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APPROXIMATE SOLUTIONS OF A SINGULAR INTEGRAL EQUATION WITH CAUCHY KERNELS IN THE QUARTER PLANE

Abstract. In the paper, we present explicit formulae for the solution of the singular integral equation with Cauchy kernels in the quarter plane. Next, Jacobi and Chebyshev polynomials are used to derive approximate solutions of this equation.

Keywords: singular integral equation, Cauchy kernel, Chebyshev and Jacobi polynomials.

Mathematics Subject Classification: 65R20, 45L10, 45E05.

1. INTRODUCTION

Let us consider the equation with a multiplicative Cauchy kernel on the quarter plane $D = \{(x, y) \in \mathbb{R}^2 : 0 < x < \infty,\ 0 < y < \infty\}$, of the form

$$\frac{1}{(\pi i)^2} \iint\limits_D \frac{\varphi(\sigma_1, \sigma_2)}{(\sigma_1 - x)(\sigma_2 - y)} d\sigma_1 d\sigma_2 = f(x, y), \quad (x, y) \in D, \tag{1}$$

where f is a given complex-valued Hölder continuous function ($f \in H(\mu_1, \mu_2)$) and $\varphi(x, y)$ is an unknown function.

Definition 1. *We say that a function $f(x, y)$, $x > 0$, $y > 0$, belongs to the Hölder class $H(\mu_1, \mu_2)$ if $f^*(t_1, t_2) = f\left(\frac{1+t_1}{1-t_1}, \frac{1+t_2}{1-t_2}\right)$, $(t_1, t_2) \in [-1, 1) \times [-1, 1)$ satisfies the inequality*

$$|f^*(t_1', t_2') - f^*(t_1'', t_2'')| \le K_1 |t_1' - t_1''|^{\mu_1} + K_2 |t_2' - t_2''|^{\mu_2} \tag{2}$$

where $0 < \mu_1, \mu_2 \le 1$, and $K_1 > 0$, $K_2 > 0$ are constants independent of the choice of points $(t_1', t_2') \in [-1, 1) \times [-1, 1)$, $(t_1'', t_2'') \in [-1, 1) \times [-1, 1)$.

The theory of singular integral equations of form (1) was presented in [7], where the general solutions in the class of Hölder functions were constructed and their uniqueness was considered. In the present paper, we briefly recall explicit formulae for the solution of (1) and then present numerical schemes for solving (1), based on Jacobi and Chebyshev polynomials.

Next, we introduce some function classes used in the description of (1).

Definition 2. *We write* $\varphi(x,y) \in h(\infty) \times h(\infty),\ x > 0,\ y > 0,$ *if the function*

$$\varphi^*(t_1,t_2) = \varphi\left(\frac{1+t_1}{1-t_1}, \frac{1+t_2}{1-t_2}\right),\quad (t_1,t_2) \in (-1,1) \times (-1,1),$$

belongs to the class $h(1) \times h(1)$ *([5]), i.e.,* $\varphi^*(t_1,t_2)$ *satisfies Hölder inequality* (2) *in each closed rectangle contained in the domain* $(-1,1) \times (-1,1)$*, the representation*

$$\varphi^*(t_1,t_2) = \varphi^{**}(t_1,t_2)(1+t_1)^{-\alpha}(1+t_2)^{-\beta},\ \ 0 \le \text{Re}\ \alpha, \text{Re}\ \beta < 1,\ \varphi^{**} \in H(\mu_1,\mu_2),$$

is valid near the lines $t_1 = -1,\ t_2 = -1$*, and moreover,*

$$\lim_{t_1 \to 1^{-0}} \varphi^*(t_1,t_2) = \lim_{x\to\infty} \varphi(x,y) = 0, \qquad \text{for } t_2 \in (-1,1),\ (y \in (0,+\infty)),$$
$$\lim_{t_2 \to 1^{-0}} \varphi^*(t_1,t_2) = \lim_{y\to\infty} \varphi(x,y) = 0, \qquad \text{for } t_1 \in (-1,1),\ (x \in (0,+\infty)).$$

Definition 3. *We write* $\varphi(x,y) \in h(0,\infty) \times h(0,\infty),\ x \ge 0,\ y \ge 0,$ *if the function*

$$\varphi^*(t_1,t_2) = \varphi\left(\frac{1+t_1}{1-t_1}, \frac{1+t_2}{1-t_2}\right),\quad (t_1,t_2) \in [-1,1) \times [-1,1)$$

belongs to the class $h(-1,1) \times h(-1,1)$ *([5]), i.e., satisfies Hölder inequality* (2) *in* $[-1,1) \times [-1,1)$*, and vanishes at infinity*

$$\lim_{t_1 \to 1} \varphi^*(t_1,t_2) = \lim_{x\to\infty} \varphi(x,y) = 0, \qquad \text{for } y \in [0,+\infty),$$
$$\lim_{t_2 \to 1} \varphi^*(t_1,t_2) = \lim_{y\to\infty} \varphi(x,y) = 0, \qquad \text{for } x \in [0,+\infty).$$

2. EXACT SOLUTIONS

In this section we present solutions of (1) in the classes $h(\infty) \times h(\infty)$ and $h(0,\infty) \times h(0,\infty)$.

Theorem 1. *Let a complex valued function* $f(x,y)$ *belong to the class* $\in h(0,\infty) \times h(0,\infty)$*. Then each solution of* (1) *in the class* $h(\infty) \times h(\infty)$ *takes the form*

$$\varphi(x,y) = \frac{1}{\sqrt{xy}} R_1(f;x,y) + iC_1(x)\frac{1}{\sqrt{y}} + iC_2(y)\frac{1}{\sqrt{x}},\quad x > 0,\ y > 0, \tag{3}$$

$$R_1(f;x,y) = \frac{1}{(\pi i)^2} \int_0^{+\infty}\int_0^{+\infty} \sqrt{\sigma_1\sigma_2}\frac{x+1}{\sigma_1+1}\frac{y+1}{\sigma_2+1}\frac{f(\sigma_1,\sigma_2)}{(\sigma_1-x)(\sigma_2-y)} d\sigma_1 d\sigma_2,$$

where $C_1(x)$ *and* $C_2(y)$ *are arbitrary functions of class* $h(\infty)$*.*

In addition, if it is required that the conditions

$$\frac{1}{\pi i}\int\limits_{0}^{+\infty}\varphi(x,\sigma_2)\frac{d\sigma_2}{\sigma_2+1}=g(x), \qquad x>0, \tag{4}$$

$$\frac{1}{\pi i}\int\limits_{0}^{+\infty}\varphi(\sigma_1,y)\frac{d\sigma_1}{\sigma_1+1}=h(y), \qquad y>0, \tag{5}$$

are fulfilled, where $g(x)$ *and* $h(y)$ *are given functions of class* $h(\infty)$*, such that*

$$\frac{1}{\pi i}\int_0^{+\infty}\frac{g(\sigma_1)}{\sigma_1+1}d\sigma_1=\frac{1}{\pi i}\int_0^{+\infty}\frac{h(\sigma_2)}{\sigma_2+1}d\sigma_2, \tag{6}$$

then the solution $\varphi(x,y)$ *is determined uniquely, and given by the following formula*

$$\varphi(x,y)=\frac{1}{\sqrt{xy}}R_1(f;x,y)+i\left(\frac{g(x)}{\sqrt{y}}+\frac{h(y)}{\sqrt{x}}\right)+\frac{A}{\sqrt{xy}}, \quad x>0,\ y>0,$$

where

$$A=\frac{1}{\pi i}\int_0^{+\infty}\frac{g(\sigma_1)}{\sigma_1+1}d\sigma_1=\frac{1}{\pi i}\int_0^{+\infty}\frac{h(\sigma_2)}{\sigma_2+1}d\sigma_2.$$

Proof. Taking into account that

$$\frac{1}{\sigma_1-x}=\frac{x+1}{\sigma_1+1}\frac{1}{\sigma_1-x}+\frac{1}{\sigma_1+1}, \qquad \frac{1}{\sigma_2-y}=\frac{y+1}{\sigma_2+1}\frac{1}{\sigma_2-y}+\frac{1}{\sigma_2+1},$$

and using the substitutions

$$x=\frac{1+t_1}{1-t_1}, \qquad \sigma_1=\frac{1+\tau_1}{1-\tau_1}, \qquad y=\frac{1+t_2}{1-t_2}, \qquad \sigma_2=\frac{1+\tau_2}{1-\tau_2}$$

we can rewrite equation (1) as

$$\begin{gathered}\frac{1}{(\pi i)^2}\int\limits_{-1}^{1}\int\limits_{-1}^{1}\frac{\varphi^*(\tau_1,\tau_2)}{(\tau_1-t_1)(\tau_2-t_2)}d\tau_1d\tau_2+G(t_1)+H(t_2)-G(1)=f^*(t_1,t_2),\\ (t_1,t_2)\in(-1,1)^2,\ \ f(\infty,y)=f^*(1,t_2)=0, \quad f(x,\infty)=f^*(t_1,1)=0,\end{gathered} \tag{7}$$

where

$$\varphi^*(t_1,t_2)=\varphi\left(\frac{1+t_1}{1-t_1},\frac{1+t_2}{1-t_2}\right), \quad f^*(t_1,t_2)=f\left(\frac{1+t_1}{1-t_1},\frac{1+t_2}{1-t_2}\right),$$

$$G(t_1)=\frac{1}{\pi i}\int\limits_{-1}^{1}g^*(\tau_1)\frac{d\tau_1}{\tau_1-t_1}, \quad H(t_2)=\frac{1}{\pi i}\int\limits_{-1}^{1}h^*(\tau_2)\frac{d\tau_2}{\tau_2-t_2}, \tag{8}$$

$$
\begin{aligned}
g^*(t_1) &= g\left(\frac{1+t_1}{1-t_1}\right) = \frac{1}{\pi i}\int_{-1}^{1} \varphi^*(t_1,\tau_2)\frac{d\tau_2}{1-\tau_2},\\
h^*(t_2) &= h\left(\frac{1+t_2}{1-t_2}\right) = \frac{1}{\pi i}\int_{-1}^{1} \varphi^*(\tau_1,t_2)\frac{d\tau_1}{1-\tau_1}.
\end{aligned} \tag{9}
$$

By solving (7) in the class $h(1)\times h(1)$, we obtain (cf. [9])

$$
\begin{aligned}
\varphi^*(t_1,t_2) = r_1(t_1)r_2(t_2)\Bigg[& R_1^*(f^*;t_1,t_2)\\
& - i\left\{\frac{1}{\pi i}\int_{-1}^{1}\frac{G(\tau_1)d\tau_1}{r_1(\tau_1)(\tau_1-t_1)} + \frac{1}{\pi i}\int_{-1}^{1}\frac{H(\tau_2)d\tau_2}{r_2(\tau_2)(\tau_2-t_2)}\right\} - G(1)\Bigg],
\end{aligned} \tag{10}
$$

where

$$
R_1^*(f^*;t_1,t_2) = \frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}\frac{1}{r_1(\tau_1)r_2(\tau_2)}\frac{f^*(\tau_1,\tau_2)}{(\tau_1-t_1)(\tau_2-t_2)}d\tau_1 d\tau_2, \tag{11}
$$

$$
r_1(t_1) = \sqrt{\frac{1-t_1}{1+t_1}}, \qquad r_2(t_2) = \sqrt{\frac{1-t_2}{1+t_2}}.
$$

By a straightforward computation, applying the Poincaré-Bertrand formula, we conclude that the general solution of (7) is of the form

$$
\begin{aligned}
\varphi^*(t_1,t_2) = {} & r_1(t_1)r_2(t_2)R_1^*(f^*;t_1,t_2)+\\
& + iC_1\left(\frac{1+t_1}{1-t_1}\right)r_2(t_2) + iC_2\left(\frac{1+t_2}{1-t_2}\right)r_1(t_1),
\end{aligned} \tag{12}
$$

where $C_1\left(\frac{1+t_1}{1-t_1}\right)$ and $C_2\left(\frac{1+t_2}{1-t_2}\right)$ are arbitrary functions of class $h(1)$ (cf. [5]).

To find the functions $C_1\left(\frac{1+t_1}{1-t_1}\right)$ and $C_2\left(\frac{1+t_2}{1-t_2}\right)$ appearing in (12), we supplement (7) with condition (9), where $g^*(t_1)$ and $h^*(t_2)$ are given functions of class $h(1)$ satisfying the relation

$$
\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}\frac{\varphi^*(\tau_1,\tau_2)}{(1-\tau_1)(1-\tau_2)}d\tau_1 d\tau_2 = \frac{1}{\pi i}\int_{-1}^{1}\frac{g^*(\tau_1)}{1-\tau_1}d\tau_1 = \frac{1}{\pi i}\int_{-1}^{1}\frac{h^*(\tau_2)}{1-\tau_2}d\tau_2. \tag{13}
$$

We substitute the function $\varphi^*(t_1,t_2)$ determined by (12) in (9) and take into account the fact that the integrals containing $f^*(t_1,t_2)$ vanish, thus obtaining the desired solution of the form

$$\varphi^*(t_1,t_2) = r_1(t_1)r_2(t_2)R_1^*(f^*;t_1,t_2) + i(r_2(t_2)g^*(t_1) + r_1(t_1)h^*(t_2)) + \qquad (14)$$
$$+ \gamma r_1(t_1)r_2(t_2), \qquad (t_1,t_2) \in (-1,1)\times(-1,1),$$
$$\gamma = \frac{1}{\pi i}\int_{-1}^{1} C_1^*(\tau_1)\frac{d\tau_1}{1-\tau_1} + \frac{1}{\pi i}\int_{-1}^{1} C_2^*(\tau_2)\frac{d\tau_2}{1-\tau_2},$$

where $C_1^*(t_1) = C_1\left(\frac{1+t_1}{1-t_1}\right)$, $C_2^*(t_2) = C_2\left(\frac{1+t_2}{1-t_2}\right)$. Further, in view of (14) and (13) we get

$$A = \frac{1}{\pi i}\int_0^{+\infty} \frac{g(\sigma_1)}{\sigma_1+1}d\sigma_1 = \frac{1}{\pi i}\int_{-1}^{1} g^*(\tau_1)\frac{d\tau_1}{1-\tau_1} =$$
$$= \frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1} \frac{\varphi^*(\tau_1,\tau_2)}{(1-\tau_1)(1-\tau_2)}d\tau_1 d\tau_2 = 2A - \gamma = \gamma. \qquad \square$$

Let us find the solution of (1) in the class $h(0,\infty)\times h(0,\infty)$. The following theorem holds.

Theorem 2. *Let a complex valued function $f(x,y)$ belong to the class $h(0,\infty)\times h(0,\infty)$. If a solution $\varphi(x,y)$ of (1) belongs to the class $h(0,\infty)\times h(0,\infty)$ the conditions*

$$\frac{1}{\pi i}\int_0^{+\infty} \varphi(x,\sigma_2)\frac{d\sigma_2}{\sigma_2+1} = \frac{\sqrt{x}}{\pi i}\int_0^{+\infty} \frac{d\sigma_1}{\sqrt{\sigma_1}(\sigma_1-x)}\frac{1}{\pi}\int_0^{+\infty} \frac{f(\sigma_1,\sigma_2)d\sigma_2}{\sqrt{\sigma_2}(\sigma_2+1)}, \quad x>0,$$
$$\frac{1}{\pi i}\int_0^{+\infty} \varphi(\sigma_1,y)\frac{d\sigma_1}{\sigma_1+1} = \frac{\sqrt{y}}{\pi i}\int_0^{+\infty} \frac{d\sigma_2}{\sqrt{\sigma_2}(\sigma_2-y)}\frac{1}{\pi}\int_0^{+\infty} \frac{f(\sigma_1,\sigma_2)d\sigma_1}{\sqrt{\sigma_1}(\sigma_1+1)}, \quad y>0, \qquad (15)$$

are fulfilled, then the solution $\varphi(x,y)$ is determined uniquely, and given by the following formula

$$\varphi(x,y) = \sqrt{xy}\frac{1}{(\pi i)^2}\int_0^{+\infty}\int_0^{+\infty} \frac{f(\sigma_1,\sigma_2)}{\sqrt{\sigma_1\sigma_2}\,(\sigma_1-x)(\sigma_2-y)}d\sigma_1 d\sigma_2, \quad x>0,\ y>0. \qquad (16)$$

Proof. Let us find the solution of (7) in the class $h(-1,1)\times h(-1,1)$. By [9], this equation is solvable under the conditions

$$\frac{1}{\pi i}\int_{-1}^{1} [f^*(\tau_1,t_2) - H(t_2) - G(\tau_1) - A]\frac{d\tau_1}{q_1(\tau_1)} = 0 \quad \text{for } t_2 \in (-1,1),$$
$$\frac{1}{\pi i}\int_{-1}^{1} [f^*(t_1,\tau_2) - H(\tau_2) - G(t_1) - A]\frac{d\tau_2}{q_2(\tau_2)} = 0 \quad \text{for } t_1 \in (-1,1), \qquad (17)$$

where $q_i(t_i) = \sqrt{(1+t_i)(1-t_i)}$, $i = 1, 2$, $G(t_1)$, $H(t_2)$, $g^*(t_1)$ and $h^*(t_2)$ are determined by (8) and (9) respectively, and $A = G(1) = H(1)$. Conditions (17) can be transformed to the following form ([7]):

$$\begin{aligned} g^*(t_1) &= \frac{1}{\pi i}\int_{-1}^{1}\frac{\varphi^*(t_1,\tau_2)d\tau_2}{1-\tau_2} = q_1(t_1)\frac{1}{\pi i}\int_{-1}^{1}\frac{1}{q_1(\tau_1)}\frac{d\tau_1}{\tau_1-t_1}\frac{1}{\pi}\int_{-1}^{1}\frac{f^*(\tau_1,\tau_2)}{q_2(\tau_2)}d\tau_2, \\ h^*(t_2) &= \frac{1}{\pi i}\int_{-1}^{1}\frac{\varphi^*(\tau_1,t_2)d\tau_1}{1-\tau_1} = q_2(t_2)\frac{1}{\pi i}\int_{-1}^{1}\frac{1}{q_2(\tau_2)}\frac{d\tau_2}{\tau_2-t_2}\frac{1}{\pi}\int_{-1}^{1}\frac{f^*(\tau_1,\tau_2)}{q_1(\tau_1)}d\tau_1, \end{aligned} \tag{18}$$

which coincides with (15).

The desired solution of (7) in $h(-1,1)\times h(-1,1)$ is given by the formula

$$\varphi^*(t_1,t_2) = q_1(t_1)q_2(t_2)\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}\frac{f^*(\tau_1,\tau_2)}{q_1(\tau_1)\,q_2(\tau_2)(\tau_1-t_1)(\tau_2-t_2)}d\tau_1 d\tau_2, \tag{19}$$

which coincides with (16) in the variables x, σ. □

3. APPROXIMATE SOLUTION IN THE CLASS $h(\infty)\times h(\infty)$

In this section, we derive an approximate solution of (7), (9) in the class $h(\infty)\times h(\infty)$, using Jacobi polynomials.

In view of (14), in (7) and (9), we can express $\varphi^*(t_1,t_2)$ as

$$\varphi^*(t_1,t_2) = \sqrt{\frac{1-t_1}{1+t_1}}\sqrt{\frac{1-t_2}{1+t_2}}\,u(t_1,t_2), \tag{20}$$

where $u(t_1,t_2)$ is a new unknown function. Then, by introducing the notation $r_i = \sqrt{\frac{1-t_i}{1+t_i}}$, $i = 1, 2$, we reduce (7) to the form

$$\begin{aligned} &\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1} r_1(\tau_1)r_2(\tau_2)\frac{u(\tau_1,\tau_2)d\tau_1 d\tau_2}{(\tau_1-t_1)(\tau_2-t_2)} - \frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1} r_1(\tau_1)r_2(\tau_2)\frac{u(\tau_1,\tau_2)d\tau_1 d\tau_2}{(\tau_1-t_1)(\tau_2-1)} - \\ &-\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1} r_1(\tau_1)r_2(\tau_2)\frac{u(\tau_1,\tau_2)d\tau_1 d\tau_2}{(\tau_1-1)(\tau_2-t_2)} + \frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1} r_1(\tau_1)r_2(\tau_2)\frac{u(\tau_1,\tau_2)d\tau_1 d\tau_2}{(\tau_1-1)(\tau_2-1)} = \\ &= f^*(t_1,t_2), \quad (t_1,t_2)\in(-1,1)\times(-1,1), \quad f^*(1,t_2)=0, \quad f^*(t_1,1)=0, \end{aligned} \tag{21}$$

and conditions (9) to the following form

$$\begin{aligned} &\frac{1}{\pi i}\int_{-1}^{1} r_2(\tau_2)\,u(t_1,\tau_2)\frac{d\tau_2}{1-\tau_2} = g^{**}(t_1), \\ &\frac{1}{\pi i}\int_{-1}^{1} r_1(\tau_1)\,u(\tau_1,t_2)\frac{d\tau_1}{1-\tau_1} = h^{**}(t_2), \end{aligned} \tag{22}$$

where

$$g^{*}(t_1) = r_1(t_1)\,g^{**}(t_1), \quad h^{*}(t_2) = r_2(t_2)\,h^{**}(t_2). \tag{23}$$

Now we can construct an approximate solution of (21), (22). To this end, we interpolate the function $f^{*}\,(t_1,t_2)$ at Chebyshev nodes

$$t_j^{(k)} = \cos\frac{(2k+1)\,\pi}{2\,(n+1)}, \quad j=1,2, \;\; k=1,2,\ldots,n+1, \tag{24}$$

by a polynomial $f^{*}_{nn}\,(t_1,t_2)$ of the form (cf. [6])

$$f^{*}_{nn}\,(t_1,t_2) = \sum_{j=0}^{n}\sum_{p=0}^{n}\alpha_{jp}\,T_j(t_1)T_p(t_2), \tag{25}$$

where

$$\alpha_{jp} = \frac{\delta_j}{n+1}\sum_{k=1}^{n+1} T_j(t_1^{(k)})\left\{\frac{\delta_p}{n+1}\sum_{l=1}^{n+1} T_p(t_2^{(l)})f(t_1^{(k)},t_2^{(l)})\right\}, \tag{26}$$

$$\delta_k = \begin{cases} 1, & k=0, \\ 2, & k=1,2,\ldots, \end{cases}$$

and $T_j(t_1), T_p(t_2), t_1 \in (-1,1), t_2 \in (-1,1)$ are Chebyshev polynomials of the first kind ($T_j(t_1) = \cos\,(j\arccos t_1)$). Expressing Chebyshev polynomials in terms of Jacobi polynomials, we obtain

$$T_j\,(t_1) = \sum_{l=0}^{j}\rho_{jl}P_l^{(-\alpha,-\beta)}\,(t_1)\,, \tag{27}$$

where

$$\rho_{jl}^{(-\alpha,-\beta)} = \frac{1}{h_l^{(-\alpha,-\beta)}} \frac{1}{\pi} \int_{-1}^{1} q(t_1) T_j(t_1) P_l^{(-\alpha,-\beta)}(t_1)\, dt_1 =$$

$$= -\frac{1}{h_l^{(-\alpha,-\beta)}} \frac{1}{\sin \pi\alpha} \mathop{Res}_{z=\infty} \left\{ q(z) T_j(z) P_l^{(-\alpha,-\beta)}(z) \right\},$$

$$q(t_1) = (1-t_1)^{-\alpha} (1+t_1)^{-\beta}, \qquad \alpha+\beta = 0,$$

$$h_l^{(-\alpha,-\beta)} = \frac{1}{\pi} \int_{-1}^{1} q(t_1) \left[P_l^{(-\alpha,-\beta)}(t_1) \right]^2 dt_1 = -\frac{1}{\sin \pi\alpha} \mathop{Res}_{z=\infty} \left\{ q(z) \left(P_l^{(-\alpha,-\beta)}(z) \right)^2 \right\}.$$

Using (27), interpolation polynomial (25) takes the form

$$f_{nn}^*(t_1,t_2) = \sum_{k=0}^{n} \sum_{j=0}^{n} f_{kj}^* P_k^{(-\alpha,-\beta)}(t_1) P_j^{(-\alpha,-\beta)}(t_2), \tag{28}$$

where

$$f_{kj}^* = \sum_{i=k}^{n} \rho_{ik}^{(-\alpha,-\beta)} \sum_{p=j}^{n} \rho_{pj}^{(-\alpha,-\beta)} \alpha_{ip}, \quad 0 \le k, j \le n. \tag{29}$$

The approximate solution $u_{nn}(t_1,t_2)$ of problem (21), (22) is defined as the solution of the following problem

$$\frac{1}{(\pi i)^2} \int_{-1}^{1}\int_{-1}^{1} \frac{r_1(\tau_1) r_2(\tau_2) u_{nn}(\tau_1,\tau_2) d\tau_1 d\tau_2}{(\tau_1 - t_1)(\tau_2 - t_2)} - \frac{1}{(\pi i)^2} \int_{-1}^{1}\int_{-1}^{1} \frac{r_1(\tau_1) r_2(\tau_2) u_{nn}(\tau_1,\tau_2) d\tau_1 d\tau_2}{(\tau_1 - t_1)(\tau_2 - 1)} -$$

$$- \frac{1}{(\pi i)^2} \int_{-1}^{1}\int_{-1}^{1} \frac{r_1(\tau_1) r_2(\tau_2) u_{nn}(\tau_1,\tau_2) d\tau_1 d\tau_2}{(\tau_1 - 1)(\tau_2 - t_2)} + \frac{1}{(\pi i)^2} \int_{-1}^{1}\int_{-1}^{1} \frac{r_1(\tau_1) r_2(\tau_2) u_{nn}(\tau_1,\tau_2) d\tau_1 d\tau_2}{(\tau_1 - 1)(\tau_2 - 1)} =$$

$$= f_{nn}^*(t_1,t_2) - f_{nn}^*(t_1,1) - f_{nn}^*(1,t_2) + f_{nn}^*(1,1), \quad (t_1,t_2) \in (-1,1) \times (-1,1), \tag{30}$$

$$\frac{1}{\pi i} \int_{-1}^{1} r_2(\tau_2)\, u_{nn}(t_1,\tau_2) \frac{d\tau_2}{1-\tau_2} = g_n^{**}(t_1), \tag{31}$$

$$\frac{1}{\pi i} \int_{-1}^{1} r_1(\tau_1)\, u_{nn}(\tau_1,t_2) \frac{d\tau_1}{1-\tau_1} = h_n^{**}(t_2), \tag{32}$$

where

$$u_{nn}(t_1,t_2) = \sum_{k=0}^{n} \sum_{j=0}^{n} c_{kj} P_k^{(\alpha,\beta)}(t_1) P_j^{(\alpha,\beta)}(t_2), \tag{33}$$

with unknown coefficients c_{kj}, and $g_n^{**}(t_1)$, $h_n^{**}(t_2)$ are interpolation polynomials of the form

$$g^{**}(t_1) \approx g_n^{**}(t_1) = \sum_{k=0}^{n} g_k\, P_k^{(\alpha,\beta)}(t_1), \tag{34}$$

$$h^{**}(t_2) \approx h_n^{**}(t_2) = \sum_{j=0}^{n} h_j\, P_j^{(\alpha,\beta)}(t_2). \tag{35}$$

The coefficients g_k of polynomial (34) can be written as:

$$\begin{aligned} g_0 &= \frac{1}{n+1}\sum_{j=0}^{n}\delta_j\left(\sum_{i=1}^{n+1}T_j\left(t_1^{(i)}\right)g^{**}\left(t_1^{(i)}\right)\right)\rho_{j0}^{(\alpha,\beta)}, \\ g_k &= \frac{2}{n+1}\sum_{j=k}^{n}\left(\sum_{i=1}^{n+1}T_j\left(t_1^{(i)}\right)g^{**}\left(t_1^{(i)}\right)\right)\rho_{jk}^{(\alpha,\beta)}, \quad k=1,2,\dots,n, \\ \delta_j &= \begin{cases} 1, & j=0, \\ 2, & j=1,2,\dots \end{cases} \end{aligned} \tag{36}$$

The coefficients h_j are defined in the same way.

If we use (28) and (33) into (30), and assume that

$$\alpha = \frac{1}{2}, \quad \beta = -\frac{1}{2},$$

then, using the following formula ([9]):

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\,\frac{P_k^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\tau_1)}{\tau_1-t_1}d\tau_1 = -P_k^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t_1) \tag{37}$$

for the computation of the singular integral, we get

$$\begin{aligned} &-\sum_{k=0}^{n}\sum_{j=0}^{n}c_{kj}P_k^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t_1)P_j^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t_2) + \sum_{k=0}^{n}\sum_{j=0}^{n}c_{kj}P_k^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t_1)P_j^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1)+ \\ &+\sum_{k=0}^{n}\sum_{j=0}^{n}c_{kj}P_k^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1)P_j^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t_2) - \sum_{k=0}^{n}\sum_{j=0}^{n}c_{kj}P_k^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1)P_j^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1) = \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n}f_{kj}^{*}P_k^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t_1)P_j^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t_2) - \sum_{k=0}^{n}\sum_{j=0}^{n}f_{kj}^{*}P_k^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t_1)P_j^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1)- \\ &-\sum_{k=0}^{n}\sum_{j=0}^{n}f_{kj}^{*}P_k^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1)P_j^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t_2) + \sum_{k=0}^{n}\sum_{j=0}^{n}f_{kj}^{*}P_k^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1)P_j^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1). \end{aligned}$$

Comparing coefficients of Jacobi polynomials, we derive

$$c_{kj} = -f^*_{kj}, \quad k = 1, \dots, n, \; j = 1, \dots, n. \tag{38}$$

The remaining coefficients $c_{k0},\ c_{0j},\ 0 \le k, j \le n$ can be computed from (31) and (32). Substituting (33) for $u_{nn}\,(t_1, t_2)$ and (34) for $g_n^{**}\,(t_1)$ into condition (31), we obtain

$$\sum_{k=0}^{n}\sum_{j=0}^{n} c_{kj} P_k^{\left(\frac{1}{2},-\frac{1}{2}\right)}(t_1) \frac{1}{\pi i}\int_{-1}^{1} \sqrt{\frac{1-\tau_2}{1+\tau_2}} \frac{P_j^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\tau_2)}{(1-\tau_2)} d\tau_2 = \sum_{k=0}^{n} g_k\, P_k^{\left(\frac{1}{2},-\frac{1}{2}\right)}(t_1).$$

Hence, by (37), we get

$$c_{k0} = i g_k - \sum_{j=1}^{n} c_{kj} P_j^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1), \quad k = n, \dots, 1. \tag{39}$$

Further, substituting (33) and (34) into condition (32) and taking into account (37), we obtain

$$c_{0j} = i h_j - \sum_{k=1}^{n} c_{kj}\, P_k^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1), \quad j = n, \dots, 0. \tag{40}$$

4. EXAMPLE

In this section, we present a numerical example, which illustrates the usefulness of the approximation algorithm presented in the previous section. Let

$$\begin{aligned} f(x,y) &= \frac{1}{(2x+3)(2y+3)}, && x \ge 0,\ y \ge 0,\\ g(x) &= \frac{1}{\sqrt{x}}, && x > 0,\\ h(y) &= \frac{1}{\sqrt{y}}, && y > 0. \end{aligned}$$

Then

$$\begin{aligned} f^*(t_1,t_2) &= f\left(\frac{1+t_1}{1-t_1}, \frac{1+t_2}{1-t_2}\right) = \frac{(t_1-1)(t_2-1)}{(t_1-5)(t_2-5)}, \quad (t_1,t_2) \in (-1,1)\times(-1,1),\\ g^*(t_1) &= g\left(\frac{1+t_1}{1-t_1}\right) = \sqrt{\frac{1-t_1}{1+t_1}}, \quad g^{**}(t_1) \equiv 1, \quad t_1 \in (-1,1),\\ h^*(t_2) &= h\left(\frac{1+t_2}{1-t_2}\right) = \sqrt{\frac{1-t_2}{1+t_2}}, \quad h^{**}(t_2) \equiv 1, \quad t_2 \in (-1,1),\\ R_1^*(f^*;t_1,t_2) &= -\left(\frac{2\sqrt{6}}{t_1-5}+1\right)\left(\frac{2\sqrt{6}}{t_2-5}+1\right). \end{aligned}$$

Owing to (14), the exact solution of (7), (9) in the class $h(1) \times h(1)$ is the function

$$\varphi^*(t_1, t_2) = \sqrt{\frac{1-t_1}{1+t_1}}\sqrt{\frac{1-t_2}{1+t_2}}\left(i - \left(\frac{2\sqrt{6}}{t_1 - 5} + 1\right)\left(\frac{2\sqrt{6}}{t_2 - 5} + 1\right)\right).$$

Thus the solution of problem (1),(4)–(6) is given by the formula

$$\varphi(x, y) = \frac{1}{\sqrt{xy}}\left[i - \left(-\frac{\sqrt{6}(x+1)}{2x+3} + 1\right)\left(-\frac{\sqrt{6}(y+1)}{2y+3} + 1\right)\right], \quad x > 0,\ y > 0.$$

The values of $u(t_1, t_2)$, $u_{nn}(t_1, t_2)$ and $e_n = |u(t_1, t_2) - u_{nn}(t_1, t_2)|$ for $n = 10$ are shown in Table 1.

Table 1. Comparison of the values of $u(t_1, t_2)$, $u_{nn}(t_1, t_2)$

t_1	t_2	$u(t_1, t_2)$	$u_{nn}(t_1, t_2)$	e_n
0	0	$-0.000408205773457 + 1i$	$-0.000408205774090 + 1i$	$6.326e - 13$
-0.333	0.980	$0.017812170889016 + 1i$	$0.017812170871103 + 1i$	$1.791e - 11$
0.998	0.130	$-0.001353862226909 + 1i$	$-0.001353862232197 + 1i$	$5.288e - 12$
0.935	-0.538	$0.023704873141732 + 1i$	$0.023704873145931 + 1i$	$4.199e - 12$
0.999	0.999	$-0.050427992631078 + 1i$	$-0.050427992427745 + 1i$	$2.033e - 10$

In Table 2, we tabulate the values of the exact and approximate solutions of (1) for $n = 10$.

Table 2. Comparison of the values of the exact and approximated solutions of (1) in the class $h(\infty) \times h(\infty)$

x	y	$\varphi(x, y)$	$\varphi_{nn}(x, y)$
1	1	$-0.0004082057734575 + 1i$	$-0.0004082057740900 + 1i$
0.5	100	$0.002519021364655 + 0.1414213i$	$0.002519021362122 + 0.1414213i$
1000	1.3	$-0.000037549382146 + 0.0277350i$	$-0.000037549382293 + 0.0277350i$
30	0.3	$0.007901624380577 + 0.3333333i$	$0.007901624381977 + 0.3333333i$
10 000	2007	$-0.000011256360524 + 0.0002232i$	$-0.000011256360479 + 0.0002232i$

5. APPROXIMATE SOLUTION IN THE CLASS $h(0, \infty) \times h(0, \infty)$

In this section, we construct an approximate solution of (7) in the $h(0, \infty) \times h(0, \infty)$ class, using Chebyshev polynomials. In the sequel, it will be useful to express $\varphi^*(t_1, t_2)$ as:

$$\varphi^*(t_1, t_2) = q_1(t_1) q_2(t_2)\, u(t_1, t_2), \tag{41}$$

where $u(t_1, t_2)$ is a new unknown function and $q_i(t_i) = \sqrt{1 - t_i^2},\ i = 1, 2$.

In view of (41), equation (7) can be represented in the form

$$\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}\frac{q_1(\tau_1)q_2(\tau_2)u(\tau_1,\tau_2)d\tau_1 d\tau_2}{(\tau_1-t_1)(\tau_2-t_2)}-\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}\frac{q_1(\tau_1)q_2(\tau_2)u(\tau_1,\tau_2)d\tau_1 d\tau_2}{(\tau_1-t_1)(\tau_2-1)}-$$
$$-\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}\frac{q_1(\tau_1)q_2(\tau_2)u(\tau_1,\tau_2)d\tau_1 d\tau_2}{(\tau_1-1)(\tau_2-t_2)}+\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}q_1(\tau_1)q_2(\tau_2)\frac{u(\tau_1,\tau_2)d\tau_1 d\tau_2}{(\tau_1-1)(\tau_2-1)}=$$
$$=f^*(t_1,t_2),\qquad (t_1,t_2)\in(-1,1)\times(-1,1),\quad f^*(1,t_2)=0,\quad f^*(t_1,1)=0, \tag{42}$$

and necessary and sufficient conditions (18) can be rewritten in the form

$$\begin{aligned}\frac{1}{\pi i}\int_{-1}^{1}q_2(\tau_2)u(t_1,\tau_2)\frac{d\tau_2}{1-\tau_2}&=\frac{1}{\pi i}\int_{-1}^{1}\frac{1}{q_1(\tau_1)}\frac{d\tau_1}{\tau_1-t_1}\frac{1}{\pi}\int_{-1}^{1}\frac{f^*(\tau_1,\tau_2)}{q_2(\tau_2)}d\tau_2,\\ \frac{1}{\pi i}\int_{-1}^{1}q_1(\tau_1)u(\tau_1,t_2)\frac{d\tau_1}{1-\tau_1}&=\frac{1}{\pi i}\int_{-1}^{1}\frac{1}{q_2(\tau_2)}\frac{d\tau_2}{\tau_2-t_2}\frac{1}{\pi}\int_{-1}^{1}\frac{f^*(\tau_1,\tau_2)}{q_1(\tau_1)}d\tau_1.\end{aligned} \tag{43}$$

Now, we derive an approximate solution of (42), (43). The approximate solution $u_{n-1,n-1}(t_1,t_2)$ of problem (42), (43) is defined to be the solution of the following problem:

$$\begin{aligned}&\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}q_1(\tau_1)q_2(\tau_2)\frac{u_{n-1,n-1}(\tau_1,\tau_2)d\tau_1 d\tau_2}{(\tau_1-t_1)(\tau_2-t_2)}-\\ &-\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}q_1(\tau_1)q_2(\tau_2)\frac{u_{n-1,n-1}(\tau_1,\tau_2)d\tau_1 d\tau_2}{(\tau_1-t_1)(\tau_2-1)}-\\ &-\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}q_1(\tau_1)q_2(\tau_2)\frac{u_{n-1,n-1}(\tau_1,\tau_2)}{(\tau_1-1)(\tau_2-t_2)}d\tau_1 d\tau_2+\\ &+\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{-1}^{1}q_1(\tau_1)q_2(\tau_2)\frac{u_{n-1,n-1}(\tau_1,\tau_2)}{(\tau_1-1)(\tau_2-1)}d\tau_1 d\tau_2=\\ &=f^*_{nn}(t_1,t_2)-f^*_{nn}(1,t_2)-f^*_{nn}(t_1,1)+f^*_{nn}(1,1),\qquad (t_1,t_2)\in(-1,1)\times(-1,1),\end{aligned} \tag{44}$$

$$\frac{1}{\pi i}\int_{-1}^{1} q_2(\tau_2)\frac{u_{n-1,n-1}(t_1,\tau_2)d\tau_2}{1-\tau_2} = \frac{1}{\pi i}\int_{-1}^{1}\frac{1}{q_1(\tau_1)}\frac{d\tau_1}{\tau_1-t_1}\frac{1}{\pi}\int_{-1}^{1}\frac{f^*_{nn}(\tau_1,\tau_2)}{q_2(\tau_2)}d\tau_2, \quad (45)$$

$$\frac{1}{\pi i}\int_{-1}^{1} q_1(\tau_1)\frac{u_{n-1,n-1}(\tau_1,t_2)d\tau_1}{1-\tau_1} = \frac{1}{\pi i}\int_{-1}^{1}\frac{1}{q_2(\tau_2)}\frac{d\tau_2}{\tau_2-t_2}\frac{1}{\pi}\int_{-1}^{1}\frac{f^*_{nn}(\tau_1,\tau_2)}{q_1(\tau_1)}d\tau_1, \quad (46)$$

where f_{nn} is the polynomial interpolating the function $f^*(t_1,t_2)$, defined in the same way as in (25), and $u_{n-1,n-1}$ is the polynomial of the following form:

$$u_{n-1,n-1}(t_1,t_2) = \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} c_{kj}\, U_k(t_1)\, U_j(t_2), \quad (47)$$

where $U_k(t_1)$, $U_j(t_2)$ are Chebyshev polynomials of the second kind, and c_{kj} are the coefficients to be determined.

To make the right-hand side of (44) vanish at $t_1 = 1$ and $t_2 = 1$, we have added three components $-f^*_{nn}(t_1,1) - f^*_{nn}(1,t_2) + f^*_{nn}(1,1)$ to $f^*_{nn}(t_1,t_2)$.

Substituting (47) and (25) into (44), and using the following formula ([10]):

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-\tau^2}\,\frac{U_{j-1}(\tau)}{\tau-t}d\tau = -T_j(t) \quad (48)$$

for the computation of the singular integral, we get

$$\begin{aligned} &-\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} c_{kj}T_{k+1}(t_1)T_{j+1}(t_2) + \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} c_{kj}T_{j+1}(t_2) + \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} c_{kj}T_{k+1}(t_1) - \\ &-\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} c_{kj} = \sum_{k=0}^{n}\sum_{j=0}^{n} f^*_{kj}\, T_k(t_1)T_j(t_2) - \sum_{k=0}^{n}\sum_{j=0}^{n} f^*_{kj}\, T_j(t_2) - \\ &-\sum_{k=0}^{n}\sum_{j=0}^{n} f^*_{kj}\, T_k(t_1) + \sum_{k=0}^{n}\sum_{j=0}^{n} f^*_{kj}, \quad (t_1,t_2)\in(-1,1)\times(-1,1). \end{aligned} \quad (49)$$

Hence, by matching the coefficients of Chebyshev polynomials, we obtain

$$c_{kj} = -f^*_{k+1,j+1}, \quad 1 \le j,k \le n-1. \quad (50)$$

We find the remaining coefficients c_{k0}, c_{0j}, $0 \le k,j \le n-1$ using condition (45) and (46). If we substitute (47) for $u_{n-1,n-1}(t_1,t_2)$ and (25) for $f_{nn}(t_1,t_2)$ in (45), then using the formula ([6]):

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-\tau^2}}\frac{T_j(\tau)}{\tau-t}d\tau = U_{j-1}(t), \quad U_{-1}(t)\equiv 0, \quad (51)$$

and using the orthogonality of Chebyshev polynomials ([4, 6]), we get

$$\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} c_{kj} U_k(t_1) = \sum_{k=0}^{n-1} f_{k+1,0} U_k(t_1).$$

Hence

$$c_{k,0} = f^*_{k+1,0} - \sum_{j=1}^{n-1} c_{k,j}, \quad k = n-1, \ldots, 1. \tag{52}$$

Next, if we substitute (47) for $u_{n-1,n-1}(t_1, t_2)$ and (25) for $f_{nn}(t_1, t_2)$ into (46), then again using (48) and (51) for the computation of the singular integrals, and using the orthogonality of Chebyshev polynomials, we derive the remaining coefficients

$$c_{0,j} = f^*_{0,j+1} - \sum_{k=1}^{n-1} c_{k,j}, \quad j = n-1, \ldots, 0. \tag{53}$$

6. EXAMPLE

Let us consider the function

$$f(x, y) = \frac{4}{(x+3)(y+3)}, \quad x \geq 0, \ y \geq 0.$$

Then

$$f^*(t_1, t_2) = f\left(\frac{1+t_1}{1-t_1}, \frac{1+t_2}{1-t_2}\right) = \frac{(1-t_1)(1-t_2)}{(t_1-2)(t_2-2)}, \quad (t_1, t_2) \in (-1, 1) \times (-1, 1).$$

By (19), the solution of this problem is:

$$\varphi(x, y) = 4\sqrt{x\,y}\,\frac{1}{\pi i}\int_0^{+\infty} \frac{d\sigma_1}{\sqrt{\sigma_1}(\sigma_1+3)(\sigma_1-x)}\,\frac{1}{\pi i}\int_0^{+\infty} \frac{d\sigma_2}{\sqrt{\sigma_2}(\sigma_2+3)(\sigma_2-y)} =$$

$$= \sqrt{1-t_1^2}\sqrt{1-t_2^2}\,\frac{1}{\pi i}\int_{-1}^{1} \sqrt{\frac{1-\tau_1}{1+\tau_1}}\,\frac{d\tau_1}{(\tau_1-2)(\tau_1-t_1)}\,\frac{1}{\pi i}\int_{-1}^{1} \sqrt{\frac{1-\tau_2}{1+\tau_2}}\,\frac{d\tau_2}{(\tau_2-2)(\tau_2-t_2)} =$$

$$= -\frac{\sqrt{1-t_1^2}\sqrt{1-t_2^2}}{3(t_1-2)(t_2-2)} = -\frac{4}{3}\,\frac{\sqrt{x\,y}}{(x+3)(y+3)}, \quad x \geq 0, \ y \geq 0,$$

and the following solvability conditions (15):

$$\frac{1}{\pi i}\int_0^{+\infty} \frac{\varphi(x, \sigma_2)\,d\sigma_2}{\sigma_2+1} = \frac{i}{3}\,\frac{\sqrt{x}}{x+3}(\sqrt{3}-1), \quad \frac{1}{\pi i}\int_0^{+\infty} \frac{\varphi(\sigma_1, y)\,d\sigma_1}{\sigma_1+1} = \frac{i}{3}\,\frac{\sqrt{y}}{y+3}(\sqrt{3}-1)$$

are satisfied.

The values of $u(t_1, t_2)$, $u_{n-1,n-1}(t_1, t_2)$ and $e_n = |u(t_1, t_2) - u_{n-1,n-1}(t_1, t_2)|$ for $n = 10$ are shown in Table 3.

Table 3. Comparison of the values of $u(t_1, t_2)$, $u_{n-1,n-1}(t_1, t_2)$

t_1	t_2	$u(t_1, t_2)$	$u_{n-1,n-1}(t_1, t_2)$	e_n
0	0	$-0.08333333333333 + 0i$	$-0.08333378662841 + 0i$	$4.533e-7$
-0.333	0.980	$-0.14008321775312 + 0i$	$-0.14008238162255 + 0i$	$8.361e-7$
0.998	0.130	$-0.17793905106386 + 0i$	$-0.17793637013586 + 0i$	$2.681e-6$
0.935	-0.538	$-0.12335475971839 + 0i$	$-0.12335583044532 + 0i$	$1.071e-6$
0.999	0.999	$-0.33293507801722 + 0i$	$-0.33292316267308 + 0i$	$1.192e-5$

In Table 4, we tabulate the values of the exact and approximated solutions of (1) in the $h(0,\infty) \times h(0,\infty)$ class for $n = 10$.

Table 4. Comparison of the values of the exact and approximated solutions of (1) in the class $h(0,\infty) \times h(0,\infty)$

x	y	$\varphi(x, y)$	$\varphi_{n-1,n-1}(x, y)$
1	1	$-0.083333333333333 + 0i$	$-0.083333786628411 + 0i$
0.5	100	$-0.019810758640732 + 0i$	$-0.019810640394014 + 0i$
1000	1.3	$-0.004935141327062 + 0i$	$-0.004935066971498 + 0i$
30	0.3	$-0.041118253239465 + 0i$	$-0.041118610148441 + 0i$
10000	2007	$-0.000074316606191 + 0i$	$-0.000074313946490 + 0i$

7. CONCLUSIONS

In this paper we presented exact solutions of (1) in the class of Hölder functions. Next, Jacobi and Chebyshev polynomials were used to derive approximate solutions of this equation. Numerical experiments show that both methods yield very accurate results and may be useful in practice. However, investigating estimations of the errors of the approximate solutions remains by itself future research problem.

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