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ON THE CHAPLYGHIN METHOD FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Abstract. Classical solutions of initial problems for nonlinear first order partial differential equations are considered. It is shown that under natural assumptions on given functions, there exist Chaplyghin sequences and they are convergent. Error estimates for approximate solutions are given. The method of characteristics is used for the construction of approximate solutions.

Keywords: characteristics, Newton method, Chaplyghin sequences, initial problems.

Mathematics Subject Classification: 35A35, 35F25, 65J15.

1. INTRODUCTION

We are interested in working out a method of approximation of solutions to nonlinear first order partial differential equations by solutions of associated linear differential equations and in estimating the difference between the exact and approximate solutions. This is precisely what the Chaplyghin method accomplishes. This method of approximating solutions of ordinary differential equations by their linearization was initiated in [5, 12]. In [13] it has been applied to partial differential equations. In [8] this method was applied to ordinary functional differential equations. The Chaplyghin method for classical or generalized solutions of semilinear functional differential equations with initial or initial boundary conditions was considered in [6,7,9,14]. The theory of the Chaplyghin method for parabolic differential or functional differential equations was developed in [1–4]. Theorems on differential or functional differential inequalities are used in the investigation into the Chaplyghin sequences.

The aim of the paper is to give a further contribution to the theory of Chaplyghin method. We give comments on relations between results known in literature and our theory. For any metric spaces X and Y, by $C(X,Y)$ we denote the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let $\mathbb{H}$ be the Haar pyramid

$$\mathbb{H} = \big\{(t,x) \in \mathbb{R}^{1+n} : t \in [0,a], x \in [-b+Mt, b-Mt]\big\}$$

where $a > 0$, $M = (M_1, \ldots, M_n) \in \mathbb{R}^n_+$, $\mathbb{R}_+ = [0, +\infty)$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $b > Ma$. Suppose that

$$\Phi\colon [0,a] \times \mathbb{R}^n \to \mathbb{R}^n, \qquad \Phi = (\Phi_1, \ldots, \Phi_n),$$
$$\Psi\colon [0,a] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \qquad \varphi\colon [-b,b] \to \mathbb{R}$$

are given functions. Write

$$L[z](t,x) = \partial_t z(t,x) + \sum_{i=1}^{n} \Phi_i(t,x)\partial_{x_i} z(t,x) \tag{1}$$

and consider the almost linear differential equation

$$L[z](t,x) = \Psi(t,x,z(t,x)) \tag{2}$$

with the initial condition

$$z(0,x) = \varphi(x) \qquad \text{for} \quad x \in [-b,b]. \tag{3}$$

Suppose that the functions Φ and Ψ in the variables (t,x) and (t,x,p) are continuous and that there exist $\partial_p\Psi$ on $[0,a] \times \mathbb{R}^n \times \mathbb{R}$ and $\partial_p\Psi \in C\big([0,a] \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}\big)$. We assume also that for every $(t,x) \in [0,a] \times \mathbb{R}^n$ we have the following estimates

$$|\Phi_i(t,x)| \le M_i \qquad \text{for} \quad i = 1, \ldots, n.$$

Consider a sequence $\{z^{(k)}\}$, $z^{(k)}\colon \mathbb{H} \to \mathbb{R}$ for $k \ge 0$, such that:

1) $z^{(0)} \in C(\mathbb{H}, \mathbb{R})$;
2) if $z^{(k)} \in C(\mathbb{H}, \mathbb{R})$ is a known function then $z^{(k+1)}$ is a solution of the linear differential equation

$$L[z](t,x) = \Psi(t,x,z^{(k)}(t,x)) + \partial_p\Psi(t,x,z^{(k)}(t,x))\big(z(t,x) - z^{(k)}(t,x)\big)$$

with initial condition (3).

The above sequence $\{z^{(k)}\}$ is called a Chaplyghin sequence for (2), (3). Sufficient conditions for the convergence

$$\lim_{k\to\infty} z^{(k)}(t,x) = \tilde{u}(t,x) \quad \text{uniformly on } \mathbb{H}, \tag{4}$$

where $\tilde{u}$ is a solution of (2), (3), can be found in [13]. The convergence that we get in (4) is of the Newton type, which means that

$$|\tilde{u}(t,x) - z^{(k)}(t,x)| \le 2^{-k+1}(2h)^{2^k-1}\tilde{\eta}, \quad (t,x) \in \mathbb{H}, \quad k \ge 0,$$

where $\tilde{\eta} \in \mathbb{R}_+$ and $0 < h \leq \frac{1}{2}$. A method of differential inequalities is used in [13] for the investigation of Chaplyghin sequences.

The above classical result is extended in [6,7,9] to differential functional equations with initial or initial boundary conditions.

Results on the Chaplyghin method in these papers concern differential functional equations, where an unknown function is a functional variable and the coefficients $(\Phi_1, \ldots, \Phi_n)$ of the operator L given by (1) depend on (t, x) only.

In our paper we investigate the Chaplyghin method for quasilinear differential equations. That is, we assume that operator L is given by

$$L[z](t,x) = \partial_t z(t,x) + \sum_{i=1}^{n} \Phi_i(t,x,z(t,x))\partial_{x_i} z(t,x).$$

Hence, functions Φ_i may depend on (t, x, p) for $i = 1, \ldots, n$. Our results are based on the following idea. The problem of the existence of classical solutions to differential equations with partial derivatives of the first order is strictly connected with the problem of solving of characteristic systems of ordinary differential equations. In the paper, we construct Chaplyghin sequences for characteristic systems corresponding to quasilinear differential equations. Furthermore we extend the above idea onto differential equations which are nonlinear with respect to partial derivatives $(\partial_{x_1} z, \ldots, \partial_{x_n} z)$. We prove that, under natural assumptions on given functions, sequences of approximate solutions are convergent and we establish estimates for the difference between exact and approximate solutions of characteristic systems. It is important that we do not assume monotonicity conditions for given functions.

The paper is organized as follows. In Section 2 we prove a theorem on Chaplyghin sequences corresponding to quasilinear differential equations. In Section 3 we study a general class of nonlinear differential equations with partial derivatives.

In the paper we use general ideas concerning the Chaplyghin method introduced in [15].

2. CHAPLYGHIN SEQUENCES FOR QUASILINEAR DIFFERENTIAL EQUATIONS

By $M_{k\times n}$ we denote the space of all matrices with real elements. For $x \in \mathbb{R}^n$ and $X \in M_{k\times n}$, where $x = (x_1, \ldots, x_n)$ and $X = [x_{ij}]_{i=1,\ldots,k, j=1,\ldots,n}$, we define the norms

$$\|x\| = |x_1| + \cdots + |x_n|, \qquad \|X\| = \max\{\sum_{j=1}^{n} |x_{ij}| : 1 \leq i \leq k\}.$$

If $X \in M_{k\times n}$ then X^T is its transpose matrix. The inner product in $\mathbb{R}^n$ is denoted by "$\circ$".

Let $\Omega = [-a, a] \times \mathbb{R}^n \times \mathbb{R}$ and

$$f\colon \Omega \to \mathbb{R}^n \quad f = (f_1, \ldots, f_n)^T \quad g\colon \Omega \to \mathbb{R} \quad \omega\colon \mathbb{R}^n \to \mathbb{R}$$

be given functions. Consider the Cauchy problem:

$$\partial_t z(t,x) + \sum_{i=1}^{n} f_i(t,x,z(t,x))\partial_{x_i} z(t,x) = g(t,x,z(t,x)), \quad t \in [-a,a], \tag{5}$$

$$z(0,x) = \omega(x) \qquad \text{for } x \in \mathbb{R}^n. \tag{6}$$

Classical solution of (5), (6) are generated by characteristics in the following meaning. Let us consider a system of differential equations:

$$y'(t) = f(t,y(t),\xi(t)), \qquad \xi'(t) = g(t,y(t),\xi(t)) \tag{7}$$

where $y = (y_1,\ldots,y_n)^T$. Assume that solutions of (7) are defined on $[-a,a]$. If $u\colon [-a,a]\times\mathbb{R}^n \to \mathbb{R}$ is a solution of (5) and u is of class C^1 and $(t^*,x^*) \in [-a,a]\times\mathbb{R}^n$, then there exists a solution $(\tilde{y},\tilde{\xi})\colon [-a,a] \to \mathbb{R}^{n+1}$ of (7) such that $\tilde{y}(t^*) = x^*$ and $u(t,\tilde{y}(t)) = \tilde{\xi}(t)$ for $t \in [-a,a]$.

We prove that for each $\eta \in \mathbb{R}^n$ there are sequences of functions $\{y^{(k)}(\cdot,\eta),\xi^{(k)}(\cdot,\eta)\}$, where $y^{(k)}(\cdot,\eta)\colon [-a,a] \to \mathbb{R}^n$, $\xi^{(k)}(\cdot,\eta)\colon [-a,a] \to \mathbb{R}$, $\eta \in \mathbb{R}^n$, satisfying the conditions:

1) for every $k \geq 1$ the functions $(y^{(k)}(\cdot,\eta),\xi^{(k)}(\cdot,\eta))$ form the solution of an initial value problem for a linear system of differential equations;
2) for every $\eta \in \mathbb{R}^n$, the limits

$$\lim_{k\to\infty} y^{(k)}(t,\eta) = y(t,\eta), \quad \lim_{k\to\infty} \xi^{(k)}(t,\eta) = \xi(t,\eta) \tag{8}$$

exist uniformly on $[-a,a]$, where $(y(\cdot,\eta),\xi(\cdot,\eta))$ is a solution of (7) satisfying the initial condition:

$$y(0) = \eta, \quad \xi(0) = \omega(\eta); \tag{9}$$

3) the convergence in (8) is of the Newton type, which means that the difference between the k-th term of any of the approximating sequence and solution of (7), (9) can be estimated by $2^{-k+1}(2h)^{2^k-1}\overline{\eta}$ where $\overline{\eta} \in \mathbb{R}_+$ and $0 < h \leq \frac{1}{2}$.

The sequences $\{y^{(k)}(\cdot,\eta),\xi^{(k)}(\cdot,\eta)\}$ with the above properties are called Chaplyghin sequences for (5), (6).

Our considerations are based on the following idea. We transform an initial value problem for characteristic system (7) into an abstract equation $\Lambda[u] = 0$ where $\Lambda\colon X \to X$ and X is a Banach space. Then we consider the Newton method for the above equation and we prove that the Chaplyghin method for (5), (6) generates a Newton sequence for the above abstract equation.

Suppose that $(X,\|\cdot\|_X)$ is a Banach space and

$$S = \left\{u \in X : \|u - u^{(0)}\|_X \leq \delta\right\}$$

where $u^{(0)} \in X$ and $\delta > 0$. Let a map $\Lambda\colon S \to X$ be given and let there exist the Fréchet derivative for $u \in S$. We consider the equation

$$\Lambda[u] = 0 \tag{10}$$

and the Newton method

$$u^{(0)} \in S, \qquad u^{(k+1)} = u^{(k)} - \left[\Lambda'[u^{(k)}]\right]^{-1}\Lambda[u^{(k)}] \quad for \quad k \geq 0. \tag{11}$$

Let us denote the norm of $\Lambda'[u]$ by $\|\Lambda'[u]\|_*$. The following theorem will be needed in our considerations.

Theorem 2.1. *Suppose that $\Lambda\colon S \to X$ and following assumptions hold:*

1) $\Lambda'[u]$ *exists for $u \in S$ and there is $K \in \mathbb{R}_+$ such that*

$$\|\Lambda'[u] - \Lambda'[\bar{u}]\|_* \leq K\|u - \bar{u}\|_X, \qquad for \quad u, \bar{u} \in S;$$

2) *there exists the bounded inverse operator $\left[\Lambda'[u^{(0)}]\right]^{-1}$ such that*

$$\|\left[\Lambda'[u^{(0)}]\right]^{-1}\|_* \leq B$$

where $B \in \mathbb{R}_+$;

3) *the following initial inequality*

$$\|\left[\Lambda'[u^{(0)}]\right]^{-1}\Lambda[u^{(0)}]\|_X \leq \bar{\eta}$$

holds;

4) *the constants $B, K, \bar{\eta}$ satisfy the conditions $h = BK\bar{\eta} \leq \frac{1}{2}$ and*

$$\frac{1 - \sqrt{1 - 2h}}{KB} \leq \delta.$$

Then equation (10) *has a solution $u^* \in S$ and:*

i) *there exists Newton sequence* (11) *exists such that*

$$\lim_{k\to\infty} u^{(k)} = u^*,$$

ii) *the following error estimation hold*

$$\|u^* - u^{(k)}\|_X \leq 2^{-k+1}(2h)^{2^k-1}\bar{\eta} \qquad for \quad k \geq 0.$$

A proof of the above theorem can be found in [10, 11, 16].

We need the following assumptions on f and g.

Assumption $H_0[f, g]$. The functions $(f, g)\colon \Omega \to \mathbb{R}^{n+1}$ in the variables (t, x, p) are continuous and there exist the partial derivatives:

$$\partial_x f = \left[\partial_{x_j} f_i\right]_{i,j=1,\ldots,n}, \partial_p f = (\partial_p f_1, \ldots, \partial_p f_n), \partial_x g = (\partial_{x_1} g, \ldots, \partial_{x_n} g),\ \partial_p g$$

and $\partial_x f \in C(\Omega, M_{n\times n})$, $\partial_p f, \partial_x g \in C(\Omega, \mathbb{R}^n)$, $\partial_p g \in C(\Omega, \mathbb{R})$.

Suppose that $\eta \in \mathbb{R}^n$ is fixed. We define the Chaplyghin sequence as follows: $(y^{(0)}(\,\cdot\,, \eta), \xi^{(0)}(\,\cdot\,, \eta)) \in C([-a, a], \mathbb{R}^{n+1})$. If the functions $(y^{(k)}(\,\cdot\,, \eta), \xi^{(k)}(\,\cdot\,, \eta))$ are known, then $(y^{(k+1)}(\,\cdot\,,\eta),\xi^{(k+1)}(\,\cdot\,,\eta))$ is the solution of the system of differential equations:

$$y'(t)=f(P_k(t,\eta))+\partial_x f(P_k(t,\eta))(y(t)-y^{(k)}(t,\eta))+\partial_p f(P_k(t,\eta))(\xi(t)-\xi^{(k)}(t,\eta)),$$
$$\xi'(t)=g(P_k(t,\eta))+\partial_x g(P_k(t,\eta))(y(t)-y^{(k)}(t,\eta))+\partial_p g(P_k(t,\eta))(\xi(t)-\xi^{(k)}(t,\eta))$$

where $P_k(t,\eta)=(t,y^{(k)}(t,\eta),\xi^{(k)}(t,\eta))$, and $y(0)=\eta,\quad \xi(0)=\omega(\eta),\quad \eta\in\mathbb{R}^n$.

Assumption H$[f,g]$**.** Assumption $H_0[f,g]$ holds and:

1) there exists $A\in\mathbb{R}_+$ such that for $(t,x,p)\in\Omega$ there is

$$\|\partial_x f(t,x,p)\|+|\partial_p f(t,x,p)|\leqslant A,\quad \|\partial_x g(t,x,p)\|+|\partial_p g(t,x,p)|\leqslant A;$$

2) there exists $L\in\mathbb{R}_+$ such that the expressions

$$\|\partial_x f(t,x,p)-\partial_x f(t,\overline{x},\overline{p})\|+\|\partial_p f(t,x,p)-\partial_p f(t,\overline{x},\overline{p})\|,$$

$$\|\partial_x g(t,x,p)-\partial_x g(t,\overline{x},\overline{p})\|+\|\partial_p g(t,x,p)-\partial_p g(t,\overline{x},\overline{p})\|$$

are bounded from above by $L[\|x-\overline{x}\|+|p-\overline{p}|]$;

3) a number $\overline{\eta}\in\mathbb{R}_+$ is defined by the relation

$$\Big[\Big\|y^{(0)}(t,\eta)-\eta-\int_0^t f(P_0(s,\eta))ds\Big\|+ + |\xi^{(0)}(t,\eta)-\omega(\eta)-\int_0^t g(P_0(s,\eta))ds|\Big]\le\overline{\eta}e^{-Aa},t\in[-a,a];$$

4) constants $K,B,\overline{\eta}\ge 0$ satisfy the conditions

$$B=e^{Aa},\quad K=La,\quad \overline{\eta}>0,\quad h=KB\overline{\eta}\le\frac{1}{2}.$$

We will use the following notations. The maximum norm in the space $C([-a,a],\mathbb{R}^{n+1})$ is given by

$$\|(h,h_0)\|_C=\max\{\|h(t)\|+|h_0(t)|:t\in[-a,a]\}.$$

Let $CL([-a,a],\mathbb{R}^{n+1})$ be the set of linear and continuous maps from $C([-a,a],\mathbb{R}^{n+1})$ into the same space. The norm in the space $CL([-a,a],\mathbb{R}^{n+1})$ generated by the norm $\|\cdot\|_C$ in $C([-a,a],\mathbb{R}^{n+1})$ will be denoted by $\|\cdot\|_{CL}$.

Theorem 2.2. *If assumption $H[f,g]$ holds, then the Chaplyghin sequence $\{y^{(k)}(\,\cdot\,,\eta),\xi^{(k)}(\,\cdot\,,\eta)\}$ exists and*

$$\|y^{(k)}(t,\eta)-y(t,\eta)\|+|\xi^{(k)}(t,\eta)-\xi(t,\eta)|\le 2^{-k+1}(2h)^{2^k-1}\overline{\eta},\quad t\in[-a,a]. \tag{12}$$

Proof. A proof proceeds in a sequence of steps.
I. Our considerations start with the observation that the integral equations for the Chaplyghin sequence have the form:

$$\begin{aligned} y(t)-\eta = &\int_0^t f(P_k(s,\eta))ds + \int_0^t \partial_x f(P_k(s,\eta))(y(s)-y^{(k)}(s,\eta))ds \\ &+ \int_0^t \partial_p f(P_k(s,\eta))(\xi(s)-\xi^{(k)}(s,\eta))ds, \end{aligned} \tag{13}$$

$$\begin{aligned} \xi(t)-\omega(\eta) = &\int_0^t g(P_k(s,\eta)ds + \int_0^t \partial_x g(P_k(s,\eta))(y(s)-y^{(k)}(s,\eta))ds \\ &+ \int_0^t \partial_p g(P_k(s,\eta))(\xi(s)-\xi^{(k)}(s,\eta))ds. \end{aligned} \tag{14}$$

We consider the operator $U{:}\,C([-a,a],\mathbb{R}^{n+1})\to C([-a,a],\mathbb{R}^{n+1})$, $U=(F,G)$ defined by

$$F[y,\xi](t) = y(t)-\eta-\int_0^t f(s,y(s),\xi(s))ds,$$

$$G[y,\xi](t) = \xi(t)-\omega(\eta)-\int_0^t g(s,y(s),\xi(s))ds,$$

where $\eta \in \mathbb{R}^n$. From Assumption $H[f,g]$, it follows that for each $(y,\xi)\in C([-a,a],\mathbb{R}^{n+1})$ there exists the derivative $U'[y,\xi]=(F'[y,\xi],G'[y,\xi])$ and

$$F'[y,\xi](h,h_0)(t) = h(t)-\int_0^t \partial_x f(P(y,\xi;s))h(s)ds - \int_0^t \partial_p f(P(y,\xi;s))h_0(s)ds,$$

$$G'[y,\xi](h,h_0)(t) = h_0(t)-\int_0^t \partial_x g(P(y,\xi;s))h(s)ds - \int_0^t \partial_p g(P(y,\xi;s))h_0(s)ds,$$

where $(h,h_0)\in C([-a,a],\mathbb{R}^{n+1})$ and $P(y,\xi;s)=(s,y(s),\xi(s))$.

Now we prove that there exists the operator $(U'[y,\xi])^{-1}=\big((F'[y,\xi])^{-1},(G'[y,\xi])^{-1}\big)$, where $(y,\xi)\in C([-a,a],\mathbb{R}^{n+1})$. Suppose that $(u,u_0)\in C([-a,a],\mathbb{R}^{n+1})$. Consider the system of equations:

$$(F'[y,\xi])(h,h_0)=u, \qquad (G'[y,\xi])(h,h_0)=u_0,$$

where (h,h_0) are unknown. The above system is equivalent to integral equation

$$h(t) = u(t)+\int_0^t \partial_x f(P(y,\xi;s))h(s)ds + \int_0^t \partial_p f(P(y,\xi;s))h_0(s)ds, \tag{15}$$

$$h_0(t) = u_0(t)+\int_0^t \partial_x g(P(y,\xi;s))h(s)ds + \int_0^t \partial_p g(P(y,\xi;s))h_0(s)ds. \tag{16}$$

From Assumption $H[f,g]$ and the Banach fixed point theorem, there follows that there exists exactly one solution $(h,h_0)\in C([-a,a],\mathbb{R}^{n+1})$ of (15), (16). This shows that:

$$(U'[y,\xi])^{-1} = \big((F'[y,\xi])^{-1},(G'[y,\xi])^{-1}\big)$$

exists.

II. It is clear that the equation $U[y,\xi]=0$ in the space $C([-a,a],\mathbb{R}^{n+1})$ is equivalent to the system

$$F[y,\xi]=0,\ G[y,\xi]=0. \tag{17}$$

We consider the Newton sequence $\{\tilde{y}^{(k)}(\,\cdot\,,\eta),\tilde{\xi}^{(k)}(\,\cdot\,,\eta)\}$ for (17) defined as follows: for $k=0$, we assume that

$$\tilde{y}^{(0)}(\,\cdot\,,\eta)\in C([-a,a],\mathbb{R}^n)\ ,\ \tilde{\xi}^{(0)}(\,\cdot\,,\eta)\in C([-a,a],\mathbb{R}),\qquad \eta\in\mathbb{R}^n.$$

If $(\tilde{y}^{(k)}(\,\cdot\,,\eta),\tilde{\xi}^{(k)}(\,\cdot\,,\eta))$ is already defined then $(\tilde{y}^{(k+1)}(\,\cdot\,,\eta),\tilde{\xi}^{(k+1)}(\,\cdot\,,\eta))$ solve the equations

$$\big(F'[\tilde{y}^{(k)}(\,\cdot\,,\eta),\tilde{\xi}^{(k)}(\,\cdot\,,\eta)]\big)\big(y-\tilde{y}^{(k)}(\,\cdot\,,\eta),\xi-\tilde{\xi}^{(k)}(\,\cdot\,,\eta)\big)=-F\big[\tilde{y}^{(k)}(\,\cdot\,,\eta),\tilde{\xi}(k)(\,\cdot\,,\eta)\big],$$

$$\big(G'[\tilde{y}^{(k)}(\,\cdot\,,\eta),\tilde{\xi}^{(k)}(\,\cdot\,,\eta)]\big)\big(y-\tilde{y}^{(k)}(\,\cdot\,,\eta),\xi-\tilde{\xi}^{(k)}(\,\cdot\,,\eta)\big)=-G\big[\tilde{y}^{(k)}(\,\cdot\,,\eta),\tilde{\xi}(k)(\,\cdot\,,\eta)\big].$$

Since $((F'[y,\xi])^{-1},(G'[y,\xi])^{-1})$ exists for $(y,\xi)\in C([-a,a],\mathbb{R}^{n+1})$, it follows that the functions $(\tilde{y}^{(k+1)}(\,\cdot\,,\eta),\tilde{\xi}^{(k+1)}(\,\cdot\,,\eta))$ exist and are unique.

III. Suppose that $\tilde{y}^{(0)}(\,\cdot\,,\eta)=y^{(0)}(\,\cdot\,,\eta)$ and $\tilde{\xi}^{(0)}(\,\cdot\,,\eta)=\xi^{(0)}(\,\cdot\,,\eta)$ for $\eta\in\mathbb{R}^n$. We prove that the Newton sequence for (17) and the Chaplyghin sequence are the same:

$$\{\tilde{y}^{(k)}(\,\cdot\,,\eta),\tilde{\xi}^{(k)}(\,\cdot\,,\eta)\}=\{y^{(k)}(\,\cdot\,,\eta),\xi^{(k)}(\,\cdot\,,\eta)\},\quad \eta\in\mathbb{R}^n. \tag{18}$$

Assume that

$$\tilde{y}^{(k)}(\,\cdot\,,\eta)=y^{(k)}(\,\cdot\,,\eta)\quad\text{ and }\quad\tilde{\xi}^{(k)}(\,\cdot\,,\eta)=\xi^{(k)}(\,\cdot\,,\eta)\quad\text{ for }\quad\eta\in\mathbb{R}^n.$$

Then the functions $(\tilde{y}^{(k+1)}(\,\cdot\,,\eta),\tilde{\xi}^{(k+1)}(\,\cdot\,,\eta))$ satisfy the relations:

$$\tilde{y}^{(k+1)}(t,\eta)-\eta=\int_0^t f\big(P_k(s,\eta)\big)ds+\int_0^t\partial_x f\big(P_k(s,\eta)\big)\big(\tilde{y}^{(k+1)}(s,\eta)-\tilde{y}^{(k)}(s,\eta)\big)ds+$$

$$+\int_0^t\partial_p f\big(P_k(s,\eta)\big)\big(\tilde{\xi}^{(k+1)}(s,\eta)-\tilde{\xi}^{(k)}(s,\eta)\big)ds,$$

$$\tilde{\xi}^{(k+1)}(t,\eta)-\omega(\eta)=\int_0^t g(P_k\big(s,\eta)\big)ds+\int_0^t\partial_x g\big(P_k(s,\eta)\big)\big(\tilde{y}^{(k+1)}(s,\eta)-\tilde{y}^{(k)}(s,\eta)\big)ds+$$

$$+\int_0^t\partial_p g\big(P_k(s,\eta)\big)\big(\tilde{\xi}^{(k+1)}(s,\eta)-\tilde{\xi}^{(k)}(s,\eta)\big)ds.$$

From the above conditions, (13) and (14), it follows that $\tilde{y}^{(k+1)}(\,\cdot\,,\eta)=y^{(k+1)}(\,\cdot\,,\eta)$, $\tilde{\xi}^{(k+1)}(\,\cdot\,,\eta)=\xi^{(k+1)}(\,\cdot\,,\eta)$ for $\eta\in\mathbb{R}^n$. Thus, by induction, relations (18) are true for all $k\in\mathbb{N}$.

IV. We prove that

$$\|\big(U'[y^{(0)}(\,\cdot\,,\eta),\xi^{(0)}(\,\cdot\,,\eta)]\big)^{-1}\|_{CL}\le e^{Aa}. \tag{19}$$

Suppose that $(u, u_0) \in C([-a,a], \mathbb{R}^{n+1})$ and $\big(U'[y^{(0)}(\,\cdot\,,\eta), \xi^{(0)}(\,\cdot\,,\eta)]\big)^{-1}(u, u_0) = (h, h_0)$, where $(h, h_0) \in C([-a,a], \mathbb{R}^{n+1})$. Then

$$h(t) = u(t) + \int_0^t \partial_x f\big(P_0(s,\eta)\big)h(s)ds + \int_0^t \partial_p f\big(P_0(s,\eta)\big)h_0(s)ds,$$

$$h_0(t) = u_0(t) + \int_0^t \partial_x g\big(P_0(s,\eta)\big)h(s)ds + \int_0^t \partial_p g\big(P_0(s,\eta)\big)h_0(s)ds.$$

From Assumption $H[f,g]$, it follows that the integral inequality

$$\|h(t)\| + |h_0(t)| \le \|(u,u_0)\|_C + A\int_0^{|t|} \big[\|h(s)\| + |h_0(s)|\big]ds$$

is satisfied on $[-a,a]$. By virtue of the Gronwall inequality,

$$\|(h,h_0)\|_C \le e^{Aa}\|(u,u_0)\|_C$$

and inequality (19) follows. The following estimate holds:

$$\|U'[y,\xi] - U'[\bar{y},\bar{\xi}]\|_{CL} \le La\|(y,\xi) - (\bar{y},\bar{\xi})\|_C$$

where $(y,\xi), (\bar{y},\bar{\xi}) \in C([-a,a], \mathbb{R}^{n+1})$.

V. Now we prove that for $\eta \in \mathbb{R}^n$ there is

$$\|(U'[y^{(0)}(\,\cdot\,,\eta), \xi^{(0)}(\,\cdot\,,\eta)])^{-1}U[y^{(0)}(\,\cdot\,,\eta), \xi^{(0)}(\,\cdot\,,\eta)](t)\|_C \le \overline{\eta}. \tag{20}$$

Let $\eta \in \mathbb{R}^n$ be fixed and

$$\big(F'[y^{(0)}(\,\cdot\,,\eta), \xi^{(0)}(\,\cdot\,,\eta)]\big)^{-1}F\big[y^{(0)}(\,\cdot\,,\eta), \xi^{(0)}(\,\cdot\,,\eta)\big](t) = u(t),$$

$$\big(G'[y^{(0)}(\,\cdot\,,\eta), \xi^{(0)}(\,\cdot\,,\eta)]\big)^{-1}G\big[y^{(0)}(\,\cdot\,,\eta), \xi^{(0)}(\,\cdot\,,\eta)\big](t) = u_0(t).$$

Write $(v, v_0) = \big(F[y^{(0)}(\,\cdot\,,\eta), \xi^{(0)}(\,\cdot\,,\eta)], G[y^{(0)}(\,\cdot\,,\eta), \xi^{(0)}(\,\cdot\,,\eta)]\big)$. It follows that

$$v(t,\eta) = u(t) - \int_0^t \partial_x f\big(P_0(s,\eta)\big)u(s)ds - \int_0^t \partial_p f\big(P_0(s,\eta)\big)u_0(s)ds,$$

$$v_0(t,\eta) = u_0(t) - \int_0^t \partial_x g\big(P_0(s,\eta)\big)u(s)ds - \int_0^t \partial_p g\big(P_0(s,\eta)\big)u_0(s)ds$$

and consequently,

$$\|u(t)\| + |u_0(t)| \le \|(v,v_0)\|_C + A\int_0^{|t|} [\|u(s)\| + |u_0(s)|]ds, \quad t \in [-a,a].$$

From the Gronwall inequality we conclude that

$$\|(u,u_0)\|_C \le \|(v,v_0)\|_C e^{Aa},$$

which proves (20).

From the above conditions it follows that all the assumptions of Theorem 2.1 are satisfied and estimate (12) follows. □

Remark 1. *Theorem 2.1 can be extended on the local Cauchy problem for* (5) *and for initial boundary value problems for* (5) *and the estimation* (12) *follows.*

3. APPROXIMATE SOLUTIONS OF NONLINEAR EQUATIONS

Write $\Xi = [-a,a] \times \mathbb{R}^{n+1+n}$ where $a > 0$ and suppose that $F\colon \Xi \to \mathbb{R}$ is a given function in the variables (t,x,p,q) , $q = (q_1, \ldots q_n)$. Given $\omega\colon \mathbb{R}^n \to \mathbb{R}$, we consider the Cauchy problem:

$$\partial_t z(t,x) = F(t,x,z(t,x),\partial_x z(t,x)), \tag{21}$$

$$z(0,x) = \omega(x) \qquad \text{for } x \in \mathbb{R}^n, \tag{22}$$

where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$. Suppose that there exist the partial derivatives $\partial_x \omega = (\partial_{x_1}\omega, \ldots, \partial_{x_n}\omega)$ and $\partial_x F = (\partial_{x_1} F, \ldots, \partial_{x_n} F), \partial_p F, \partial_q F = (\partial_{q_1} F, \ldots, \partial_{q_n} F)$ and $\partial_x \omega \in C(\mathbb{R}^n, \mathbb{R}^n)$, $\partial_x F, \partial_q F \in C(\Xi, \mathbb{R}^n)$, $\partial_p F \in C(\Xi, \mathbb{R})$. Differential equations with partial derivatives of the first order have the following property: the problem of existence of their classical solutions and properties of solutions of (21)–(22) are strictly connected with initial problems for characteristic systems of ordinary differential equations. The following property of solutions of (21)–(22) is important in these study. Let us denote by $(y(\cdot,\eta), \xi(\cdot,\eta), \lambda(\cdot,\eta))$ a solution of the characteristic system

$$\begin{aligned}
&y'(t) = -\partial_q F(t,y(t),\xi(t),\lambda(t)),\\
&\xi'(t) = F(t,y(t),\xi(t),\lambda(t)) - \lambda(t) \circ \partial_q F(t,y(t),\xi(t),\lambda(t)),\\
&\lambda'(t) = \partial_x F(t,y(t),\xi(t),\lambda(t)) + \partial_p F(t,y(t),\xi(t),\lambda(t))\lambda(t)
\end{aligned} \tag{23}$$

with the initial condition

$$y(0) = \eta, \qquad \xi(0) = \omega(\eta), \qquad \lambda(0) = \partial_x \omega(\eta) \tag{24}$$

where $\eta \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$.

Suppose that solutions of (23)–(24) exist on $[-a,a]$. If $u\colon [-a,a] \times \mathbb{R}^n \to \mathbb{R}$ is a classical solution of (21)–(22) and the function $\partial_x u$ satisfies the Lipschitz condition with respect to x on $[-a,a] \times \mathbb{R}^n$. Then

$$u(t,y(t,\eta)) = \xi(t,\eta), \qquad \partial_x u(t,y(t,\eta)) = \lambda(t,\eta), \qquad t \in [-a,a].$$

We use the above property to construct approximate solutions to (21)–(22).

3.1. CHAPLYGHIN SEQUENCES FOR NONLINEAR EQUATIONS

Suppose that $\eta \in \mathbb{R}^n$ is fixed. We prove that there is a sequence of functions $\{y^{(k)}(\,\cdot\,,\eta), \xi^{(k)}(\,\cdot\,,\eta), \lambda^{(k)}(\,\cdot\,,\eta)\}$ where $y^{(k)}(\,\cdot\,,\eta), \lambda^{(k)}(\,\cdot\,,\eta)\colon [-a,a] \to \mathbb{R}^n$, $\xi^{(k)}(\,\cdot\,,\eta)\colon [-a,a] \to \mathbb{R}$, such that:

1) for every $k \geq 1$ the functions $(y^{(k)}(\,\cdot\,,\eta), \xi^{(k)}(\,\cdot\,,\eta), \lambda^{(k)}(\,\cdot\,,\eta))$ are solutions of an initial value problem for linear systems of differential equations;
2) the limits
$$\lim_{k\to\infty} y^{(k)}(t,\eta) = y(t,\eta), \quad \lim_{k\to\infty} \xi^{(k)}(t,\eta) = \xi(t,\eta), \quad \lim_{k\to\infty} \lambda^{(k)}(t,\eta) = \lambda(t,\eta) \tag{25}$$
exist uniformly on $[-a,a]$;
3) the convergence in (25) is of the Newton type.

The sequence $\{y^{(k)}(\,\cdot\,,\eta), \xi^{(k)}(\,\cdot\,,\eta), \lambda^{(k)}(\,\cdot\,,\eta)\}$ with the above properties is considered as a Chaplyghin sequence for (21), (22).

We write integral equations corresponding to (23)–(24). Let $u = (y, \xi, \lambda)$, $\mathbb{F}$=($\mathbb{Y}$,$\mathbb{Z}$,$\mathbb{Q}$),
$\mathbb{F}[u] = (\mathbb{Y}[u], \mathbb{Z}[u], \mathbb{Q}[u])$ and

$$\mathbb{Y}[u](t) = y(t) - \eta + \int_0^t \partial_q F(s, u(s))ds,$$
$$\mathbb{Z}[u](t) = \xi(t) - \omega(\eta) - \int_0^t F(s, u(s))ds + \int_0^t \lambda(s) \circ \partial_q F(s, u(s))ds,$$
$$\mathbb{Q}[u](t) = \lambda(t) - \partial_x\omega(\eta) - \int_0^t \partial_x F(s, u(s))ds - \int_0^t \lambda(t)\partial_p F(s, u(s))ds.$$

Then Cauchy problem (23)–(24) is equivalent to the integral equation

$$\mathbb{F}[u](t) = 0. \tag{26}$$

Suppose that $\eta \in \mathbb{R}^n$ is fixed. We consider a sequence of functions $\{u^{(k)}(\,\cdot\,,\eta)\}$,

$$u^{(k)}(\,\cdot\,,\eta) = (y^{(k)}(\,\cdot\,,\eta), \xi^{(k)}(\,\cdot\,,\eta), \lambda^{(k)}(\,\cdot\,,\eta))$$

defined in the following way. For $k = 0$ we put $u^{(0)}(t, \eta) = (\eta, \omega(\eta), \partial_x\omega(\eta))$. If $u^{(k)}(\,\cdot\,,\eta)$ is a known function then $u^{(k+1)}(\,\cdot\,,\eta)$ is defined by the relation

$$\mathbb{F}'[u^{(k)}(\,\cdot\,,\eta)]\big(u^{(k+1)}(\,\cdot\,,\eta) - u^{(k)}(\,\cdot\,,\eta)\big) + \mathbb{F}[u^{(k)}(\,\cdot\,,\eta)] = 0.$$

It is clear that $\{u^{(k)}(\,\cdot\,,\eta)\}$ is a Newton sequence for (26). We consider $\{u^{(k)}(\,\cdot\,,\eta)\}$ as a Chaplyghin sequence for initial problem (23)–(24). We prove that the above Chaplyghin sequence exists and converges to a solution of (23)–(24).

3.2. CONVERGENCE OF CHAPLYGHIN SEQUENCES

Suppose that $\omega\colon \mathbb{R}^n \to \mathbb{R}$ is of class C^1. Write

$$\Sigma[\eta] = \{(t, x, p, q) \in \Xi : \|x - \eta\| + |p - \omega(\eta)| + \|q - \partial_x\omega(\eta)\| \le \delta\},$$

$$S[\eta] = \{u = (y, \xi, \lambda) \in C([-a, a], \mathbb{R}^{n+1+n}) : \\ \|y(t) - \eta\| + |\xi(t) - \omega(\eta)| + \|\lambda(t) - \partial_x\omega(\eta)\| \le \delta, t \in [-a, a]\}.$$

We denote by Y^* the class of all linear operators defined on $\mathbb{R}^{n+1+n}$ and taking values in $\mathbb{R}^{n+1+n}$. For $\kappa = (x, p, q) \in \mathbb{R}^{n+1+n}$, we write $\|\kappa\|_* = \|x\| + |p| + \|q\|$. The norm of $\Psi \in Y^*$ generated by the norm $\|\cdot\|_*$ in $\mathbb{R}^{n+1+n}$ will by denoted by $\|\Psi\|_*$. This space is a Banach space.

Assumption $H_0[F]$**.** The function $F\colon \Xi \to \mathbb{R}$ is continuous, there exist the partial derivatives

$$\begin{aligned} \partial_p F, \qquad \partial_q F &= (\partial_{q_1} F, \ldots, \partial_{q_n} F), & \partial_x F &= (\partial_{x_1} F, \ldots, \partial_{x_n} F), \\ \partial_p\partial_q F &= (\partial_p\partial_{q_1} F, \ldots, \partial_p\partial_{q_n} F)^T, & \partial_p\partial_x F &= (\partial_p\partial_{x_1} F, \ldots, \partial_p\partial_{x_n} F)^T, \\ \partial_q\partial_q F &= \big[\partial_{q_j}\partial_{q_i} F\big]_{i,j=1,\ldots,n}, & \partial_q\partial_x F &= \big[\partial_{q_j}\partial_{x_i} F\big]_{i,j=1,\ldots,n}, \\ \partial_x\partial_q F &= \big[\partial_{x_j}\partial_{q_i} F\big]_{i,j=1,\ldots,n}, \end{aligned} \tag{27}$$

and the above functions are continuous on Ξ.

Suppose that Assumption $H_0[F]$ is satisfied. For each $(t,x,p,q) \in \Xi$, we consider the linear operator

$$W^*(t,x,p,q)\colon \mathbb{R}^{n+1+n} \to \mathbb{R}^{n+1+n},$$
$$W^*(t,x,p,q) = \big(U(t,x,p,q), W(t,x,p,q), V(t,x,p,q)\big)$$

defined in the following way. For $(\tilde{x},\tilde{p},\tilde{q}) \in \mathbb{R}^{n+1+n}$, $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n)^T$, $\tilde{q} = (\tilde{q}_1,\ldots,\tilde{q}_n)^T$ we write

$$\begin{aligned} U(t,x,p,q)(\tilde{x},\tilde{p},\tilde{q}) &= \partial_x\partial_q F(t,x,p,q)\tilde{x} + \partial_p\partial_q F(t,x,p,q)\tilde{p} + \partial_q\partial_q F(t,x,p,q)\tilde{q}, \\ W(t,x,p,q)(\tilde{x},\tilde{p},\tilde{q}) &= \partial_x F(t,x,p,q) \circ \tilde{x} + \partial_p F(t,x,p,q)\tilde{p} - \\ &\quad - q \circ \big[\partial_x\partial_q F(t,x,p,q)\tilde{x} + \partial_p\partial_q F(t,x,p,q)\tilde{p} + \\ &\qquad + \partial_q\partial_q F(t,x,p,q)\tilde{q}\big], \\ V(t,x,p,q)(\tilde{x},\tilde{p},\tilde{q}) &= \partial_x\partial_x F(t,x,p,q)\tilde{x} + \partial_p\partial_x F(t,x,p,q)\tilde{p} + \partial_q\partial_x F(t,x,p,q)\tilde{q} + \\ &\quad + q\big[\partial_x\partial_p F(t,x,p,q) \circ \tilde{x} + \partial_p\partial_p F(t,x,p,q)\tilde{p} + \\ &\qquad + \partial_q\partial_p F(t,x,p,q) \circ \tilde{q}\big] + \partial_p F(t,x,p,q)\tilde{q}. \end{aligned}$$

We formulate next assumptions on F and ω.

Assumption $H[F,\omega]$**.** Assumption $H_0[F]$ is satisfied and:

1) a constant $A \in \mathbb{R}_+$ is defined by the relation

$$\|W^*(t,x,p,q)\|_* \le A, \qquad (t,x,p,q) \in \Sigma[\eta];$$

2) there exists $L \in \mathbb{R}_+$ such that for $(t,x,p,q), (\tilde{x},\tilde{p},\tilde{q}) \in \Sigma[\eta]$ there holds

$$\|W^*(t,x,p,q) - W^*(\tilde{x},\tilde{p},\tilde{q})\|_* \le L\big[\|x-\tilde{x}\| + |p-\tilde{p}| + \|q-\tilde{q}\|\big]; \tag{28}$$

3) a constant $\bar{\eta} \in \mathbb{R}_+$ is defined by the relation

$$\|\mathbb{Y}[u^{(0)}](t)\| + |\mathbb{Z}[u^{(0)}](t)| + \|\mathbb{Q}[u^{(0)}](t)\| \le \bar{\eta}e^{-Aa}$$

where $u^{(0)} = (y^{(0)}, \xi^{(0)}, \lambda^{(0)}) \in S[\eta]$;

4) constants $K, B > 0$ satisfy the conditions:

$$B = e^{Aa}, \quad K = La, \quad \bar{\eta} > 0, \quad h = KB\bar{\eta} > 0, \quad h \le \frac{1}{2}, \quad \frac{1-\sqrt{1-2h}}{KB} \in [0,\delta];$$

5) a function $\omega\colon \mathbb{R}^n \to \mathbb{R}$ is of class C^1.

Remark 2. *If function* (27) *satisfies the Lipschitz condition with respect to* (x, p, q), *then there is* $L \in \mathbb{R}_+$ *such that relation* (28) *holds.*

We start with the formulation of the following property of the operator

$$\mathbb{F}\colon C([-a,a],\mathbb{R}^{n+1+n}) \to C([-a,a],\mathbb{R}^{n+1+n}).$$

If Assumption $H[F,\omega]$ is satisfied, then there exists $\mathbb{F}'[u] = (\mathbb{Y}'[u], \mathbb{Z}'[u], \mathbb{Q}'[u])$, where $u = (y,\xi,\lambda) \in C([-a,a],\mathbb{R}^{n+1+n})$, and for $v = (h,\mu,\gamma) \in C([-a,a],\mathbb{R}^{n+1+n})$ there is

$$\mathbb{Y}'[u](v)(t) = h(t) - \int_0^t U\big(s,u(s)\big)v(s)ds,$$
$$\mathbb{Z}'[u](v)(t) = \mu(t) - \int_0^t W\big(s,u(s)\big)v(s)ds,$$
$$\mathbb{Q}'[u](v)(t) = \gamma(t) - \int_0^t V(s,u(s))v(s)ds.$$

Lemma 1. *Let Assumption* $H[F,\omega]$ *hold. Then for every function* $u = (y,\xi,\lambda) \in S[\eta]$ *there exists* $(\mathbb{F}'[u])^{-1}$.

Proof. Let $\bar v = (\bar h, \bar\mu, \bar\gamma) \in S[\eta]$. Then equation $\mathbb{F}'[u]v = \bar v$ is equivalent to the system of equations:

$$\mathbb{Y}'[u] = \bar h, \quad \mathbb{Z}'[u] = \bar\mu, \quad \mathbb{Q}'[u] = \bar\gamma.$$

It is a linear system of integral equations of Volterra type. From the Banach fixed point theorem there follows that this system of equations has exactly one solution $v \in C([-a,a],\mathbb{R}^{n+1+n})$. That completes the proof. □

Lemma 2. *If Assumption* $H[F,\omega]$ *holds, then*

$$\max\{\|(\mathbb{Y}'[u^{(0)}])^{-1}\| + |(\mathbb{Z}'[u^{(0)}])^{-1}| + \|(\mathbb{Q}'[u^{(0)}])^{-1}\| : t \in [-a,a]\}$$

is estimated by e^{Aa}

Proof. Let us assume that $\bar v = (\bar h, \bar\mu, \bar\gamma) \in S[\eta]$. Consider the equation $\mathbb{F}'[u^{(0)}]v = \bar v$, which is equivalent to system of equations:

$$\mathbb{Y}'[u^{(0)}]v = \bar h, \quad \mathbb{Z}'[u^{(0)}]v = \bar\mu, \quad \mathbb{Q}'[u^{(0)}]v = \bar\gamma.$$

From Assumption $H[F,\omega]$ it follows that the function v satisfies the integral inequality:

$$\|v(t)\|_* \le \|\bar v(t)\|_* + \left|\int_0^t A\|v(s)\|_* ds\right|.$$

From the Gronwall inequality there follows:

$$\|v(t)\|_* \le \|\bar v(t)\|_* e^{A|t|} \le \|\bar v(t)\|_* e^{Aa},$$

which completes the proof. □

Theorem 3.1. *If Assumption $H[F,\omega]$ holds, then there exists the Chaplyghin sequence for* (23) *and the sequence*

$$\{u^{(k)}(\,\cdot\,,\eta)\} = \{y^{(k)}(\,\cdot\,,\eta), \xi^{(k)}(\,\cdot\,,\eta), \lambda^{(k)}(\,\cdot\,,\eta)\},$$

converges to the solution $u(\,\cdot\,,\eta) = (y(\,\cdot\,,\eta), \xi(\,\cdot\,,\eta), \lambda(\,\cdot\,,\eta))$ of (23)–(24),

$$\lim_{k\to+\infty} u^{(k)}(t,\eta) = u(t,\eta) = (y(t,\eta), \xi(t,\eta), \lambda(t,\eta))$$

uniformly on $[-a,a]$ and the following estimate:

$$\|u(t,\eta) - u^{(k)}(t,\eta)\|_* \le 2^{-k+1}(2h)^{2^k-1}\overline{\eta} \tag{29}$$

hold for $k \in \mathbb{N}$ and $t \in [-a,a]$.

Proof. We apply Theorem 2.1 to prove the above properties of the Chaplyghin sequence. The proof will be divided into two steps.
I. From Lemma 1 it follows that the operator $\mathbb{F}'[u]\colon C([-a,a],\mathbb{R}^{n+1+n}) \to C([-a,a],\mathbb{R}^{n+1+n})$ exists and is linear. Let $CL([-a,a],\mathbb{R}^{n+1+n})$ be a set of linear and continuous maps from $C([-a,a],\mathbb{R}^{n+1+n})$ into itself. The norm in this space, generated by the norm $\|\cdot\|_*$ in $C([-a,a],\mathbb{R}^{n+1+n})$, will be denoted by $\|\cdot\|_{CL}$. Then it is obvious that:

$$\|\mathbb{F}'[u] - \mathbb{F}'[\bar{u}]\|_{CL} \le La\Bigg[\|u-\bar{u}\|_*\Bigg] \quad for \quad t \in [-a,a].$$

From Lemma 2, it follows that $\|(\mathbb{F}'[u^{(0)}(\,\cdot\,,\eta)])^{-1}\|_{CL}$ is bounded by e^{Aa}.
II. We will show that $\|(\mathbb{F}'[u^{(0)}(\,\cdot\,,\eta)])^{-1}\mathbb{F}[u^{(0)}(\,\cdot\,,\eta)]\|_{CL} \le \bar{\eta}$. Write

$$h(t,\eta) = (\mathbb{Y}'[u^{(0)}(\,\cdot\,,\eta)])^{-1}\mathbb{Y}[u^{(0)}(\,\cdot\,,\eta)](t),$$
$$\mu(t,\eta) = (\mathbb{Z}'[u^{(0)}(\,\cdot\,,\eta)])^{-1}\mathbb{Z}[u^{(0)}(\,\cdot\,,\eta)](t),$$
$$\gamma(t,\eta) = (\mathbb{Q}'[u^{(0)}(\,\cdot\,,\eta)])^{-1}\mathbb{Q}[u^{(0)}(\,\cdot\,,\eta)](t),$$

and $v(\,\cdot\,,\eta) = (\bar{h}(\,\cdot\,,\eta), \bar{\mu}(\,\cdot\,,\eta), \bar{\gamma}(\,\cdot\,,\eta)), \tilde{u}(\,\cdot\,,\eta) = (h(\,\cdot\,,\eta), \mu(\,\cdot\,,\eta), \gamma(\,\cdot\,,\eta))$ From the definition of the operator F, we conclude that

$$\|\tilde{u}(t,\eta)\|_* \le \|v(t,\eta)\|_* + \int_0^t A\Bigg[\|\tilde{u}(s,\eta)\|_*\Bigg]ds.$$

From the Gronwall inequality there follows:

$$\|\tilde{u}(t)\|_* \le \Bigg[\|v(t,\eta)\|_*\Bigg]e^{Aa}.$$

From Assumption $H[F,\omega]$ we infer that $\|(\mathbb{F}'[u^{(0)}(\,\cdot\,,\eta)])^{-1}\mathbb{F}[u^{(0)}(\,\cdot\,,\eta)]\|_{CL} \le \bar{\eta}$.
From Theorem 2.1, the conclusion of the Theorem follows. □

Remark 3. *Theorems on differential or differential-functional inequalities are used in to prove the convergence of Chaplyghin sequences in* [1–4] *and* [7–9]. *Then assumptions on monotonicity of given functions are needed there. It is important in our considerations that we do not assume monotonicity conditions for the given functions in* (7) *and* (23).

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