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**ASYMPTOTIC PROPERTIES
OF SOLUTIONS OF SOME ITERATIVE
FUNCTIONAL INEQUALITIES**

Abstract. Continuous solutions of iterative linear inequalities of the first and second order are considered, belonging to a class \mathcal{F}_T of functions behaving at the origin as a prescribed function T .

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1. INTRODUCTION

The iterative functional inequality of second order (studied, among others, in [6])

$$\psi[f^2(x)] \leq (p(x) + q[f(x)])\psi[f(x)] - p(x)q(x)\psi(x), \quad (1)$$

where ψ is the unknown function, thanks to its specific form, is equivalent to the system consisting of the inequality of first order

$$z[f(x)] \leq p(x)z(x) \quad (2)$$

(for the unknown z) and of the linear inhomogeneous functional equation

$$\varphi[f(x)] = q(x)\varphi(x) + z(x). \quad (3)$$

This means that every solution φ of equation (3) with a z satisfying inequality (2) is a solution of inequality (1) and vice versa: given a solution ψ of inequality (1), insert it to (3) in place of φ and calculate z from (3) to get a solution z of inequality (1).

The aim of this paper is to investigate these continuous solutions of inequalities (2) and (1) that behave at the fixed point of f like a prescribed “test” function T , in particular, like any of the functions p, q or f . Basic facts from the theory of iterative

functional inequalities, originated by the first author in [1] (cf. also Chapter 12 in [5]) are recalled, as well as those on the asymptotic behavior of solutions of equation (3), found in the papers by the remaining authors: [2] and [3].

The functions f, p, q and T meet the following general assumptions:

- (H_1) The function $f : I \rightarrow I$ is continuous and strictly increasing in an interval $I = [0, a[$ ($a > 0$ may belong to I or not). Moreover, $0 < f(x) < x$ for $x \in I^* = I \setminus \{0\}$.
- (H_2) The function $p : I \rightarrow \mathbf{R}$ is continuous in I and $p(x) > 0$ for $x \in I^*$.
- (H_3) The function $q : I \rightarrow \mathbf{R}$ is continuous in I and $q(x) \neq 0$ for $x \in I^*$.
- (H_4) The function $T : I \rightarrow \mathbf{R}$ is continuous in I and $T(x) \neq 0$ for $x \in I^*$, $T(0) = 0$.

With f^n denoting the n -th iterate of the function f , hypotheses (H_1) imply (see [5]) that

$$\lim_{n \rightarrow \infty} f^n(x) = 0 \quad \text{for every } x \in I. \quad (4)$$

Moreover, 0 is the only fixed point of f in I .

Assuming that (H_4) is satisfied, we introduce the following class of functions

$$\mathcal{F}_T := \left\{ \varphi : I \rightarrow \mathbf{R} : \varphi \text{ is continuous on } I \text{ and the limit } L_\varphi^T := \lim_{x \rightarrow 0^+} \frac{\varphi(x)}{T(x)} \text{ is finite} \right\}.$$

Note that if $T(0) \neq 0$, the class \mathcal{F}_T would consist of all functions continuous in I .

We are interested in solutions of inequalities (2) and (1) belonging to the class \mathcal{F}_T .

2. CONTINUOUS SOLUTIONS OF INEQUALITY (2)

It is known (see [5]) that the number of continuous solutions of inequality (2) as well as of the equation

$$\alpha[f(x)] = p(x)\alpha(x), \quad x \in I, \quad (5)$$

associated to inequality (2), depends on the behavior of the functional sequence

$$P_n(x) = \prod_{i=0}^{n-1} p[f^i(x)], \quad x \in I, \quad n \in \mathbf{N}. \quad (6)$$

We shall need some results from [4] (cf. also [5], Section 3.1) on continuous solutions α of equation (5). They are quoted below as a lemma, and concern the following two cases:

(A) The limit

$$P(x) = \lim_{n \rightarrow \infty} P_n(x) \quad (7)$$

exists in I and P is continuous in I . Moreover, $P(x) > 0$ for $x \in I^*$.

(B) There exists an interval $J \subset I$ such that, uniformly in J ,

$$\lim_{n \rightarrow \infty} P_n(x) = 0.$$

Lemma 1. *Let hypotheses (H_1) and (H_2) be fulfilled.*

i) *In case (A) all continuous solutions $\alpha : I \rightarrow \mathbf{R}$ of equation (5) are given by*

$$\alpha(x) = \frac{c}{P(x)} \quad \text{for } x \in I,$$

where c is any real number.

ii) *If, moreover, $p(0) = 0$, then in case (B) equation (5) has in I continuous solution depending on an arbitrary function. (This means that for any $x_0 \in I^*$ and every continuous function $\alpha_0 : [f(x_0), x_0] \rightarrow \mathbf{R}$ fulfilling the boundary condition $\alpha_0[f(x_0)] = p(x_0)\alpha_0(x_0)$ there is the unique continuous solution $\alpha : I \rightarrow \mathbf{R}$ of equation (5) such that*

$$\alpha(x) = \alpha_0(x) \quad \text{for } x \in [f(x_0), x_0] \quad \text{and} \quad \alpha(0) = 0,$$

and every continuous solution to (5) may be obtained in this manner.)

For the sake of completeness, we now quote from [1] (cf. also [5], Chapter 12.) as next lemmas, some representation theorems which will be useful in the sequel. We start with the notions of an $\{f\}$ -monotonic function.

Definition 1. *A continuous function $\eta : I \rightarrow \mathbf{R}$ [resp. $\vartheta : I \rightarrow \mathbf{R}$] is said to be $\{f\}$ -decreasing in I [resp. $\{f\}$ -increasing in I], if*

$$\eta[f(x)] \leq \eta(x), \quad x \in I \quad [\text{resp.}, \quad \vartheta[f(x)] \geq \vartheta(x), \quad x \in I]. \quad (8)$$

The family of all continuous $\{f\}$ -decreasing [resp., $\{f\}$ -increasing] functions will be denoted by \mathcal{D}_f [resp., \mathcal{I}_f].

Theorem 1. *Let hypotheses (H_1) , (H_2) be satisfied and case (A) occur. Then the general continuous solution z of inequality (2) is given by*

$$z(x) = \frac{\eta(x)}{P(x)}, \quad x \in I \quad (9)$$

where η is an arbitrary function from \mathcal{D}_f .

Note that, by Lemma 1i), formula (9) says that the function z is the product of a function from \mathcal{D}_f and of a solution of equation (5). Similar assertion remains true for case (B), but in a narrower class of solutions than that of continuous ones, namely, solutions called *regular*.

Definition 2. *A continuous solution z of inequality (2) is said to be regular [a CR-solution, for short] if there exists a continuous solution α of equation (5) such that $\alpha(x) \leq z(x)$, $x \in I$, and the function α_z defined by formula*

$$\alpha_z(x) = \lim_{n \rightarrow \infty} \frac{z[f^n(x)]}{P_n(x)} \quad \text{for } x \in I^*, \quad \alpha_z(0) = 0, \quad (10)$$

is a continuous solution of equation (5) in I .

Some necessary and sufficient conditions for a solution of (2) to be *regular* are collected in [1], cf. also [5], Section 12.4. A representation theorem based on the results found therein reads:

Theorem 2. *Let hypotheses $(H_1), (H_2)$ be satisfied, $p(0) = 0$, and let case (\mathbf{B}) occur.*

i) *The general CR-solution z of inequality (2) such that (cf. (10))*

$$\alpha_z(x) \neq 0, \quad x \in I^*, \quad (11)$$

is given by the formula (valid for $x \in I^$)*

$$z(x) = \begin{cases} \eta(x)\alpha(x), & \text{if } \alpha_z(x) > 0, x \in I^*, \\ \vartheta(x)\alpha(x), & \text{if } \alpha_z(x) < 0, x \in I^*, \end{cases} \quad (12)$$

where α is an arbitrary continuous solution of equation (5) vanishing at $x = 0$ only, $\eta \in \mathcal{D}_f$ and $\vartheta \in \mathcal{I}_f$, both are arbitrary and such that

$$\eta(0) = 1, \quad \vartheta(0) = -1. \quad (13)$$

ii) *The general CR-solution z of inequality (2) such that*

$$\alpha_z(x) = 0 \quad \text{for } x \in I, \quad (14)$$

is given by the formula

$$z(x) = \eta(x)\alpha_0(x), \quad x \in I, \quad (15)$$

where $\eta \in \mathcal{D}_f$ is arbitrary with $\eta(0) = 0$ and α_0 is a positive on I^ continuous solution of (2).*

Finally, we put

$$p_T(x) = \frac{p(x)T(x)}{T[f(x)]}, \quad x \in I^*, \quad (16)$$

and introduce an auxiliary equation.

Lemma 2. *Let hypotheses $(H_1), (H_2)$ and (H_4) be fulfilled and let the limit*

$$p_T(0) := \lim_{x \rightarrow 0^+} p_T(x) \quad (17)$$

exist. Then there is a one-to-one correspondence between the continuous solutions $\beta : I \rightarrow \mathbf{R}$ of the auxiliary equation

$$\beta[f(x)] = p_T(x)\beta(x), \quad x \in I, \quad (18)$$

where p_T is given by (16) and (17), and the solutions α of equation (5), belonging to \mathcal{F}_T .

Proof. The equivalence claimed in the lemma is established as follows. If a function $\alpha \in \mathcal{F}_T$ satisfies (5), then it is easily verified that the function $\beta : I \rightarrow \mathbf{R}$ given by

$$\beta(x) = \frac{\alpha(x)}{T(x)}, \quad x \in I^*, \quad \beta(0) = L_\alpha^T \quad (19)$$

(cf. the definition of \mathcal{F}_T), is a continuous in I solution of (18). Conversely, if a continuous $\beta : I \rightarrow \mathbf{R}$ satisfies (18) then $\alpha := T \cdot \beta$ is a solution to (5), belonging to \mathcal{F}_T . \square

3. SOLUTIONS OF INEQUALITY (2) ASYMPTOTICALLY COMPARABLE WITH T

We start with introducing the functional sequence corresponding to (6) (with p replaced by p_T), defined by

$$P_n^T(x) = \prod_{i=0}^{n-1} p_T[f^i(x)], \quad x \in I, n \in \mathbf{N} \quad (20)$$

It is easy to verify the following formula

$$P_n^T(x) = \frac{T(x)P_n(x)}{T[f^n(x)]}, \quad x \in I^*, \quad P_n^T(0) = [p_T(0)]^n. \quad (21)$$

Case (A)

We note that $p(0) = 1$ in this case (see [3]) and, consequently, $P(0) = 1$, where the function P is defined by (6). Moreover, since P_n^T tends to ∞ when n does (cf. (4) and $T(0) = 0$), the zero function (defined on I) is the only continuous solution of equation (18) in I , cf. [4]. Thus, by virtue of Lemma 2, this zero function is also the only solution of equation (2) in the class \mathcal{F}_T .

As a simple consequence of Theorem 2*i*) we obtain

Theorem 3. *Let hypotheses (H_1) , (H_2) be satisfied and let case (A) occur. Then the general solution $z \in \mathcal{F}_T$ of inequality (2) is given by formula (9), where $\eta \in \mathcal{D}_f \cap \mathcal{F}_T$ is arbitrary.*

Proof. Refer to (9), (6) and (7) (with $P(0) = 1$), and the definition of the class \mathcal{F}_T to obtain the relations

$$\lim_{x \rightarrow 0^+} \frac{\eta(x)}{T(x)} = \lim_{x \rightarrow 0^+} \frac{z(x)P(x)}{T(x)} = \lim_{x \rightarrow 0^+} \frac{z(x)}{T(x)},$$

whenever the functions z and η have the properties stated in the theorem. \square

Remark 1. *Note that if $k : I \rightarrow \mathbf{R}$ is any nonnegative function from \mathcal{D}_f , such that $\lim_{x \rightarrow 0^+} k(x)$ exists (in particular, if k is increasing), then the function $\eta := k \cdot T \in \mathcal{D}_f \cap \mathcal{F}_T$.*

Case (B)

We start with a theorem which describes all *CR*-solutions $z \in \mathcal{F}_T$ of (2) in the case of the function α_z defined by (10) vanishing at zero only.

Theorem 4. *Let hypotheses (H_1) , (H_2) be satisfied, $p(0) = 0$ and let case (B) occur. Then the general *CR*-solution $z \in \mathcal{F}_T$ of inequality (2) such that (11) holds is given by formula (12), where $\alpha \in \mathcal{F}_T$ is an arbitrary continuous solution of (5) vanishing at $x = 0$ only and $\eta \in \mathcal{D}_f$ if $\alpha_z > 0$ [resp. $\theta \in \mathcal{I}_f$ if $\varphi_z < 0$] is an arbitrary function satisfying (13).*

When $\alpha_z = 0$ in I we have only a sufficient condition for z to be in \mathcal{F}_T :

Theorem 5. *Let hypotheses (H_1) , (H_2) be satisfied, $p(0) = 0$ and let case (B) occur. Moreover let α_0 be a continuous solution of (5) such that (14) holds and let $\eta \in \mathcal{D}_f$ be arbitrary with (13). If either $\alpha_0 \in \mathcal{F}_T$ or $\eta \in \mathcal{F}_T$, then the function z defined by formula (15) is a *CR*-solution of (2) in the class \mathcal{F}_T .*

Theorems 5 and 6 follow directly from formula (12), resp. (16), and the definition of \mathcal{F}_T .

Finally we consider the following two subcases of (B):

(**BA^T**) The case (B) occurs, the limit

$$P^T(x) = \lim_{n \rightarrow \infty} P_n^T(x) \quad (22)$$

exists in I and P^T is continuous in I . Moreover, $P^T(x) > 0$, for $x \in I^*$.

(**BB^T**) The case (B) occurs and there exists an $x_0 \in I^*$ such that, uniformly in $[f(x_0), x_0]$,

$$\lim_{n \rightarrow \infty} P_n^T(x) = 0.$$

We shall present the following

Theorem 6. *Let hypotheses (H_1) , (H_2) be fulfilled and let the limit (17) exist. In cases (**BA^T**) or (**BB^T**), every *CR*-solution z of inequality (2) such that (11) or (14) holds belongs to the class \mathcal{F}_T .*

Moreover, in case (**BA^T**), the general *CR*-solution $z \in \mathcal{F}_T$ of inequality (2) such that (11) is satisfied, is given by the formulae (for $x \in I$)

$$z(x) = c \cdot \frac{\eta(x)T(x)}{P^T(x)} \quad \text{with } c > 0; \quad z(x) = c \cdot \frac{\vartheta(x)T(x)}{P^T(x)} \quad \text{with } c < 0 \quad (23)$$

where P_T is defined by (22), $\eta \in \mathcal{D}_f$ and $\vartheta \in \mathcal{I}_f$ satisfy condition (13).

Proof. Case (**BA^T**). If $z \in \mathcal{F}_T$ is a *CR*-solution of inequality (2) and (11) holds, then by Theorem 4 it is given by formula (12), where $\alpha \in \mathcal{F}_T$ is any continuous solution of equation (5) vanishing at $x = 0$ only and η and ϑ are some $\{f\}$ -monotonic functions described in the assertion of the theorem. Define β by (19). Lemma 2 says that this β is a continuous solution of equation (18). Consequently from Lemma 1, applied to equation (18), we obtain $\beta(x) = c/P^T(x)$, where $c \neq 0$. Thanks to (**BA^T**), (12) and (19), we get (23). Of course, the function z defined by (23) is a *CR*-solution of

inequality (2). Moreover, $z \in \mathcal{F}_T$, because (we take the first equality in (23); for the other, the proof is the same)

$$\lim_{x \rightarrow 0^+} \frac{z(x)}{T(x)} = \lim_{x \rightarrow 0^+} c \cdot \frac{\eta(x)}{P^T(x)} = c \eta(0) = c.$$

Case (**BB^T**). Given a continuous solution α of equation (5) vanishing at $x = 0$ only, in virtue of Theorem 5, it is enough to check whether it belongs to \mathcal{F}_T . For, we know that a continuous solution of both equations (5) and (18) depends on an arbitrary function. Let us then take an $x_0 \in I^*$ and define

$$\beta_0(x) = \frac{\alpha(x)}{T(x)} \quad \text{for } x \in [f(x_0), x_0].$$

Since $\alpha[f(x_0)] = p(x_0)\alpha(x_0)$, the relation $\beta_0[f(x_0)] = p_T(x_0)\beta_0(x_0)$ also holds (see (19) and (16)). Consequently, there exists the (unique) continuous solution β of (18) such that $\beta(x) = \beta_0(x)$ for $x \in [f(x_0), x_0]$ and $\beta(0) = 0$. Obviously, $\alpha^* := \beta \cdot T$ satisfies (5), is continuous, vanishes at $x = 0$ only and is in \mathcal{F}_T . And $\alpha^*(x) = \alpha(x)$ for $x \in [f(x_0), x_0]$. This means that $\alpha^* = \alpha$, i.e., $\alpha \in \mathcal{F}_T$, as claimed. \square

To conclude the section, we supply some examples concerning solutions of inequality (2) that behave at the origin like the given functions p or f , occurring in (2).

Example 1. Take $I = [0, 1)$ and consider the inequality

$$z(x^2) \leq \frac{1}{1+x} z(x). \quad (24)$$

Here

$$P_n(x) = \prod_{i=0}^{n-1} (1+x^{2^i})^{-1} = \frac{1-x}{1-x^{2^n}}, \quad \text{whence } P(x) = 1-x.$$

Consequently, case (**A**) occurs and the continuous solutions of inequality (24) are of the form $z(x) = \eta(x)/(1-x)$, $x \in I$, where $\eta \in \mathcal{D}_f$ is arbitrary. Let $T(x) = f(x) = x^2$, $x \in I$. By Theorem 3, $z \in \mathcal{F}_f$ if and only if there exists the finite limit $\lim_{x \rightarrow 0^+} x^{-2} \eta(x)$ (which is the case, for instance, when $\eta(x) = x^\gamma$, $\gamma \geq 2$ or $\eta(x) = e^{x^2} - 1$).

Example 2. Consider the inequality

$$z\left(\frac{x}{2}\right) \leq \frac{1}{2} z(x) \quad (25)$$

in the interval $I = [0, 1)$ and take $T(x) = f(x) = x/2$, $x \in I$. Since $\lim_{n \rightarrow \infty} P_n(x) = \lim_{n \rightarrow \infty} 2^{-n} = 0$ and $P_n^T(x) = 1$, $x \in I$, case (**BA^T**) occurs. In virtue of Theorem 6, the general *CR*-solution $z \in \mathcal{F}_f$ has the form

$$z(x) = x\eta(x), \quad \eta(0) > 0 \quad \text{or} \quad z(x) = x\vartheta(x), \quad \vartheta(0) < 0.$$

Take the function $z : I \rightarrow R$, given by $z(x) = e^x - 1$, $x \in I$, which is a particular continuous solution of (25), and the identity function α_0 (on I), which is a particular continuous solution of the equation

$$\alpha\left(\frac{x}{2}\right) = \frac{1}{2}\alpha(x), \quad (26)$$

corresponding to inequality (25). Since $\lim_{x \rightarrow 0^+} [(e^x - 1)/x] = 1$, then $z \in \mathcal{F}_f$. By Theorem 12.4.3 from [5], z is an CR -solution of (25) and $\alpha_z(x) = x$ for $x \in I$. The function η given by

$$\eta(x) = \frac{e^x - 1}{x}, \quad x \in I^*, \quad \eta(0) = 1,$$

belongs to \mathcal{D}_f and our solution z of (25) is of the form (12) with η as above and $\alpha(x) = x$ for $x \in I$.

For another continuous solution z of (25) given by $z(x) = ax^2$, $a > 0$, we calculate $\lim_{x \rightarrow 0^+} (z[f^n(x)]/P_n(x)) = \lim_{x \rightarrow 0^+} (ax^2 2^{-n}) = 0$. Consequently, this solution z is *regular* with $\alpha_z(x) = 0$ for $x \in I$; moreover, it is actually given by formula (9) with $\eta \in \mathcal{D}_f$, $\eta(x) = ax$, $x \in I$, and a positive in I^* continuous solution α_0 of (26) defined by $\alpha_0(x) = x$, $x \in I$.

4. SOLUTIONS OF INEQUALITY (1) ASYMPTOTICALLY COMPARABLE WITH T

Thanks to the equivalence of inequality (1) and the system consisting of inequality (2) and equation (3), we may use a uniqueness result from [4], which is adapted to equation (3) and quoted below as the last lemma.

Given a continuous function $r : I^* \rightarrow \mathbf{R}$, define on I^* the (continuous) functions q_T and z_T as follows

$$q_T(x) = \frac{q(x)T(x)}{T[f(x)]}, \quad \text{and} \quad z_T(x) = \frac{z(x)}{T[f(x)]}, \quad \text{for } x \in I^*. \quad (27)$$

Now, we are in a position to formulate

Lemma 3. *Assume that hypotheses $(H_1) - (H_4)$ are fulfilled, and that there exist the finite limits*

$$q_T(0) := \lim_{x \rightarrow 0^+} q_T(x); \quad z_T(0) := \lim_{x \rightarrow 0^+} z_T(x). \quad (28)$$

i) *If*

$$|q_T(0)| > 1, \quad (29)$$

then equation (3) has the unique solution φ in \mathcal{F}_T which is given by the formula

$$\varphi(x) = - \sum_{n=0}^{\infty} \frac{z[f^n(x)]}{Q_{n+1}(x)}, \quad x \in I^*; \quad \varphi(0) = 0, \quad (30)$$

where (cf. (6))

$$Q_{n+1} = \prod_{i=0}^n q \circ f^i, \quad n \in N \cup \{0\}. \quad (31)$$

- ii) If $|q_T(0)| < 1$, then for any $x_0 \in I^*$ every continuous function $\varphi_0 : [f(x_0), x_0] \rightarrow \mathbf{R}$ satisfying the condition: $\varphi(f(x_0)) = q(x_0\varphi(x_0) + z(x_0))$ can be uniquely extended to a solution $\varphi : I \rightarrow \mathbf{R}$ of (3) belonging to \mathcal{F}_T .

Since (2) with (3) are equivalent to (1), directly from Lemma 3i) we get the following

Theorem 7. Assume that hypotheses $(H_1) - (H_4)$ are fulfilled, inequality (2) has a continuous solution $z : I \rightarrow \mathbf{R}$ and there exist limits (28) and condition (29) is satisfied.

Then all solutions $\psi \in \mathcal{F}_T$ (3) (with this z , thus also of inequality (1)) are given by the formula

$$\psi(x) = - \sum_{n=0}^{\infty} \frac{z[f^n(x)]}{Q_{n+1}(x)}, \quad x \in I^*; \quad \psi(0) = 0. \quad (32)$$

In the case of z a CR-solution (cf. Definition 2) to (2), formula (32) may be written in another form, see [6].

Theorem 8. Assume that hypotheses $(H_1) - (H_4)$ are fulfilled, z is a CR-solution to (2), case **(A)** or **(B)** occurs and there exist the second limit in (28) and a finite limit

$$t := \lim_{x \rightarrow 0^+} \frac{T(x)}{T[f(x)]}. \quad (33)$$

Moreover,

$$|q(0) t| > 1. \quad (34)$$

If $\psi \in \mathcal{F}_T$ solves equation (3) (with this z , hence also of inequality (1)), then

$$\psi(x) = -S(x) \sum_{n=0}^{\infty} \frac{\zeta[f^n(x)]P_n(x)}{Q_{n+1}(x)}, \quad x \in I^*, \quad \psi(0) = 0, \quad (35)$$

where:

- in case **(A)** there is $S = 1/P$ (a continuous solution of equation (5) when P is defined by (7), cf. Lemma 1), and $\zeta = \eta \in \mathcal{D}_f \cap \mathcal{F}_T, \eta(0) = 1$;
- in case **(B)**, when $\alpha_z \neq 0$ for $x \in I^*$ (cf. Definition 2), there is $S = \alpha \in \mathcal{F}_T$ solving (5) and vanishing at zero only, and $\zeta = \eta \in \mathcal{D}_f, \eta(0) = 1$ (when $\alpha_z > 0$), whereas $\zeta = \vartheta \in \mathcal{I}_f, \vartheta(0) = -1$ (when $\alpha_z < 0$);
- in case **(B)**, when $\alpha_z = 0$ for $x \in I$, there is $S = \alpha_0$, which is a continuous solution of equation (5), positive in I^* (provided it does exist), $\zeta = \eta \in \mathcal{D}_f, \eta(0) = 1$, and either α_0 or η belongs to \mathcal{F}_T .

Proof. Note first that because of (27) and (33), for $z \in \mathcal{F}_T$, from (28) we derive

$$q_T(0) = \lim_{x \rightarrow 0^+} \left(q(x) \frac{T(x)}{T[f(x)]} \right) = q(0)t; \quad z_T(0) = \lim_{x \rightarrow 0^+} \left(\frac{z(x)}{T(x)} \frac{T(x)}{T[f(x)]} \right) = L_z^T t.$$

Thus, thanks to inequality (34), Lemma 3 applies and formula (30) determines the solutions $\varphi \in \mathcal{F}_T$ of equation (3), whence, as $\psi = \varphi$, those of inequality (1).

It remains to check formulae (35). Observe first that if α is a solution to (5), then it also satisfies, for every $n \in \mathbf{N}$, the equations below, resulting from equation (5) on iterating it n times,

$$\alpha[f^n(x)] = P_n(x)\alpha(x), \quad x \in I. \quad (36)$$

- a) Because of Theorem 3, the solutions $z \in \mathcal{F}_T$ of (2) are given by (9), and the function $\alpha = 1/P$ satisfies (3). Using (9) and (36) in (30), we get (35).
- b) This results from Theorem 4 and formula (12) with (36), when used in (30).
- c) By Theorem 5, we obtain formula (15) for z and (35) follows from (36).

The applicability of Theorems 7 and 8 in the case of the “test function” $T = q$ or $T = f$ will be shown in next two examples. \square

Example 3. Take $I = [0, 1)$ and consider the inequality

$$\psi(x^4) \leq \left(\frac{1}{1+x} + 2x^2 \right) \psi(x^2) - \frac{2x}{1+x} \psi(x), \quad x \in I. \quad (37)$$

This inequality is of the form (1), with $f(x) = x^2$, $p(x) = (1+x)^{-1}$, $q(x) = 2x$; all for $x \in I$. These functions fulfil hypotheses $(H_1) - (H_3)$. The function $\eta : I \rightarrow \mathbf{R}$, given by $\eta(x) = \ln(1+x^2)$, $x \in I$, satisfies (8), so that $\eta \in \mathcal{D}_f$, and it produces the solution $z : I \rightarrow \mathbf{R}$;

$$z(x) = \frac{\ln(1+x^2)}{1-x}, \quad x \in I, \quad (38)$$

of inequality (24) (cf. (2)), related to (37). For $T(x) = q(x) = 2x$, the limits (28) are:

$$q_T(0) = \lim_{x \rightarrow 0^+} \frac{[q(x)]^2}{q[f(x)]} = 2; \quad z_T(0) = \lim_{x \rightarrow 0^+} \frac{z(x)}{q[f(x)]} = \lim_{x \rightarrow 0^+} \frac{1}{2} \frac{\ln(1+x^2)}{x^2} = \frac{1}{2},$$

so that condition (29) is fulfilled. Consequently, Theorem 7 works, and in the class \mathcal{F}_q inequality (37) has the solution $\psi : I \rightarrow \mathbf{R}$ given by formula (32), in which $f^n(x) = x^{2^n}$,

$$Q_{n+1}(x) = \prod_{i=0}^n (2x^{2^i}) = 2^{n+1} x^{2^{n+1}-1}$$

and z is given by (38). Thus (35) now takes the form

$$\psi(x) = - \sum_{n=0}^{\infty} 2^{-n-1} x^{1-2^{n+1}} \frac{\ln(1+x^{2^{n+1}})}{1-x^{2^n}}, \quad \psi(0) = 0.$$

Example 4. Take $I = [0, 1)$ and consider the inequality

$$\psi\left(\frac{1}{4}x\right) \leq \left(\frac{1}{2} + \cos\left(\frac{1}{4}x\right)\right)\psi\left(\frac{1}{2}x\right) - \frac{1}{2}\cos\left(\frac{1}{2}x\right)\psi(x), \quad x \in I. \quad (39)$$

The given functions $f(x) = \frac{1}{2}x$, $p(x) = \frac{1}{2}$, $q(x) = \cos(\frac{1}{2}x)$, and $T(x) = f(x) = \frac{1}{2}x$, $x \in I$, satisfy hypotheses $(H_1) - (H_4)$. We note that the function $z : I \rightarrow \mathbf{R}$ given by $z(x) = e^x - 1$, $x \in I$, is a regular solution of inequality (25) (related to (39), cf. (2)) and $\alpha_z(x) = x$, $x \in I^*$. Moreover, $z \in \mathcal{F}_f$ (see Example 2). In turn, since $P_n(x) = 2^{-n}$, case **(B)** occurs. Finally, we have $f^i(x) = 2^{-i}x$, whence

$$Q_{n+1}(x) = \prod_{i=0}^n \cos(2^{-i}x) = \frac{\sin x \cos x}{2^n \sin(2^{-n}x)}.$$

(see (31)). We calculate the limits (see (28) and (33)):

$$t = \lim_{x \rightarrow 0^+} \frac{f(x)}{f^2(x)} = 2; \quad z_T(0) = \lim_{x \rightarrow 0^+} \frac{z(x)}{f^2(x)} = 4 \lim_{x \rightarrow 0^+} \frac{e^x - 1}{x} = 4.$$

Since $q(0) = 1$, (34) holds. By Theorem 8 formula (35) represents all solutions $\psi \in \mathcal{F}_f$ of inequality (39). With $S = \alpha_z = id|_I$ (which satisfies equation (25) and vanishes at zero only), formula (35) now takes the form

$$\psi(x) = -\frac{x}{\sin x \cos x} \sum_{n=0}^{\infty} \eta(2^{-n}x) \sin(2^{-n}x), \quad x \in I^*, \quad \psi(0) = 0,$$

where $\eta \in \mathcal{D}_f$, $\eta(0) = 1$, is arbitrary. We may take $\eta(x) = (e^x - 1)/x$, $x \in I^*$, $\eta(0) = 1$ (cf. Example 2), to get a particular solution $\psi \in \mathcal{F}_f$ of inequality (1).

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