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3-BIPLACEMENT OF BIPARTITE GRAPHS

Abstract. Let $G = (L, R; E)$ be a bipartite graph with color classes L and R and edge set E. A set of two bijections $\{\varphi_1, \varphi_2\}$, $\varphi_1, \varphi_2 : L \cup R \to L \cup R$, is said to be a *3-biplacement* of G if $\varphi_1(L) = \varphi_2(L) = L$ and $E \cap \varphi_1^*(E) = \emptyset$, $E \cap \varphi_2^*(E) = \emptyset$, $\varphi_1^*(E) \cap \varphi_2^*(E) = \emptyset$, where φ_1^*, φ_2^* are the maps defined on E, induced by φ_1, φ_2, respectively.

We prove that if $|L| = p$, $|R| = q$, $3 \leq p \leq q$, then every graph $G = (L, R; E)$ of size at most p has a 3-biplacement.

Keywords: bipartite graph, packing of graphs, placement, biplacement.

Mathematics Subject Classification: 05C70.

1. INTRODUCTION

1.1. BASIC DEFINITIONS

Throughout the paper we will only consider finite, undirected graphs without loops and multiple edges.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The cardinality of the set $V(G)$ is called the *order* of G and is denoted by $|G|$, while the cardinality of the edge set $E(G)$ is the *size* of G, denoted by $\|G\|$.

For a vertex $x \in V(G)$, $N(x, G)$ denotes the set of its neighbors in G. The *degree* $d(x, G)$ of the vertex x in G is the cardinality of the set $N(x, G)$. A vertex x of G is said to be *pendent* (resp. *isolated*) if $d(x, G) = 1$ (resp. $d(x, G) = 0$).

A set of pairwise non-incident edges in a graph G is called a *matching*.

Let G_1 and G_2 be vertex disjoint graphs. The *union* $G = G_1 \cup G_2$ is a graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph G is the union of k disjoint copies of a graph H, then we write $G = kH$.

Let $G = (L, R; E)$ be a bipartite graph with vertex set $V(G) = L \cup R$ and edge set $E(G) = E$. We denote then $L(G) = L$ and $R(G) = R$, and we call these sets the *left* and *right set of bipartition* of the vertex set of G.
We denote by $\Delta_L(G)$ (resp. $\Delta_R(G)$) the maximum vertex degree in the set L (resp. R).

If $|L| = p$ and $|R| = q$, we say that G is a (p,q)-*bipartite graph.* $K_{p,q}$ stands for the complete (p,q)-bipartite graph. $\overline{G}^{bip}$ is the complement of G in $K_{p,q}$. Thus $\overline{G}^{bip} = (L, R; E')$, where E' consists of all the edges joining L with R which are not in E.

1.2. 2-PLACEMENT AND 3-PLACEMENT OF SIMPLE GRAPHS

Definition 1. *Let G be a simple graph. We say that G is 2-placeable if there exists a bijection $\varphi : V(G) \to V(G)$ such that*

$$\text{if } xy \in E(G), \quad \text{then } \varphi(x)\varphi(y) \notin E(G).$$

The bijection φ will be called a 2-placement of G.

The study of placing problems was initiated by a series of papers published in the late 1970s. The following theorem, proved by Sauer and Spencer [3], was the first result in this area.

Theorem A. *Let G be a graph of order n. If $\|G\| \le n-2$, then G is 2-placeable.*

This theorem can be generalized in a great variety of ways. Woźniak and Wojda [5] showed that under the assumptions of Theorem A there exists a 3-placement of a given graph G, unless G is an exception (see Theorem B below).

A 3-placement of a given graph can be defined analogously to a 2-placement.

Definition 2. *Let G be a simple graph of order n. A graph G is 3-placeable if there exist bijections $\varphi_1, \varphi_2 : V(G) \to V(G)$ such that $E(G) \cap \varphi_1^*(E(G)) = \emptyset$, $E(G) \cap \varphi_2^*(E(G)) = \emptyset$, $\varphi_1^*(E(G)) \cap \varphi_2^*(E(G)) = \emptyset$, where the map φ_i^* defined on $E(G)$ is induced by φ_i $(i = 1, 2)$, that is $\varphi_i^*(xy) = \varphi_i(x)\varphi_i(y)$.*
The set $\{\varphi_1, \varphi_2\}$ is called a 3-placement of G.

Woźniak and Wojda proved the following theorem.

Theorem B. *Let G be a simple graph of order n. If $\|G\| \le n-2$, then either G is 3-placeable or G is isomorphic to $K_3 \cup 2K_1$ or to $K_4 \cup 4K_1$.*

Exhaustive surveys of the results concerning the problems of placing of simple graphs are given in [1, Chapter 8] and [4]. However, we would like to focus on placements of bipartite graphs, the so-called biplacements, defined by Fouquet and Wojda [2] in 1993.

1.3. 2-BIPLACEMENT AND 3-BIPLACEMENT OF BIPARTITE GRAPHS

Definition 3. *Let $G = (L, R; E)$ be a bipartite graph. We say that G is 2-biplaceable if there exists a bijection $\varphi : L \cup R \to L \cup R$ such that $\varphi(L) = L$ and*

$$if \ xy \in E, \ then \ \varphi(x)\varphi(y) \notin E.$$

The bijection φ is called a 2-biplacement of G.

Fouquet and Wojda [2] proved the following theorem, which is an analogue of Theorem A for bipartite graphs.

Theorem C. *Let $G = (L, R; E)$ be a (p, q)-bipartite graph such that either $p \geq 3$, $q \geq 3$ and $\|G\| \leq p+q-3$, or $2 = p \leq q$ and $\|G\| \leq p+q-2$. Then G is 2-biplaceable.*

The aim of this paper is to find a sufficient condition for a bipartite graph to be 3-biplaceable; in other words, find an analogue of Theorem B for bipartite graphs.

By analogy to a 2-biplacement we consider a 3-biplacement of a bipartite graph.

Let $G = (L, R; E)$ be a (p, q)-bipartite graph. Then G can be considered as a subgraph of the complete bipartite graph $K_{p,q}$.

Definition 4. *The graph $G = (L, R; E)$ is 3-biplaceable if there exist bijections $\varphi_1, \varphi_2 : L \cup R \to L \cup R$ such that $\varphi_1(L) = \varphi_2(L) = L$ and $E \cap \varphi_1^*(E) = \emptyset$, $E \cap \varphi_2^*(E) = \emptyset$, $\varphi_1^*(E) \cap \varphi_2^*(E) = \emptyset$, where the maps $\varphi_1^*, \varphi_2^* : E \to E(K_{p,q})$ are induced by φ_1, φ_2, respectively (i.e., $\varphi_i^*(xy) = \varphi_i(x)\varphi_i(y)$ for $i = 1, 2$).*
The set $\{\varphi_1, \varphi_2\}$ is called a 3-biplacement of G.

It is easy to see that a (p, q)-bipartite graph G is 3-biplaceable if and only if we can find two edge-disjoint copies of G, say G_r and G_b, in the graph $\overline{G}^{bip}$. We then call the edges of G black, the edges of G_r red, the edges of G_b blue, and there is $L(G) = L(G_r) = L(G_b)$, $R(G) = R(G_r) = R(G_b)$, $E(G) \cap E(G_r) = \emptyset$, $E(G) \cap E(G_b) = \emptyset$, $E(G_r) \cap E(G_b) = \emptyset$.

Now we are ready to formulate the main result of this paper.

2. MAIN RESULT

Let G_1 denote a (2,3)-bipartite graph such that $\|G_1\| = 2$ and $\Delta_L(G_1) = 2$.

Our goal is to prove the following theorem.

Theorem 1. *Let $G = (L, R; E)$ be a (p, q)-bipartite graph, $p \leq q$ and $q \geq 3$. If $\|G\| \leq p$ then either G is 3-biplaceable or G is isomorphic to G_1.*

Proof. We will proceed by induction on $p + q$.

The assertion is easy to check for $p \leq 3$ and $q = 3$ (see Fig. 1), and hence for all $q \geq 3$.

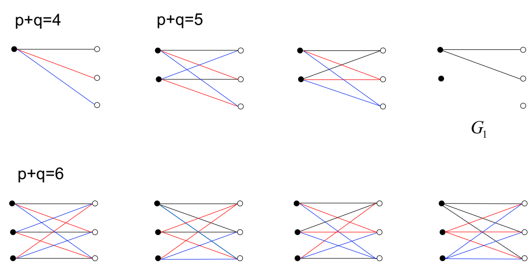

Fig. 1

Now assume that $p+q \geq 8$, $q \geq p \geq 4$, and the theorem holds for all integers $p' \geq 1$, $q' \geq 3$, such that $p' \leq q'$ and $p'+q' < p+q$.

Let $G = (L, R; E)$ be a (p,q)-bipartite graph with p and q as above. Without loss of generality, we can assume that $\|G\| = p$. We will show that G is 3-biplaceable.

In the proof, we shall consider three cases.

Case 1. $\Delta_L(G) \geq 3$.

Let $v \in L$ be a vertex such that $d(v,G) = \Delta_L(G)$. It is evident that there are at least two isolated vertices, say x and y, in L.

We define a new graph $G' := G \setminus \{v, x, y\}$. G' is (p',q')-bipartite, where $p' = p-3 \geq 1$, $q' = q \geq 4$, $p' \leq q'$. Thus $G' \neq G_1$ and $\|G'\| \leq p-3 = p'$. Hence, by the inductive hypothesis, G' is 3-biplaceable. Let $\{\varphi_1', \varphi_2'\}$ be a 3-biplacement of G'.
We define a 3-biplacement $\{\varphi_1, \varphi_2\}$ of G as follows:
$\varphi_1(v) = x$, $\varphi_1(x) = v$, $\varphi_1(y) = y$, $\varphi_1(w) = \varphi_1'(w)$ $\forall w \in V(G')$,
$\varphi_2(v) = y$, $\varphi_2(x) = x$, $\varphi_2(y) = v$, $\varphi_2(w) = \varphi_2'(w)$ $\forall w \in V(G')$.

Case 2. $\Delta_L(G) = 2$.

Pick $v \in L$ with $d(v,G) = 2$. We need to consider several subcases.

Subcase 2.1. *There is a pendent vertex in L, say x, such that $N(x,G) \cap N(v,G) = \emptyset$.*

Let $N(v,G) = \{w_1, w_2\} \subset R$, $N(x,G) = \{w_3\} \subset R$, and let y be an isolated vertex in L. We have to consider three subcases depending on the degrees of the vertices w_1, w_2, w_3.

Subcase 2.1.1. $d(w_3, G) = 1$.

Put $G' := G \setminus \{v, x, y, w_3\}$. G' is a (p',q')-bipartite graph with $p' = p-3 \geq 1$, $q' = q-1 \geq 3$, $p' \leq q'$, $\|G'\| = p'$. Obviously, G' is not isomorphic with G_1, for otherwise $p = 5$ and $q = 4$, which contradicts the assumption $p \leq q$. By the inductive hypothesis, there is a 3-biplacement of G', say $\{\varphi_1', \varphi_2'\}$. We define bijections φ_1 and φ_2 in the following way:
$\varphi_1(v) = y$, $\varphi_1(x) = v$, $\varphi_1(y) = x$, $\varphi_1(w_3) = w_3$, $\varphi_1(w) = \varphi_1'(w)$ $\forall w \in V(G')$,
$\varphi_2(v) = x$, $\varphi_2(x) = y$, $\varphi_2(y) = v$, $\varphi_2(w_3) = w_3$, $\varphi_2(w) = \varphi_2'(w)$ $\forall w \in V(G')$.
$\{\varphi_1, \varphi_2\}$ is a 3-biplaceament of G.

Subcase 2.1.2. $d(w_3, G) > 1$ *and* $d(w_1, G) = d(w_2, G) = 1$.

In the case of $p = q = 4$, we get one graph only. Obviously, it is 3-biplaceable (see Fig. 2).

Thus we can assume that $q \geq 5$. Then we define a graph $G' := G \setminus \{v, x, y, w_1, w_2\}$, which is (p',q')-bipartite with $p' = p-3 \geq 1$, $q' = q-2 \geq 3$, $p' \leq q'$. Since $\|G'\| = p'$, there exists a 3-biplacement of G', unless $G' = G_1$.

In the case of $G' = G_1$, the graph G is 3-biplaceable (see Fig. 3).

In the case of $G' \neq G_1$, let $\{\varphi_1', \varphi_2'\}$ be a 3-biplacement of G'.
To get a 3-biplacement $\{\varphi_1, \varphi_2\}$ of G, put:
$\varphi_1(v) = y$, $\varphi_1(x) = v$, $\varphi_1(y) = x$, $\varphi_1(w_1) = w_1$, $\varphi_1(w_2) = w_2$,
$\varphi_1(w) = \varphi_1'(w)$ $\forall w \in V(G')$,
$\varphi_2(v) = x$, $\varphi_2(x) = y$, $\varphi_2(y) = v$, $\varphi_2(w_1) = w_1$, $\varphi_2(w_2) = w_2$,
$\varphi_2(w) = \varphi_2'(w)$ $\forall w \in V(G')$.

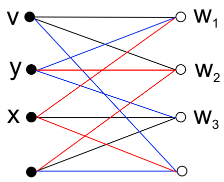

Fig. 2

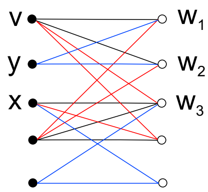

Fig. 3

Subcase 2.1.3. *$d(w_3, G) > 1$; $d(w_1, G) > 1$ or $d(w_2, G) > 1$.*

These assumptions imply that $p \geq 5$. It is easy to check that, for $q \geq p = 5$, G is 3-biplaceable. Therefore, we may assume that $q \geq p \geq 6$.

Let u_1, u_2 be isolated vertices in R and $G' := G \setminus \{v, x, y, w_3, u_1, u_2\}$. Again, G' is 3-biplaceable; let $\{\varphi_1', \varphi_2'\}$ be a 3-biplacement of G'.
A set of bijections $\{\varphi_1, \varphi_2\}$ such that
$\varphi_1(v) = y$, $\varphi_1(x) = v$, $\varphi_1(y) = x$, $\varphi_1(w_3) = u_1$, $\varphi_1(u_1) = w_3$, $\varphi_1(u_2) = u_2$, $\varphi_1(w) = \varphi_1'(w)$ $\forall w \in V(G')$,
$\varphi_2(v) = x$, $\varphi_2(x) = y$, $\varphi_2(y) = v$, $\varphi_2(w_3) = u_2$, $\varphi_2(u_1) = u_1$, $\varphi_2(u_2) = w_3$, $\varphi_2(w) = \varphi_2'(w)$ $\forall w \in V(G')$,
is then a 3-biplacement of G.

Subcase 2.2. *There is a pendent vertex in L, say x, such that $N(x, G) \cap N(v, G) \neq \emptyset$.*

Without loss of generality, we put $N(v, G) = \{w_1, w_2\}$ and $N(x, G) = \{w_2\}$.

Consequently, for all $z \in L$ of degree 2, there is $N(z, G) \supset \{w_2\}$, and for all $y \in L$ of degree 1, there is $N(y, G) \subset \{w_1, w_2\}$. Otherwise, we get Subcase 2.1.

We have to consider the following subcases.

Subcase 2.2.1. *For all $z \in L$ of degree 2, there is $N(z, G) = \{w_1, w_2\}$.*

In this case all (p, q)-bipartite graphs for $p + q = 8, 9, 10$ are 3-biplaceable, which is easily verifiable. Hence we can assume that $q \geq 6$. If so, there are at least four isolated vertices in R, say u_1, u_2, u_3, u_4.
A 3-biplacement $\{\varphi_1, \varphi_2\}$ of G is defined as follows:
$\varphi_1(w_1) = u_1$, $\varphi_1(w_2) = u_2$, $\varphi_1(u_1) = w_1$, $\varphi_1(u_2) = w_2$,
$\varphi_1(w) = w$ $\forall w \in V(G) \setminus \{w_1, w_2, u_1, u_2\}$,
$\varphi_2(w_1) = u_3$, $\varphi_2(w_2) = u_4$, $\varphi_2(u_3) = w_1$, $\varphi_2(u_4) = w_2$,
$\varphi_2(w) = w$ $\forall w \in V(G) \setminus \{w_1, w_2, u_3, u_4\}$.

Subcase 2.2.2. *There exists $z \in L$ of degree 2 such that $N(z, G) = \{w_2, w_3\}$ and $w_3 \neq w_1$.*

It follows that $p \geq 5$. Moreover, every pendent vertex in L is joined with w_2, for otherwise we would get Subcase 2.1. Consequently, all non-isolated vertices in L are joined with w_2.

Firstly, suppose that $d(w_3, G) = 1$.
A trivial verification shows that the theorem is true for $q \geq p = 5$. Therefore, assume that $p \geq 6$. Let $y_1, y_2 \in L, u \in R$ be isolated vertices in G.
Consider a graph $G' := G \setminus \{v, x, z, y_1, y_2, w_2, w_3, u\}$. $G' \neq G_1$ and by the inductive hypothesis G' is 3-biplaceable.
A 3-biplacement of G is given by the maps φ_1, φ_2 defined as:
$\varphi_1(v) = z$, $\varphi_1(x) = x$, $\varphi_1(z) = v$, $\varphi_1(y_1) = y_1$, $\varphi_1(y_2) = y_2$, $\varphi_1(w_2) = u$, $\varphi_1(w_3) = w_3$, $\varphi_1(u) = w_2$, $\varphi_1(w) = \varphi'_1(w)$ $\forall w \in V(G')$,
$\varphi_2(v) = y_1$, $\varphi_2(x) = x$, $\varphi_2(z) = y_2$, $\varphi_2(y_1) = v$, $\varphi_2(y_2) = z$, $\varphi_2(w_2) = w_3$, $\varphi_2(w_3) = u$, $\varphi_2(u) = w_2$, $\varphi_2(w) = \varphi'_2(w)$ $\forall w \in V(G')$,
where $\{\varphi'_1, \varphi'_2\}$ is a 3-biplacement of G'.

Secondly, suppose that $d(w_3, G) \geq 2$.
It follows that $d(w_1, G) \geq 2$, for if not, we would replace w_1 with w_3, and get the case proved above. Since all non-isolated vertices in L are joined with w_2, then $d(w_2, G) \geq 5$.
We conclude that $q \geq p \geq 9$ and there are at least three isolated vertices in L and six isolated vertices in R. Let us denote by y_1, y_2, y_3 isolated vertices in L and by u_1, u_2, u_3, u_4 isolated vertices in R. Consider a graph $G' := G \setminus \{v, x, z, y_1, y_2, y_3, w_2, w_3, u_1, u_2, u_3, u_4\}$. As $p \geq 9$, there is $G' \neq G_1$. Thus G' has a 3-biplacement, say $\{\varphi'_1, \varphi'_2\}$.
A 3-biplacement $\{\varphi_1, \varphi_2\}$ of G is defined below:
$\varphi_1(v) = z$, $\varphi_1(x) = x$, $\varphi_1(z) = v$, $\varphi_1(y_i) = y_i$ for $i = 1, 2, 3$, $\varphi_1(w_2) = u_1$, $\varphi_1(w_3) = u_2$, $\varphi_1(u_1) = w_2$, $\varphi_1(u_2) = w_3$, $\varphi_1(u_3) = u_3$, $\varphi_1(u_4) = u_4$, $\varphi_1(w) = \varphi'_1(w)$ $\forall w \in V(G')$,
$\varphi_2(v) = y_1$, $\varphi_2(x) = x$, $\varphi_2(z) = y_2$, $\varphi_2(y_1) = v$, $\varphi_2(y_2) = z$, $\varphi_2(y_3) = y_3$, $\varphi_2(w_2) = u_3$, $\varphi_2(w_3) = u_4$, $\varphi_2(u_1) = u_1$, $\varphi_2(u_2) = u_2$, $\varphi_2(u_3) = w_2$, $\varphi_2(u_4) = w_3$, $\varphi_2(w) = \varphi'_2(w)$ $\forall w \in V(G')$.

Subcase 2.3. *There are no pendent vertices in L.*

It follows that all vertices in L are of degree 0 or 2. Three subcases need to be considered.

Subcase 2.3.1. *There are no pendent vertices in R.*

Then we define sets:
$A := \{w \in L : \ d(w, G) = 2\}$, $B := \{w \in L : \ d(w, G) = 0\}$,
$C := \{w \in R : \ d(w, G) \geq 2\}$, $D := \{w \in R : \ d(w, G) = 0\}$.
We have $A \subset L$, $B \subset L$, $|A| = |B|$ (since $\|G\| = p$) and $C \subset R$, $D \subset R$, $|C| \leq |A|$, $|C| \leq |B|$, $|C| \leq |D|$.

It is easy to see that G is 3-biplaceable (see Fig. 4).

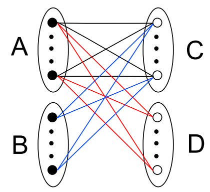

Fig. 4

Subcase 2.3.2. *There are no vertices in R of degree greater than 1.*

Set $N(v,G) = \{w_1, w_2\} \subset R$. There is $d(w_1,G) = d(w_2,G) = 1$. We deduce that there are at least two isolated vertices in L, say y_1, y_2, and, apart from v, at least one other vertex of degree 2, say x.

It is a simple matter to show that G is 3-biplaceable in the case of $q \geq p = 4$. Therefore, we assume that $p \geq 5$ and apply the inductive hypothesis to the graph $G' := G \setminus \{v, x, y_1, y_2, w_1, w_2\}$. We extend bijections φ_1' and φ_2' of a 3-biplacement of G' to φ_1 and φ_2, maps of a 3-biplacement of G, in the following way:
$\varphi_1(v) = x$, $\varphi_1(x) = v$, $\varphi_1(y_1) = y_1$, $\varphi_1(y_2) = y_2$, $\varphi_1(w_1) = w_1$, $\varphi_1(w_2) = w_2$, $\varphi_1(w) = \varphi_1'(w)\ \forall w \in V(G')$,
$\varphi_2(v) = y_2$, $\varphi_2(x) = y_1$, $\varphi_2(y_1) = x$, $\varphi_2(y_2) = v$, $\varphi_2(w_1) = w_1$, $\varphi_2(w_2) = w_2$, $\varphi_2(w) = \varphi_2'(w)\ \forall w \in V(G')$.

Subcase 2.3.3. *There is a vertex of degree 2 in L such that one of its neighbors has degree 1 and the other has degree at least 2.*

Without loss of generality, we can choose our v to be this vertex. Put $N(v,G) = \{w_1, w_2\}$ with $d(w_1,G) = 1,\ d(w_2,G) \geq 2$.

It follows that there exists a vertex $x \in L$ such that $N(x,G) = \{w_2, w_3\}$, $w_3 \neq w_1$, and there exist isolated vertices, say $y_1, y_2 \in L$ and $u \in R$.

The case of $q \geq p = 4$ is left to the reader. We assume that $q \geq p \geq 5$. In fact, since every non-isolated vertex in L has degree 2 and $\|G\| = p$, it implies that $p \geq 6$.

Let $G' := G \setminus \{v, x, y_1, y_2, w_1, w_2, u\}$. If $G' = G_1$, then G is one of the two graphs which are 3-biplaceable, which is easy to check. If $G' \neq G_1$, then by the inductive hypothesis there exists a 3-biplacement $\{\varphi_1', \varphi_2'\}$ of G'.
A 3-biplacement of G is given by the maps φ_1, φ_2 defined as
$\varphi_1(v) = x$, $\varphi_1(x) = v$, $\varphi_1(y_1) = y_1$, $\varphi_1(y_2) = y_2$, $\varphi_1(w_1) = w_1$, $\varphi_1(w_2) = u$, $\varphi_1(u) = w_2$, $\varphi_1(w) = \varphi_1'(w)\ \forall w \in V(G')$,
$\varphi_2(v) = y_1$, $\varphi_2(x) = y_2$, $\varphi_2(y_1) = v$, $\varphi_2(y_2) = x$, $\varphi_2(w_1) = w_2$, $\varphi_2(w_2) = w_1$, $\varphi_2(u) = u$, $\varphi_2(w) = \varphi_2'(w)\ \forall w \in V(G')$.

Case 3. $\Delta_L(G) = 1$.

By the assumption $\|G\| = p$, all vertices in L are pendent.

We shall consider three subcases depending on the maximum vertex degree in the set R.

Subcase 3.1. $\Delta_R(G) = 1$.

The theorem is evident in this case, since the edges of G define a matching $pK_{1,1}$.

Subcase 3.2. $\Delta_R(G) \geq 3$.

It is easily seen that the theorem is true for $q \leq 5$. For this reason, assume that $q \geq 6$. Let u be a vertex in R such that $d(u, G) = \Delta_R(G)$ and let v_1, v_2, v_3 be neighbors of u. There are at least two isolated vertices in R, say w_1, w_2. We define a graph $G' := G \setminus \{w_1, w_2, u, v_1, v_2, v_3\}$. Obviously, $G' \neq G_1$, since all vertices in L are pendent. Consequently, we may define a 3-biplacement $\{\varphi_1, \varphi_2\}$ of G as follows:
$\varphi_1(w_1) = u$, $\varphi_1(w_2) = w_2$, $\varphi_1(u) = w_1$, $\varphi_1(v_i) = v_i$ for $i = 1, 2, 3$,
$\varphi_1(w) = \varphi_1'(w)$ $\forall w \in V(G')$,
$\varphi_2(w_1) = w_1$, $\varphi_2(w_2) = u$, $\varphi_2(u) = w_2$, $\varphi_2(v_i) = v_i$ for $i = 1, 2, 3$,
$\varphi_2(w) = \varphi_2'(w)$ $\forall w \in V(G')$,
where $\{\varphi_1', \varphi_2'\}$ is a 3-biplacement of G'.

Subcase 3.3. $\Delta_R(G) = 2$.

In this case, we have to consider the two situations: either there is a pendent vertex in R or all non-isolated vertices in R are of degree 2.

Subcase 3.3.1. *There is a pendent vertex in R, say w_1.*

If $q \leq 5$, then G is 3-biplaceable, which is easy to check. Assume that $q \geq 6$. Let $w_2 \in R$ be of degree 2 and let u be an isolated vertex in R. Let $N(w_1, G) = \{v_1\}$ and $N(w_2, G) = \{v_2, v_3\}$. We may apply the inductive hypothesis to the graph $G' := G \setminus \{w_1, w_2, u, v_1, v_2, v_3\}$. Again, $G' \neq G_1$ and, in consequence, G' has a 3-biplacement, say $\{\varphi_1', \varphi_2'\}$.
A 3-biplacement $\{\varphi_1, \varphi_2\}$ of G is defined below:
$\varphi_1(w_1) = w_2$, $\varphi_1(w_2) = u$, $\varphi_1(u) = w_1$, $\varphi_1(v_i) = v_i$ for $i = 1, 2, 3$,
$\varphi_1(w) = \varphi_1'(w)$ $\forall w \in V(G')$,
$\varphi_2(w_1) = u$, $\varphi_2(w_2) = w_1$, $\varphi_2(u) = w_2$, $\varphi_2(v_i) = v_i$ for $i = 1, 2, 3$,
$\varphi_2(w) = \varphi_2'(w)$ $\forall w \in V(G')$.

Subcase 3.3.2. *There are no pendent vertices in R.*

A trivial verification shows that in the cases of $p + q = 8, 9, 10, 11$ the theorem is true. For $q \geq p \geq 6$, we define a graph $G' := G \setminus \{w_1, w_2, u, v_1, v_2, v_3, v_4\}$, where $w_1, w_2 \in R$ are vertices of degree 2, u is an isolated vertex in R, v_1, v_2 and v_3, v_4 are neighbors of w_1 and w_2, respectively. G' is 3-biplaceable, hence so is G: put $\{\varphi_1, \varphi_2\}$ to be:
$\varphi_1(w_1) = w_2$, $\varphi_1(w_2) = u$, $\varphi_1(u) = w_1$, $\varphi_1(v_i) = v_i$ for $i = 1, 2, 3, 4$,
$\varphi_1(w) = \varphi_1'(w)$ $\forall w \in V(G')$,
$\varphi_2(w_1) = u$, $\varphi_2(w_2) = w_1$, $\varphi_2(u) = w_2$, $\varphi_2(v_i) = v_i$ for $i = 1, 2, 3, 4$,
$\varphi_2(w) = \varphi_2'(w)$ $\forall w \in V(G')$,
where $\{\varphi_1', \varphi_2'\}$ is a 3-biplacement of G'. $\square$

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