Opuscula Mathematica • Vol. 28 • No. 1 • 2008

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SINGULAR INTEGRAL EQUATION WITH A MULTIPLICATIVE CAUCHY KERNEL IN THE HALF-PLANE

Abstract. In this paper the explicit solutions of singular integral equation with a multiplicative Cauchy kernel in the half-plane are presented.

Keywords: singular integral equation, exact solution, Cauchy kernel, multiplicative kernel.

Mathematics Subject Classification: 45E05.

1. INTRODUCTION

In the literature [2, 5–7] formulae describing a solution of the following equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(\tau)}{\tau - t}dt = f(t), \quad -1 < x < 1,$$

are very well known. Some problems of aeroelasticity [1] can be reduced to the equation of the form

$$\frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\frac{\varphi(\sigma_1,\sigma_2)}{(\sigma_1 - x)(\sigma_2 - y)}d\sigma_1 d\sigma_2 = f(x,y), \quad -1 < x, y < 1.$$

Theory of this equation is well developed in [3,4,9]. In paper [8], a theory of following equations

$$\frac{1}{(\pi i)^2}\iint_{D_1}\frac{\varphi(\sigma_1,\sigma_2)}{(\sigma_1 - x)(\sigma_2 - y)}d\sigma_1 d\sigma_2 = f(x,y), \quad (x,y) \in D_1,$$

$$\frac{1}{(\pi i)^2}\iint\limits_{D_2}\frac{\varphi(\sigma_1,\sigma_2)}{(\sigma_1-x)(\sigma_2-y)}d\sigma_1 d\sigma_2=f(x,y)\,,\quad (x,y)\in D_2,$$

where D_1, D_2 are the quarter-plane and the whole complex plane, respectively, is presented. We have not found in the literature any study of the equation in which the surface of integration is the half-plane. In this paper, we consider the equation

$$\frac{1}{(\pi i)^2}\iint\limits_{D}\frac{\varphi(\sigma_1,\sigma_2)}{(\sigma_1-x)(\sigma_2-y)}d\sigma_1 d\sigma_2=f(x,y)\,,\tag{1}$$

where $(x,y)\in D=\{(x,y):0<\operatorname{Re}z<\infty,\,-\infty<\operatorname{Im}z<\infty,\,z=x+iy\}$, $f(x,y)$ is a given function and $\varphi(x,y)$ is an unknown function.

2. FUNCTION CLASSES

Let us introduce function classes that will be used in this paper.

We write $\varphi(x)\in h(\infty)$, $x>0$, if the function

$$\varphi^*(t)=\varphi\left(\frac{1+t}{1-t}\right),\quad t\in(-1,1)\,,$$

satisfies the inequality

$$|\varphi^*(t')-\varphi^*(t'')|\le K\,|t'-t''|^{\mu}\,,\tag{2}$$

where $K>0$, $0<\mu\le 1$ are constants independent of the arrangement of the points t',t'' in each closed interval contained in $(-1,1)$, and in a neighbourhood of the point $t=-1$ the following condition is satisfied:

$$\varphi^*(t)=\varphi^{**}(t)\,|t+1|^{-\alpha}\,,\quad 0\le\operatorname{Re}\alpha<1.$$

Here $\varphi^{**}(t)$ is a Hölder continuous function on the interval $[-1,1)$, and

$$\lim_{t\to 1^{-0}}\varphi^*(t)=\lim_{x\to\infty}\varphi(x)=0.\tag{3}$$

We write $\varphi(x)\in h(\infty)$, $-\infty<x<\infty$, if the function

$$\varphi^*(t)=\varphi\left(i\frac{1+t}{1-t}\right),\quad |t|=1,$$

satisfies inequality (2).

We write $\varphi(x,y)\in h(\infty)\times h(\infty)$, $x>0$, $-\infty<y<\infty$, if the function

$$\varphi^*(t_1,t_2)=\varphi\left(\frac{1+t_1}{1-t_1},i\frac{1+t_2}{1-t_2}\right),\quad (t_1,t_2)\in(-1,1)\times L,\quad L=\{t_2:|t_2|=1\}\,,$$

satisfies the inequality

$$\left|\varphi^*\left(t_1', t_2'\right) - \varphi^*\left(t_1'', t_2''\right)\right| \leq K_1 \left|t_1' - t_1''\right|^{\mu_1} + K_2 \left|t_2' - t_2''\right|^{\mu_2}, \tag{4}$$

$K_1,\ K_2 > 0$, $0 < \mu_1,\ \mu_2 \leq 1$, in each closed domain contained in $(-1,1) \times L$, and

$$\varphi^*\left(t_1, t_2\right) = \varphi^{**}\left(t_1, t_2\right) \left|t_1 + 1\right|^{-\alpha}, \quad 0 \leq \operatorname{Re} \alpha < 1,$$

where $\varphi^{**}\left(t_1, t_2\right)$ satisfies the Hölder condition with respect to both variables on $[-1,1) \times L$, and moreover,

$$\lim_{t_1 \to 1-0} \varphi^*\left(t_1, t_2\right) = \lim_{x \to \infty} \varphi\left(x, y\right) = 0 \quad \text{for} \quad t_2 \in L \quad (\text{for} \quad y \in (-\infty, \infty)). \tag{5}$$

We write $\varphi(x) \in h(0,\infty)$, $x \geq 0$, if the function

$$\varphi^*(t) = \varphi\left(\frac{1+t}{1-t}\right), \quad t \in [-1,1),$$

satisfies (2) and the condition of the form (3).

We write $\varphi(x,y) \in h(0,\infty) \times h(\infty)$, $0 \leq x \leq \infty$, $-\infty \leq y \leq \infty$, if the function

$$\varphi^*\left(t_1, t_2\right) = \varphi\left(\frac{1+t_1}{1-t_1}, i\frac{1+t_2}{1-t_2}\right), \quad \left(t_1, t_2\right) \in [-1,1) \times L, \quad L = \left\{t_2 : \left|t_2\right| = 1\right\},$$

satisfies conditions (4) and (5).

3. SOLUTION IN THE CLASS $h(\infty) \times h(\infty)$

Theorem 3.1. *Let $f(x,y) \in h(0,\infty) \times h(\infty)$ and let*

$$\lim_{|y| \to \infty} f(x,y) = 0, \quad x \in [0,\infty). \tag{6}$$

Then each solution $\varphi(x,y)$ of (1) in the function class $h(\infty) \times h(\infty)$ is given by the formula

$$\varphi(x,y) = R(f;x,y) + C_1(x) + \frac{C_2(y)\, i}{\sqrt{x}}, \tag{7}$$

where

$$R(f;x,y) = \frac{(x+1)(y+i)}{\sqrt{x}\,(\pi i)^2} \int_0^\infty \int_{-\infty}^\infty \frac{\sqrt{\sigma_1} f\left(\sigma_1, \sigma_2\right)}{\left(\sigma_1+1\right)\left(\sigma_2+i\right)\left(\sigma_1-x\right)\left(\sigma_2-y\right)} d\sigma_1 d\sigma_2,$$

$C_1(x)$, $x > 0$, $C_2(y)$, $-\infty < y < \infty$, *are arbitrary functions of class $h(\infty)$.*

If we seek for a solution $\varphi(x,y)$ in the class of functions satisfying the following conditions:

$$\frac{1}{\pi i} \int_{-\infty}^\infty \frac{\varphi\left(x, \sigma_2\right)}{\sigma_2 + i} d\sigma_2 = p(x), \tag{8}$$

$$\frac{1}{\pi i} \int_0^\infty \frac{\varphi\left(\sigma_1, y\right)}{\sigma_1 + 1} d\sigma_1 = q(y), \tag{9}$$

where $p(x) \in h(\infty)$, $x > 0$, $q(y) \in h(\infty)$, $-\infty < y < \infty$ are the functions fulfilling the relation

$$\frac{1}{\pi i}\int_{0}^{\infty}\frac{p\left(\sigma_1\right)}{\sigma_1+1}d\sigma_1=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{q\left(\sigma_2\right)}{\sigma_2+i}d\sigma_2=\omega, \tag{10}$$

then the solution is given by the formula

$$\varphi\left(x,y\right)=R\left(f;x,y\right)-p\left(x\right)+\frac{q\left(y\right)i}{\sqrt{x}}+\frac{\omega i}{\sqrt{x}}. \tag{11}$$

Proof. We can rewrite (1) in the form

$$\frac{1}{\pi i}\int_{0}^{\infty}\frac{d\sigma_1}{\sigma_1-x}\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi\left(\sigma_1,\sigma_2\right)}{\sigma_2-y}d\sigma_2=f\left(x,y\right), \tag{12}$$

or

$$\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{d\sigma_2}{\sigma_2-y}\frac{1}{\pi i}\int_{0}^{\infty}\frac{\varphi\left(\sigma_1,\sigma_2\right)}{\sigma_1-x}d\sigma_2=f\left(x,y\right). \tag{13}$$

Let us consider equation (12). It can be represented in the following form

$$\frac{1}{\pi i}\int_{0}^{\infty}\frac{\psi_1\left(\sigma_1,y\right)}{\sigma_1-x}d\sigma_1=f\left(x,y\right), \tag{14}$$

where

$$\psi_1\left(\sigma_1,y\right)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi\left(\sigma_1,\sigma_2\right)}{\sigma_2-y}d\sigma_2. \tag{15}$$

Let us find the function $\psi_1\left(x,y\right)$ appearing in (14). We solve (14) in the function class $h\left(\infty\right), 0<x<\infty$. By [8], we obtain

$$\psi_1\left(x,y\right)=\frac{x+1}{\sqrt{x}}\frac{1}{\pi i}\int_{0}^{\infty}\frac{\sqrt{\sigma_1}}{\sigma_1+1}\frac{f\left(\sigma_1,y\right)}{\sigma_1-x}d\sigma_1+\frac{C_3\left(y\right)i}{\sqrt{x}},$$

where $C_3\left(y\right), y\in\left(-\infty,\infty\right)$, is an arbitrary function of class $h\left(\infty\right)$.

Next, solving equation (15), we obtain the solution $\varphi\left(x,y\right)$:

$$\begin{aligned}\varphi(x,y)&=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{y+i}{\sigma_2+i}\frac{\psi_1\left(x,\sigma_2\right)}{\sigma_2-y}d\sigma_2-C_4\left(x\right)=\\&=\frac{1}{\left(\pi i\right)^2}\frac{\left(x+1\right)\left(y+i\right)}{\sqrt{x}}\int_{0}^{\infty}\int_{-\infty}^{\infty}\frac{\sqrt{\sigma_1}f\left(\sigma_1,\sigma_2\right)}{\left(\sigma_1+1\right)\left(\sigma_2+i\right)\left(\sigma_1-x\right)\left(\sigma_2-y\right)}d\sigma_1d\sigma_2+\\&+\frac{\left(y+i\right)i}{\pi i\sqrt{x}}\int_{-\infty}^{\infty}\frac{C_3\left(\sigma_2\right)}{\left(\sigma_2+i\right)\left(\sigma_2-y\right)}d\sigma_2-C_4\left(x\right),\end{aligned} \tag{16}$$

where $C_4\left(x\right), 0<x<\infty$, is an arbitrary function of class $h\left(\infty\right)$.

Now we consider equation (13). We rewrite it in the form

$$\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{\psi_2(x,\sigma_2)}{\sigma_2-y}d\sigma_2=f(x,y), \tag{17}$$

where

$$\psi_2(x,\sigma_2)=\frac{1}{\pi i}\int\limits_{0}^{\infty}\frac{\varphi(\sigma_1,\sigma_2)}{\sigma_1-x}d\sigma_1. \tag{18}$$

Solving (17) in the function class $h(\infty)$ with respect to the second variable, we obtain

$$\psi_2(x,y)=\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{y+i}{\sigma_2+i}\frac{f(x,\sigma_2)}{\sigma_2-y}d\sigma_2-C_5(x), \tag{19}$$

where $C_5(x)$, $x>0$, is an arbitrary function of class $h(\infty)$.

Using (19), we solve equation (18). We derive

$$\begin{aligned}\varphi(x,y)&=\frac{x+1}{\sqrt{x}}\frac{1}{\pi i}\int\limits_{0}^{\infty}\frac{\sqrt{\sigma_1}}{\sigma_1+1}\frac{\psi_2(\sigma_1,y)}{\sigma_1-x}d\sigma_1+\frac{C_6(y)\,i}{\sqrt{x}}=\\&=\frac{(x+1)(y+i)}{\sqrt{x}}\frac{1}{(\pi i)^2}\int\limits_{0}^{\infty}\int\limits_{-\infty}^{\infty}\frac{\sqrt{\sigma_1}f(\sigma_1,\sigma_2)}{(\sigma_1+1)(\sigma_2+i)(\sigma_1-x)(\sigma_2-y)}d\sigma_1d\sigma_2-\\&-\frac{x+1}{\sqrt{x}}\frac{1}{\pi i}\int\limits_{0}^{\infty}\sqrt{\sigma_1}\frac{C_5(\sigma_1)}{(\sigma_1+1)(\sigma_1-x)}d\sigma_1+\frac{C_6(y)\,i}{\sqrt{x}},\end{aligned} \tag{20}$$

where $C_6(y)$, $-\infty<y<\infty$, is an arbitrary function of class $h(\infty)$.

Owing to (16) and (20), the general solution $\varphi(x,y)$ of (1) in the considered class of functions is given by (7). Let us check it by substituting (7) into (1). Substituting $R(f;x,y)$ into equation (1) and using the Poincaré-Bertrand formula for an infinite surface of integration:

$$\begin{aligned}&\frac{1}{(\pi i)^2}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\frac{\varphi(\sigma_1,\sigma_1')}{(\sigma_1-x)(\sigma_1'-\sigma_1)}d\sigma_1'd\sigma_1=\\&\qquad=\varphi(x,x)+\frac{1}{(\pi i)^2}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\frac{\varphi(\sigma_1,\sigma_1')}{(\sigma_1-x)(\sigma_1'-\sigma_1)}d\sigma_1d\sigma_1'-\varphi(\infty,\infty),\end{aligned}$$

and its similar version for the whole plane as the surface of integration, and the following formulae

$$\frac{1}{\pi i}\int\limits_{0}^{\infty}\frac{\sigma_1+1}{\sqrt{\sigma_1}(\sigma_1'-\sigma_1)(\sigma_1-x)}d\sigma_1=0,\quad\frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\frac{\sigma_2+i}{(\sigma_2-y)(\sigma_2'-\sigma_2)}d\sigma_2=0,$$

we get

$$
\begin{aligned}
&\tfrac{1}{(\pi i)^2}\int\limits_0^\infty\int\limits_{-\infty}^\infty \tfrac{R(f;\sigma_1,\sigma_2)}{(\sigma_1-x)(\sigma_2-y)}d\sigma_1 d\sigma_2 = \tfrac{1}{(\pi i)^2}\int\limits_0^\infty\int\limits_{-\infty}^\infty \tfrac{1}{(\pi i)^2}\tfrac{(\sigma_1+1)(\sigma_2+i)}{\sqrt{\sigma_1}(\sigma_1-x)(\sigma_2-y)}\times\\
&\quad\times\int\limits_0^\infty\int\limits_{-\infty}^\infty \tfrac{\sqrt{\sigma_1'}f\left(\sigma_1',\sigma_2'\right)}{\left(\sigma_1'+1\right)\left(\sigma_2'+i\right)\left(\sigma_1'-\sigma_1\right)\left(\sigma_2'-\sigma_2\right)}d\sigma_1' d\sigma_2' d\sigma_1 d\sigma_2 =\\
&= \tfrac{1}{(\pi i)^2}\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty \tfrac{\sigma_2+i}{\left(\sigma_2'+i\right)(\sigma_2-y)\left(\sigma_2'-\sigma_2\right)}\times\\
&\quad\times\tfrac{1}{(\pi i)^2}\int\limits_0^\infty\int\limits_0^\infty \tfrac{(\sigma_1+1)\sqrt{\sigma_1'}f\left(\sigma_1',\sigma_2'\right)}{\sqrt{\sigma_1}\left(\sigma_1'+1\right)\left(\sigma_1'-\sigma_1\right)(\sigma_1-x)}d\sigma_1' d\sigma_1 d\sigma_2' d\sigma_2 =\\
&= \tfrac{1}{(\pi i)^2}\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty \tfrac{\sigma_2+i}{\left(\sigma_2'+i\right)(\sigma_2-y)\left(\sigma_2'-\sigma_2\right)}\times\\
&\quad\times\left(f\left(x,\sigma_2'\right)+\tfrac{1}{(\pi i)^2}\int\limits_0^\infty\left(\int\limits_0^\infty \tfrac{\sigma_1+1}{\sqrt{\sigma_1}\left(\sigma_1'-\sigma_1\right)(\sigma_1-x)}d\sigma_1\right)\tfrac{\sqrt{\sigma_1'}f\left(\sigma_1',\sigma_2'\right)}{\sigma_1'+1}d\sigma_1'\right)d\sigma_2' d\sigma_2 =\\
&= \tfrac{1}{(\pi i)^2}\int\limits_{\infty}^\infty\int\limits_{-\infty}^\infty \tfrac{(\sigma_2+i)f\left(x,\sigma_2'\right)}{\left(\sigma_2'+i\right)(\sigma_2-y)\left(\sigma_2'-\sigma_2\right)}d\sigma_2' d\sigma_2 =\\
&= f(x,y)+\tfrac{1}{(\pi i)^2}\int\limits_{-\infty}^\infty\left(\int\limits_{-\infty}^\infty \tfrac{\sigma_2+i}{(\sigma_2-y)\left(\sigma_2'-\sigma_2\right)}d\sigma_2\right)\tfrac{f\left(x,\sigma_2'\right)}{\sigma_2'+i}d\sigma_2' = f(x,y).
\end{aligned}
$$

Now we substitute the function $C_1(x)$, $x>0$ appearing in (7) into (1). We obtain

$$\tfrac{1}{(\pi i)^2}\int\limits_0^\infty\int\limits_{-\infty}^\infty \tfrac{C_1(\sigma_1)}{(\sigma_1-x)(\sigma_2-y)}d\sigma_1 d\sigma_2 = \tfrac{1}{\pi i}\int\limits_0^\infty \tfrac{C_1(\sigma_1)}{\sigma_1-x}d\sigma_1 \tfrac{1}{\pi i}\int\limits_{-\infty}^\infty \tfrac{d\sigma_2}{\sigma_2-y} = 0.$$

Finally, substituting $\frac{C_2(y)i}{\sqrt{x}}$, $-\infty<y<\infty$ into (1) we get

$$\tfrac{1}{(\pi i)^2}\int\limits_0^\infty\int\limits_{-\infty}^\infty \tfrac{C_2(\sigma_2)i}{\sqrt{\sigma_1}(\sigma_1-x)(\sigma_2-y)}d\sigma_1 d\sigma_2 = \tfrac{1}{\pi i}\int\limits_{-\infty}^\infty \tfrac{C_2(\sigma_2)i}{\sigma_2-y}\tfrac{1}{\pi i}\int\limits_0^\infty \tfrac{d\sigma_1}{\sqrt{\sigma_1}(\sigma_1-x)}d\sigma_2 = 0.$$

The above calculations justify formula (7). Now we prove formula (11). To this end, we substitute (7) into conditions (8), (9). We derive

$$\frac{1}{\pi i}\int\limits_{-\infty}^\infty \frac{R(f;x,\sigma_2)}{\sigma_2+i}d\sigma_2 + \frac{1}{\pi i}\int\limits_{-\infty}^\infty \frac{C_1(x)}{\sigma_2+i}d\sigma_2 + \frac{1}{\pi i}\int\limits_{-\infty}^\infty \frac{C_2(\sigma_2)i}{\sqrt{x}(\sigma_2+i)}d\sigma_2 = p(x), \tag{21}$$

$$\frac{1}{\pi i}\int\limits_0^\infty \frac{R(f;\sigma_1,y)}{\sigma_1+1}d\sigma_1 + \frac{1}{\pi i}\int\limits_0^\infty \frac{C_1(\sigma_1)}{\sigma_1+1}d\sigma_1 + \frac{1}{\pi i}\int\limits_0^\infty \frac{C_2(y)i}{\sqrt{\sigma_1}(\sigma_1+1)} = q(y). \tag{22}$$

Since

$$\frac{1}{\pi i}\int\limits_{-\infty}^\infty \frac{R(f;x,\sigma_2)}{\sigma_2+i}d\sigma_2 = 0, \qquad \frac{1}{\pi i}\int\limits_{-\infty}^\infty \frac{C_1(x)}{\sigma_2+i}d\sigma_2 = -C_1(x),$$

$$\frac{1}{\pi i}\int\limits_0^\infty \frac{R(f;\sigma_1,y)}{\sigma_1+1}d\sigma_1 = 0, \qquad \frac{C_2(y)i}{\pi i}\int\limits_0^\infty \frac{d\sigma_1}{\sqrt{\sigma_1}(\sigma_1+1)} = C_2(y),$$

it follows that (21) and (22) take the forms

$$C_1(x) = \frac{i}{\sqrt{x}} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{C_2(\sigma_2)}{\sigma_2 + i} d\sigma_2 - p(x),$$

$$C_2(y) = q(y) - \frac{1}{\pi i} \int_{0}^{\infty} \frac{C_1(\sigma_1)}{\sigma_1 + 1} d\sigma_1.$$

Hence

$$\varphi(x,y) = R(f;x,y) - p(x) + \frac{q(y)\, i}{\sqrt{x}} + \frac{i}{\sqrt{x}} \left(\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{C_2(\sigma_2)}{\sigma_2 + i} d\sigma_2 - \frac{1}{\pi i} \int_{0}^{\infty} \frac{C_1(\sigma_1)}{\sigma_1 + 1} d\sigma_1 \right).$$

Let us denote

$$\gamma = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{C_2(\sigma_2)}{\sigma_2 + i} d\sigma_2 - \frac{1}{\pi i} \int_{0}^{\infty} \frac{C_1(\sigma_1)}{\sigma_1 + 1} d\sigma_1.$$

Taking into account (10), we obtain

$$\begin{aligned}
\frac{1}{\pi i} \int_{0}^{\infty} \frac{p(\sigma_1)}{\sigma_1 + 1} d\sigma_1 &= \frac{1}{\pi i} \int_{0}^{\infty} \left(\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi(\sigma_1, \sigma_2)}{\sigma_2 + i} d\sigma_2 \right) \frac{d\sigma_1}{\sigma_1 + 1} = \\
&= \frac{1}{\pi i} \int_{0}^{\infty} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{R(f;\sigma_1,\sigma_2)}{\sigma_2 + i} d\sigma_2 \frac{d\sigma_1}{\sigma_1 + 1} - \\
&\quad - \frac{1}{\pi i} \int_{0}^{\infty} \frac{p(\sigma_1)}{\sigma_1 + 1} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{d\sigma_2}{\sigma_2 + i} + \\
&\quad + \frac{i}{\pi i} \int_{0}^{\infty} \frac{d\sigma_1}{\sqrt{\sigma_1}(\sigma_1 + 1)} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{q(\sigma_2)}{\sigma_2 + i} d\sigma_2 + \\
&\quad + \frac{\gamma i}{\pi i} \int_{0}^{\infty} \frac{d\sigma_1}{\sqrt{\sigma_1}(\sigma_1 + 1)} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{d\sigma_2}{\sigma_2 + i} = \\
&= \omega + \omega - \gamma.
\end{aligned}$$

Therefore $\gamma = \omega$, and formula (11) is proved. □

Example 3.1. *Let the functions $f(x,y)$, $p(x)$, $q(y)$ be given by the following formulae*

$$f(x,y) = \frac{1}{x+2} \frac{1}{y+1+i}, \quad p(x) = 0, \quad q(y) = \frac{1}{y+1+i}.$$

Then the solution of (1) *in the function class* $h(\infty)\times h(\infty)$ *has the form*

$$\varphi(x,y)=\frac{i\sqrt{2}\,(x+1)}{\sqrt{x}\,(x+2)\,(y+1+i)}.$$

4. SOLUTION IN THE CLASS $h(0,\infty)\times h(\infty)$

Theorem 4.1. *Let* $f(x,y)\in h(0,\infty)\times h(\infty)$ *satisfy condition* (6). *Then a solution* $\varphi(x,y)$ *of* (1) *in the function class* $h(0,\infty)\times h(\infty)$*, satisfying the relations*

$$\varphi(x,\infty)=0,\quad x\in[0,\infty), \tag{23}$$

and

$$\frac{1}{\pi i}\int_0^\infty \frac{\varphi(\sigma_1,y)}{\sigma_1+1}d\sigma_1=\frac{i\,(y+i)}{(\pi i)^2}\int_0^\infty\int_{-\infty}^\infty \frac{f(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2}{\sqrt{\sigma_1}\,(\sigma_1+1)\,(\sigma_2+i)\,(\sigma_2-y)}, \tag{24}$$

is given by the following formula:

$$\varphi(x,y)=\frac{\sqrt{x}\,(y+i)}{(\pi i)^2}\int_0^\infty\int_{-\infty}^\infty \frac{f(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2}{\sqrt{\sigma_1}\,(\sigma_2+i)\,(\sigma_1-x)\,(\sigma_2-y)}. \tag{25}$$

Proof. As in the proof of Theorem 3.1, equation (1) can be rewritten in form (14). Solving (14) in the function class $h(0,\infty)$, we obtain (cf. [8])

$$\psi_1(x,y)=\frac{\sqrt{x}}{\pi i}\int_0^\infty \frac{f(\sigma_1,y)}{\sqrt{\sigma_1}\,(\sigma_1-x)}d\sigma_1, \tag{26}$$

with the condition

$$\frac{1}{\pi i}\int_0^\infty \frac{\psi_1(\sigma_1,y)}{\sigma_1+1}d\sigma_1=\frac{1}{\pi}\int_0^\infty \frac{f(\sigma_1,y)}{\sqrt{\sigma_1}\,(\sigma_1+1)}d\sigma_1. \tag{27}$$

Substituting (15) into (27), we derive

$$\frac{1}{(\pi i)^2}\int_0^\infty\int_{-\infty}^\infty \frac{\varphi(\sigma_1,\sigma_2)}{(\sigma_1+1)\,(\sigma_2-y)}d\sigma_1 d\sigma_2=\frac{1}{\pi}\int_0^\infty \frac{f(\sigma_1,y)}{\sqrt{\sigma_1}\,(\sigma_1+1)}d\sigma_1. \tag{28}$$

Multiplying each side of (28) by $\frac{1}{y+i}$, on account of (23), using Hilbert transform [8], and finally multiplying both sides of the equation by $y+i$, we obtain condition (24).

Now we solve (15). By [8], there is

$$\varphi(x,y)=\frac{y+i}{\pi i}\int_{-\infty}^\infty \frac{\psi_1(x,\sigma_2)}{(\sigma_2+i)\,(\sigma_2-y)}d\sigma_2=$$

$$=\frac{\sqrt{x}\,(y+i)}{(\pi i)^2}\int_0^\infty\int_{-\infty}^\infty \frac{f(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2}{\sqrt{\sigma_1}\,(\sigma_2+i)\,(\sigma_1-x)\,(\sigma_2-y)}.$$

We can rewrite (1) in form (17), (18). We solve (17) in the function class $h(\infty)$ vanishing at infinity:

$$\psi_2(x,y) = \frac{y+i}{\pi i}\int_{-\infty}^{\infty}\frac{f(x,\sigma_2)}{(\sigma_2+i)(\sigma_2-y)}d\sigma_2.$$

Now we find the solution of (18) in the class of bounded functions:

$$\begin{aligned}\varphi(x,y) &= \frac{\sqrt{x}}{\pi i}\int_0^{\infty}\frac{\psi_2(\sigma_1,y)}{\sqrt{\sigma_1}(\sigma_1-x)}d\sigma_1 = \\ &= \frac{\sqrt{x}(y+i)}{(\pi i)^2}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{f(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2}{\sqrt{\sigma_1}(\sigma_2+i)(\sigma_1-x)(\sigma_2-y)},\end{aligned} \tag{29}$$

with the condition

$$\begin{aligned}\frac{1}{\pi i}\int_0^{\infty}\frac{\varphi(\sigma_1,y)}{\sigma_1+1}d\sigma_1 &= \frac{1}{\pi}\int_0^{\infty}\frac{\psi_2(\sigma_1,y)}{\sqrt{\sigma_1}(\sigma_1+1)}d\sigma_1 = \\ &= \frac{i(y+i)}{(\pi i)^2}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{f(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2}{\sqrt{\sigma_1}(\sigma_1+1)(\sigma_2+i)(\sigma_2-y)}.\end{aligned} \tag{30}$$

Formulae (29) and (30) coincide with (25) and (24), respectively.

As in the proof of Theorem 3.1, one can substitute (25) into (1) and check that function (25) is a solution of equation (1). □

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Received: June 25, 2007.
Revised: July 11, 2007.
Accepted: July 12, 2007.