that $\check{V}$ is non-empty.

**Proposition 2.1.** *The set $U$ is subinvariant.*

*Proof.* This follows from the inclusions

$$S(U) = S(K \cap V) \subset S(K) \cap S(V) \subset K \cap V = U.$$

□

**Proposition 2.2** (cf. [1,p. 226])**.** *If $E$ is an open set that meets $K$, then $\mu_P(K \cap E) > 0$.*

*Proof.* Let $x \in K \cap E$. As $K$ is the support of $\mu_P$, the balls $B(x, r)$, centered at $x$, with radius $r$, will have $\mu_P \circ B(x, r) > 0$, for every $r > 0$. Since $E$ is open, they will lie in $E$ when $r$ is sufficiently small. For these $r$, $0 < \mu_P \circ B(x, r) \leq \mu_P(K \cap E)$, and the assertion follows. □

**Remark 2.3.** If $V$ satisfies the SOSC, then, setting $V = E$ gives $\mu_P(V) > 0$. The converse is likewise true, for $\mu_P(V) = \mu_P(K \cap V)$, since $K$ supports $\mu_P$. Accordingly, if $\mu_P(V) > 0$ for some probability distribution $P$, then $K \cap V \neq \emptyset$, and the SOSC holds for $V$.

Suppose now that $E \subset X$ is arbitrary. For every natural number $n$ and each string of indices $i_1, i_2, \ldots, i_n$ with values in $\{1, 2, \ldots, N\}$, define the *cylinder* set

$$E_{i_1 i_2 \ldots i_n} = w_{i_1} \circ w_{i_2} \circ \ldots \circ w_{i_n}(E).$$

Then $S^n(E)$ can be expressed as

$$S^n(E) = \bigcup_{i_1, \ldots, i_n} E_{i_1 i_2 \ldots i_n},$$

where the $i_1, i_2, \ldots, i_n$ vary independently over $\{1, 2, \ldots, N\}$.

The following fundamental theorem was suggested by a folklore result, recorded in Graf [1], 226.

**Theorem 2.4.** *Let $n$ be any natural number. If $E$ is a subinvariant Borel set, and, as the indices $i_1, i_2, \ldots, i_n$ vary over $\{1, 2, \ldots, N\}$, the subsets $E_{i_1 i_2 \ldots i_n}$ are disjoint, modulo sets of $\mu_P$-measure zero, then the $E_{i_1 i_2 \ldots i_n}$ form a partition of $E$ modulo $\mu_P$-null sets, and*

$$\mu_P(E_{i_1 i_2 \ldots i_n}) = p_{i_1} p_{i_2} \cdots p_{i_n} \mu_P(E),$$

*for any string of indices $i_1, i_2, \ldots, i_n$, and any distribution $P$.*

*Proof.* By the subinvariance of $E$, $S^n(E) \subset E$, for each $n$. Hence,

$$\mu_P(E) \geq \mu_P(S^n(E)) = \sum_{i_1 i_2 \ldots i_n} \mu_P(E_{i_1 i_2 \ldots i_n}).$$

By the invariance of $\mu_P$,

$$\mu_P(E_{i_1 i_2 \ldots i_n}) = M_P \mu_P(E_{i_1 i_2 \ldots i_n}) \geq p_{i_1} \cdot \mu_P(E_{i_2 \ldots i_n}),$$

for any string of indices $i_1, i_2, \ldots, i_n$. Arguing recursively gives, ultimately,

$$\mu_P(E_{i_1 i_2 \ldots i_n}) \geq p_{i_1} p_{i_2} \cdots p_{i_n} \mu_P(E).$$

Summing over all the indices yields

$$\mu_P(E) \geq \mu_P \circ S^n(E) = \sum_{i_1 i_2 \ldots i_n} \mu_P(E_{i_1 i_2 \ldots i_n}) \geq \mu_P(E).$$

Hence, equality holds throughout, and the assertion follows. $\square$

**Corollary 2.5** (of the proof). *If the SOSC holds, then $\mu_P(U) = \mu_P(\check{U})$.*

*Proof.* The proof of Theorem 2.4 reveals that $\mu_P(E) = \mu_P \circ S^n(E)$, for every $n$. Hence setting $E = U$, and sending $n \to \infty$, gives $\mu_P(U) = \mu_P(\check{U})$, as stated. $\square$

**Proposition 2.6.** *If $U$ satisfies the SOSC, then the core $\check{U}$ is non-empty.*

*Proof.* Since $\mu_P(U) > 0$, by Remark 2.3, and $\mu_P(U) = \mu_P(\check{U})$, it must be that $\check{U} \neq \emptyset$. $\square$

**Scholium 2.7.** *The SOSC is a necessary and sufficient condition for $\check{U}$, and hence $\check{V}$, to be non-empty.*

**Remark 2.8.** The foregoing product formula shows that the probabilities act *geometrically* on $\mu_P(E)$ as *scaling factors*. The notion of a partition modulo sets of $\mu_P$-measure zero is the measure-theoretic analog of the idea of set-theoretic partition and serves the same purpose. It is valid whenever the latter condition holds, but it may hold when that condition fails, as we shall see in Theorem 5.2 and Corollary 5.5 below.

## 3. THE ZERO ONE LAW AND ITS IMPLICATIONS

In fact, when $X$ is separable, $\mu_P(\check{U}) = 1$. This is a direct consequence of a remarkable Zero-One Law, proved in [5]. There, the authors consider Markov operators of the form

$$M_\mu(E) = \int_X p(x, E)\mu(dx), \quad \text{for} \quad E \in \mathfrak{B}(X),$$

acting on Borel measures $\mu$, where the *transition kernel* $p(x, E)$ is measurable in $x$ for every $E \in \mathfrak{B}(X)$, and $p(x, \cdot)$ is a Borel probability measure for every $x \in X$. The image $M\mu$ of $\mu$ is then a Borel measure. They define the *Markov function* $\Gamma(x) = \mathrm{supp} M\delta_x = \mathrm{supp}\, p(x, \cdot)$, where supp denotes the topological support, and $\delta_x$ is the Dirac measure concentrated at $x$. When $M_P$ is taken as $M$, the transition kernel assumes either of the equivalent forms

$$p(x, \cdot) = \sum_{i=1}^{N} p_i \cdot 1_{w_i^{-1}(\cdot)}(x) = \sum_{i=1}^{N} p_i \cdot 1_{(\cdot)} \circ w_i(x),$$

where 1 denotes the indicator function, and $\Gamma(x)$ then reduces to $S(x)$.

**Theorem 3.1** **(**[5, p. 346]**).** (Zero-One Law) *Assume $X$ to be a Polish space, and let $M$ have a unique invariant measure $\mu_*$. If $E \in \mathfrak{B}(X)$ is such that $\Gamma(E) \subset E$, then $\mu_*(E) = 0$ or 1.*

*Proof.* Loc. cit. □

**Corollary 3.2.** *If $X$ is separable and $U$ satisfies the SOSC, then $\mu_P(U) = \mu_P(V) = 1$.*

*Proof.* Proposition 2.2 implies that $\mu_P(U) > 0$. By Proposition 2.1, $U$ is subinvariant. As it is open, hence Borel, and $\mu_P$ is unique, Theorem 3.1 implies that $\mu_P(U) = 1$. The result for $V$ follows from the inequalities $\mu_P(U) \le \mu_P(V) \le 1$. □

**Theorem 3.3.** *If $X$ is separable and $U$ satisfies the SOSC, then $\mu_P(\check{U}) = 1$.*

*Proof.* By Corollary 2.5, $\mu_P(U) = \mu_P(\check{U})$, so the assertion follows from Corollary 3.2. □

**Theorem 3.4.** *If $V$ satisfies the SOSC and $X$ is separable, then, for every $P$, the sets $S^n(V)$ and the core $\check{V}$ belong to the $\mu_P$-completion of $\mathfrak{B}(X)$ for each $n$, and have measure one.*

*Proof.* Since $\check{U} \subset \check{V} \subset S^n(V) \subset X$, for each $n$, while $\check{U}$ and $X$ both belong to $\mathfrak{B}(X)$ and satisfy $\mu_P(X \setminus \check{U}) = 0$, the sets $S^n(V)$ and $\check{V}$ lie in the $\mu_P$-completion of $\mathfrak{B}(X)$, for every $P$. As

$$1 = \mu_P(\check{U}) \le \overline{\mu}_P(\check{V}) \le \overline{\mu}_P S^n(V) \le \mu_P(X) = 1,$$

where $\overline{\mu}_P$ denotes the completion of $\mu_P$, the second assertion follows. □

**Remark 3.5.** The authors of [5] go on to assert that under the assumptions of Theorem 3.1, $\mu_* \circ \bigcap_{n=0}^{\infty} \Gamma^n(E) = 0$ or 1. However, their proof presupposes that the sets $\Gamma^n(E)$ belong to $\mathfrak{B}(X)$, without addressing the question of how this follows from their assumptions. Nevertheless, when $\Gamma$ reduces to the scaling operator $S$, their reasoning can be applied to $U$. Accordingly, under the OSC, there holds $\mu_P(\check{U}) = 0$ or 1. We have already seen that these two alternatives reflect the absence or presence of the SOSC, and, in fact, in the first instance, $\check{U}$ is empty.

**Proposition 3.6.** *If $V$ satisfies the SOSC, and $X$ is separable, then the subset of $K$ made up of points having multiple addresses is an $\mu_P$-null set, for every $P$.*

*Proof.* Theorem 4.4 of [2] states that the points in $\check{V}$ have unique addresses, and, therefore, so do the points in $\check{U}$. Consequently, the ones in $K$ with multiple addresses belong to the complement of $\check{U}$, and this has measure zero. $\square$

**Remark 3.7.** Proposition 3.6 implies that, under the SOSC, $\mu_P$ *vanishes on the intersection of any pair of self-images of $K$ under the scaling maps.* This generalizes a finding of Hutchinson, [3], 738, who proved it for the Hausdorff measure of suitable dimension, when the scaling maps are Euclidean similitudes. This was extended by Moràn and Rey [6], Th. 2.1, and Patzschke [7], to other invariant measures on self-similar $K$.

**Theorem 3.8.** *If $X$ is separable and $U$ satisfies the SOSC, the subset of $K$ where two or more of its self-images under the scaling maps overlap does not contain any set that is open in the relative topology of $K$.*

*Proof.* Any open set in the induced topology of $K$ is the intersection of $K$ with an open set of $X$. Since $\check{U}$ is dense in $K$, the intersection contains a point of $U$. That is impossible, because points in the overlap have multiple addresses, while those in $\check{U}$ have only one. $\square$

**Theorem 3.9.** *If the scaling maps are injective, then $\check{V}$ and $\check{U}$ coincide.*

*Proof.* By definition, $U = K \cap V$, so that

$$S(U) = S(K \cap V) = S(K) \cap S(V) = K \cap S(V),$$

by injectivity and the invariance of $K$. Iterating $n$ times, taking the intersection over $n$ and sending $n \to \infty$, gives

$$\check{U} = \bigcap_{n=0}^{\infty} S^n(U) = \bigcap_{n=0}^{\infty} S^n(K \cap V) = K \cap \bigcap_{n=0}^{\infty} S^n(V) = K \cap \check{V}.$$

By Theorem 4.2 of [2], $\check{V} \subset K$, hence, $K \cap \check{V} = \check{V}$, so $\check{V}$ and $\check{U}$ coincide. $\square$

**Corollary 3.10.** *When the scaling maps are injective, $\check{V}$ is Borel.*

**Remark 3.11.** We suspect that when $\Gamma$ reduces to the scaling operator $S$, the separability assumption can be dropped from the results of this Section.

## 4. A MEASURE-THEORETIC CHARACTERIZATION OF THE SOSC

**Proposition 4.1** (Hutchinson [3, p. 724])**.** *If $E$ is non-empty and subinvariant, then, for every non-negative integer $n$, there holds $K \subset S^n(\mathrm{cl}E)$.*

**Corollary 4.2.** *If $V$ satisfies the OSC, then $K \subset \mathrm{cl}V$.*

**Corollary 4.3.** *If $E \subset X$ is non-empty, subinvariant, and $K \cap E = \emptyset$, then $K \subset \partial E$, and thus $\mu_P(\partial E) = 1$, for every $P$.*

*Proof.* From Proposition 4.1, Since, $K \subset clE = E \cap \partial E$ by hypothesis, $K \cap E = \emptyset$, it follows that $K \subset \partial E$. As $K$ is the support of $\mu_P$, this implies that $\mu_P(\partial E) = 1$ for every $P$, as stated. □

Thus, if $V$ satisfies the OSC, but not the SOSC, it is because $K \subset \partial V$.

**Theorem 4.4.** *If $V$ satisfies the SOSC and $X$ is separable, then $\mu_P(\partial V) = 0$, for every $P$.*

*Proof.* By Corollary 3.2, $\mu_P(V) = 1$. As $\mathrm{cl}V = \partial V \cup V$, and the union is disjoint, while $K \subset \mathrm{cl}V$, there holds

$$1 = \mu_P(K) \leq \mu_P(\mathrm{cl}V) = \mu_P(\partial V) + \mu_P(V) = \mu_P(\partial V) + 1 \leq 1.$$

Hence, $\mu_P(\partial V) = 0$, as claimed. □

When the scaling maps are homeomorphisms, this result can be extended as follows.

**Theorem 4.5.** *If $V$ satisfies the SOSC, $X$ is separable, and the scaling maps are homeomorphisms, then, $\mu_P(\partial S^n(V)) = 0$, for every natural number $n$, and every $P$.*

*Proof.* Let $V' = S(V)$. Then, $V'$ is open and subinvariant, and its images under the scaling maps are disjoint. Moreover, $K \cap V' \neq \emptyset$, otherwise, for each natural number $m$,

$$\emptyset = S^m(K \cap V') = S^m(K) \cap S^m(V') = K \cap S^m(V') = K \cap S^{m+1}(V).$$

Sending $m \to \infty$ would yield $K \cap \check{V} = \check{V} = \emptyset$, in violation of Corollary 2.6. Thus, $V'$ satisfies the SOSC, and, applying Theorem 4.4 to $V'$, yields $\mu_P(\partial V') = \mu_P(\partial S(V)) = 0$. This proves the theorem in the case $n = 1$, and the rest follows by induction. □

**Definition 4.6.** *Call the* rim *of $\check{V}$, denoted $\check{V}_\partial$, the set of points given by the formula*

$$\check{V}_\partial = \bigcup_{n=0}^{\infty} \partial S^n(V).$$

**Theorem 4.7.** *If $V$ satisfies the SOSC, then $\mu_P \circ \check{V}_\partial = 0$, for every $P$.*

*Proof.* The rim is a countable union of null sets, hence it has measure zero. □

**Theorem 4.8.** *If $V$ satisfies the SOSC, and the scaling maps are homeomorphisms, then $K \setminus \check{V}_\partial = \check{V}$.*

*Proof.* For each non-negative integer $n$ there holds $S^n(\mathrm{cl}V) = S^n(V) \cup S^n(\partial V)$, where the union is disjoint. Since $S^n(\partial V) = \partial S^n(V)$, it follows that $S^n(\mathrm{cl}V) \setminus \partial S^n(V) = S^n(V)$. Taking the intersection over $n$ gives

$$\bigcap_{n=0}^{\infty} S^n(\mathrm{cl}V) \setminus \bigcup_{n=0}^{\infty} \partial S^n(V) = \bigcap_{n=0}^{\infty} S^n(V).$$

By Theorem 2.3 of [2], $K = \bigcap_{n=0}^{\infty} S^n(\mathrm{cl}V)$. Recalling the definitions of $\check{V}_\partial$ and $\check{V}$ gives $K \setminus \check{V}_\partial = \check{V}$, as stated.

□

We can define the *rim* of $\check{U}$ in a way analogous to $\check{V}_\partial$.

**Corollary 4.9.** *If $U$ satisfies the SOSC, and the scaling maps are homeomorphisms, then, $K$ admits the decomposition $K = \check{U} \cup \check{U}_\partial$, and the union is disjoint.*

*Proof.* This follows from the preceding theorem by noting that $K \cap \check{V}_\partial = \check{U}_\partial$. □

**Theorem 4.10** (Characterization of the SOSC)**.** *If $V$ satisfies the OSC, then the property $\mu_P(\partial V) = 0$ is both a necessary and sufficient condition for the SOSC to hold.*

*Proof.* Theorem 4.4 gives the necessity, while the sufficiency follows from the contrapositive of Corollary 4.3, taking $E = V$. □

We are thus led to a measure-theoretic characterization of the SOSC. It has the virtue of being expressed in terms of $V$ itself, without any need to construct the fractal $K$.

In fact, using the full strength of Corollary 4.3, yields the following dichotomy:

**Theorem 4.11** (Zero - One Law for $\partial V$)**.** *Let $V$ satisfy the OSC. Then $\mu_P(\partial V) = 0$ or 1, depending as to whether $V$ meets $K$ or not, and it is independent of $P$.*

## 5. MULTIPLICATIVE CASCADES

The results of the preceding section, along with Theorem 2.6, allow us to prove that the values assigned by invariant measures to certain cylinder sets form *multiplicative cascades*. We start with a definition.

**Definition 5.1.** *Let $n$ be a natural number, and let $\mu_P$ be an invariant probability measure on $\mathfrak{B}(X)$. Suppose $E$ is a Borel set with $\mu_P(E) = 1$, and the subsets $E_{i_1 i_2 \ldots i_n}$ form a Borel measureable partition of $E$, modulo $\mu_P$-null sets, as the indices $i_1, i_2, \ldots, i_n$ vary over $\{1, 2, \ldots, N\}$. If*

$$\mu_P(E_{i_1 i_2 \ldots i_n}) = p_{i_1} p_{i_2} \cdots p_{i_n},$$

*for any string of indices $i_1, i_2, \ldots, i_n$ and any $n$, the values of $\mu_P$ are said to form a* multiplicative cascade *on the cylinder sets associated with $E$.*

**Theorem 5.2.** *If $X$ is separable, the scaling maps are homeomorphisms, and $V$ satisfies the SOSC, then, for each natural number $n$, the sets $\mathrm{cl}V_{i_1 i_2 \ldots i_n}$ form a partition of $\mathrm{cl}V$, modulo sets of $\mu_P$-measure zero, as the indices $i_1, i_2, \ldots, i_n$ vary over $\{1, 2, \ldots, N\}$.*

*Proof.* Let $n$ be fixed. Then, for each string $i_1, i_2, \ldots, i_n$, the sets $V_{i_1 i_2 \ldots i_n}$ are open, and $\mathrm{cl}V_{i_1 i_2 \ldots i_n} = V_{i_1 i_2 \ldots i_n} \cup \partial V_{i_1 i_2 \ldots i_n}$, where the union is disjoint. Hence,

$$\mu_P(\mathrm{cl}V_{i_1 i_2 \ldots i_n}) = \mu_P(V_{i_1 i_2 \ldots i_n}) + \mu_P(\partial V_{i_1 i_2 \ldots i_n}).$$

Since

$$\partial S^n(V) = \bigcup_{i_1, \ldots, i_n} \partial V_{i_1 i_2 \ldots i_n},$$

Theorem 4.5 implies that $\mu_P(\partial V_{i_1 i_2 \ldots i_n}) = 0$, for each string $i_1, i_2, \ldots, i_n$. As the sets $V_{i_1 i_2 \ldots i_n}$ are disjoint, it follows that their closures $\mathrm{cl}V_{i_1 i_2 \ldots i_n}$ are disjoint modulo sets of $\mu_P$-measure zero, for every $P$.

Since $\mathrm{cl}V$ is subinvariant, the conditions of Theorem 2.6 are satisfied, and the assertion concerning the partitioning of $\mathrm{cl}V$ follows. □

**Corollary 5.3.** *Under the assumptions of the preceding theorem, there holds*

$$\mu_P(\mathrm{cl}V_{i_1 i_2 \ldots i_n}) = p_{i_1} p_{i_2} \cdots p_{i_n},$$

*for any string of indices $i_1, i_2, \ldots, i_n$ in $\{1, 2, \ldots, N\}$, and any distribution $P$.*

*Proof.* The set $\mathrm{cl}V$ is subinvariant, and thus, by Proposition 4.1, $K \subset \mathrm{cl}V$. Since $K$ is the support of $\mu_P$ for each $P$, $1 = \mu_P(K) \leq \mu_P(\mathrm{cl}V) \leq 1$, because $\mu_P$ is a probability measure. Hence, equality holds, and the formula results from the conclusion of Theorem 2.6. □

We have thus established that the values assigned by $\mu_P$ to the cylinder sets associated with $\mathrm{cl}V$ form a multiplicative cascade. Next, we show that an analogous result holds for $K$.

**Theorem 5.4.** *Under the assumptions of the preceding theorem, for each $n$,*

$$\mu_P(K_{i_1 i_2 \ldots i_n}) = p_{i_1} p_{i_2} \cdots p_{i_n},$$

*for any string of indices $i_1, i_2, \ldots, i_n$, and any distribution $P$.*

*Proof.* Since $K \subset \mathrm{cl}V$, there holds $K_{i_1 i_2 \ldots i_n} \subset \mathrm{cl}V_{i_1 i_2 \ldots i_n}$, and thus $\mu_P(K_{i_1 i_2 \ldots i_n}) \leq \mu_P(\mathrm{cl}V_{i_1 i_2 \ldots i_n})$, for each indices $i_1, i_2, \ldots, i_n$, and any $P$. As $K$ is the support of $\mu_P$,

$$1 = \mu_P(K) = \mu_P(S^n(K)) \leq \sum_{i_1 i_2 \ldots i_n} \mu_P(K_{i_1 i_2 \ldots i_n}) \leq$$
$$\leq \sum_{i_1 i_2 \ldots i_n} \mu_P(\mathrm{cl}V_{i_1 i_2 \ldots i_n}) = \sum_{i_1 i_2 \ldots i_n} p_{i_1} p_{i_2} \cdots p_{i_n} = 1,$$

hence equality holds throughout, and, therefore, for every string $i_1, i_2, \ldots, i_n$,

$$\mu_P(K_{i_1 i_2 \ldots i_n}) = p_{i_1} p_{i_2} \cdots p_{i_n},$$

as asserted. □

**Corollary 5.5.** *Under the assumptions of the previous theorem, the sets $K_{i_1 i_2 \ldots i_n}$ form a partition of $V$, modulo sets of $\mu_P$-measure zero, as the indices $i_1, i_2, \ldots, i_n$ vary over $\{1, 2, \ldots, N\}$.*

*Proof.* The equality $\mu_P(K) = \sum_{i_1 i_2 \ldots i_n} \mu_P(K_{i_1 i_2 \ldots i_n})$ implies that the $K_{i_1 i_2 \ldots i_n}$ are disjoint, modulo sets of $\mu_P$-measure zero, and their union has measure $\mu_P(K)$, as required. □

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