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ALGEBRAIC CONSTRUCTION OF A COBOUNDARY OF A GIVEN CYCLE

Abstract. We present an algebraic construction of the coboundary of a given cycle as a simpler alternative to the geometric one introduced in [4, 5].

Keywords: algorithm, homology theory, cycle, coboundary.

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1. INTRODUCTION

The Conley index is a very useful tool in the study of dynamical systems. It allows us to prove various results like the existence of periodic points or chaos. However, it is not that easy to compute the index because its construction is based on the quite complicated homotopy and homology theories.

A reasonable solution to this problem is to construct an algorithm for computation of the index. In a series of papers [1, 3–5], Allili and Kaczyński presented an algorithm for the construction of a chain homomorphism induced by a multivalued representable map, which was necessary to compute a homeomorphism induced in homology by a given continuous map. One of its most important parts was the computation of the coboundary of a given cycle σ, i. e., a chain τ satisfying a condition $\partial\tau = \sigma$, which was done by means of geometry (see [4, 5]). In [2], the authors also suggested that this problem may be solved by means of elementary linear algebra. The aim of this paper is to develop the idea of an alternative algebraic construction.

Some remarks on the implementation of the Allili-Kaczyński algorithm were presented in [9]. The appropriate computer program which is a part of the Computational Homology Project was written in C++ language and it is available at [14]. For applications of this algorithm, we refer the reader to [12, 13].

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2. NOTATION

In this section we recall the notation introduced in [1, 3, 4].

Definition 1. *A set $\mathcal{E}$ of cells in $\mathbf{R}^n$ is called a convex grid in $\mathbf{R}^n$ if the following conditions are satisfied:*

(1) $\mathbf{R}^n = \bigcup_{e \in \mathcal{E}} e$,
(2) $e \cap e' = \emptyset$ *if* $e \neq e'$,
(3) *each cell* $e \in \mathcal{E}$ *is a convex polyhedron without boundary,*
(4) *for each bounded set* $B \subset \mathbf{R}^n$ *the family* $stB := \{e \in \mathcal{E} | \, \overline{e} \cap \overline{B} \neq \emptyset\}$ *is finite,*
(5) *if* $B \subset \mathbf{R}^n$ *is bounded, then there exists a representable convex set containing* B *(we call a closed set representable over a grid* $\mathcal{E}$ *if it is a finite union of cells from* $\mathcal{E}$*).*

Example 2. *The following two examples are especially important for applications.*

(i) *Any simplicial triangulation.*
(ii) *The cubic grid of mesh* $\frac{1}{k}$, $k \in \mathbf{N}_*$

$$\mathcal{E}_k := \{\frac{1}{k}(x + Q) \in \mathbf{R}^n | x \in \mathbf{Z}^n, Q = I_1 \times \ldots \times I_n \textit{ and } I_j = (0,1), \{0\} \text{ or } \{1\}\}.$$

Let $\mathcal{E}$ be a convex grid in $\mathbf{R}^n$ and let X be a representable set over $\mathcal{E}$. By $\mathcal{E}(X)$ we denote the set of all cells in X and by $\mathcal{E}^q(X)$ the set of all q–dimensional cells in X.

The original geometric construction of a coboundary was presented by Allili and Kaczyński in [4] for a cubic grid only, but later it was extended to the general case (see [5]). For the sake of simplicity of the presentation, we also restrict our algorithm to the case of a cubic grid. However, unlike the geometric construction, the algebraic one is easily adaptable to wider classes of cells in a space grid. A way of such generalization is suggested in Section 3, paragraphs *The General Case*.

Henceforth we proceed under the assumption that $\mathcal{E}$ is a cubic grid of $\mathbf{R}^n$. It will not result in the loss of generality if we assume that $k = 1$, i. e., that $\mathcal{E}$ is a unitary cubic grid.

3. CONSTRUCTION OF A COBOUNDARY OF A GIVEN CYCLE

In this section, we present a construction of a coboundary $COB(\sigma)$ of a given $(q-1)$–dimensional cycle σ. The construction is based on the elementary linear algebra. To make it applicable to the Allili-Kaczyński algorithm, we require $COB(\sigma) \subset conv_{\mathcal{E}}(\sigma)$, where $conv_{\mathcal{E}}(\sigma)$ denotes the smallest representable convex set containing σ (see [3] for details). Note that representable convex sets over a cubic grid are rectangles.

If $\sigma = \overline{v} - v$ is a zero-dimensional cycle for $v = (v_1, \dots v_n)$ and $\overline{v} = (\overline{v}_1, \dots \overline{v}_n)$, we define its coboundary by the following formula

$$COB(\sigma) := \sum_{i=1}^{n} sgn(\overline{v}_i - v_i)\{\overline{v}_1\} \times \dots \times \{\overline{v}_{i-1}\} \times (min\{v_i, \overline{v}_i\}, max\{v_i, \overline{v}_i\}) \times \\ \times \{v_{i+1}\} \times \dots \times \{v_n\}.$$

Here

$$\{a_1\} \times \dots \times \{a_{i-1}\} \times (a_i, b_i) \times \{a_{i+1}\} \times \dots \times \{a_n\} := \\ := \sum_{j=a_i}^{b_i - 1} \{a_1\} \times \dots \times \{a_{i-1}\} \times (j, j+1) \times \dots \times \{a_n\}.$$

One can easily check that $\partial(COB(\sigma)) = \sigma$. So, we have constructed the coboundary of the cycle σ in the case of $q = 1$.

The General Case. In the general case constructing $COB(\sigma)$ is equivalent to finding such a path of edges in the connected graph $\mathbf{G} = (\mathcal{E}^0(conv_{\mathcal{E}}(\sigma)), \mathcal{E}^1(conv_{\mathcal{E}}(\sigma)))$ which connects v and $\overline{v}$. It is a classical problem of the graph theory and we can apply any standard algorithm to solving it (see [7]).

Now let $q > 1$. Since reduced homologies of the rectangle $conv_{\mathcal{E}}(\sigma)$ are trivial, there exists a coboundary of σ contained in $conv_{\mathcal{E}}(\sigma)$. To find it, it is enough to solve the system of linearly dependent equations $\sigma = \sum_{\{e \in \mathcal{E}^q(conv(\sigma)\}} a_e \partial e$ with respect to coefficients a_e and then put $COB(\sigma) := \sum_{\{e \in \mathcal{E}^q(conv(\sigma))\}} a_e e$.

To implement this step, first we introduce some order of cells in the sets $\mathcal{E}^q(conv(\sigma))$ and $\mathcal{E}^{q-1}(conv(\sigma))$, which allows us to create a computer representation of the above system. For simplification, we make this independent from the dimension and define a universal order in the set $\mathcal{E}(conv(\sigma))$, i.e., a function

$$\mathcal{N} : \mathcal{E}(conv(\sigma)) \to \{1, 2, \dots, d\}$$

(for some $d \in \mathbf{N}$) which is one-to-one but not onto (see Section 3.1). Then we obtain the system of d linearly dependent equations $Ax = b$, where $A = (a_{ij}) \in \mathbf{R}^{d \times d}$ is a matrix and $b \in \mathbf{R}^d$ is a vector such that for $i \in \{1, \dots, d\}$ the coordinate b_i is equal to the coefficient of $\mathcal{N}^{-1}(i)$ in σ. Here, for $i, j \in \{1, \dots, d\}$, $a_{ij} = 0$ or, if $\mathcal{N}^{-1}(j) \in \mathcal{E}^q(conv(\sigma))$, a_{ij} is equal to the coefficient of $\mathcal{N}^{-1}(i)$ in the boundary of $\mathcal{N}^{-1}(j)$.

Finally, we solve the above system using some version of the Gauss algorithm adapted to the “dependent” case (see Section 3.2).

The General Case. In Section 3.1, we define a special order of cells for the cubic grid only, which admits an implementation enabling us to calculate values of functions $\mathcal{N}$ and $\mathcal{N}^{-1}$ using simple formulas. In fact, in the general case, one can take an arbitrary order instead.

One can also notice that the construction of a coboundary depends on a computation of a boundary operator. We give neither the abstract definition of the boundary, which is useless for implementation, nor the special one for a cubic grid. However, a construction of a boundary operator is not a goal of this paper and can easily be done in a case of any standard grid. For all detailed definitions we refer the reader to [8].

3.1. DEFINITION OF AN ORDER OF CELLS IN A GIVEN REPRESENTABLE RECTANGLE

Assume that we are given a representable rectangle $R[a,b] := [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$, where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ denote its "left lower" and "right upper" corner, respectively. Let $d(R[a,b])$ be equal to the dimension of $R[a,b]$), i.e. $d(R[a,b]) = \sharp\{i \in \{1, \ldots, n\} : \ a_i < b_i\}$. If $v \in \mathcal{E}^0(R[a,b])$ is a vertex, then let $v+1$ also be a vertex of $\mathcal{E}^0(R[a,b])$ with coordinates $(v+1)_i = v_i + sgn(b_i - a_i)$ for $i \in 1, \ldots, n$.

First, we define an auxiliary function

$$\mathcal{N}^0_{c,d} : \mathcal{E}^0(R[c,d]) \to \mathbf{N}$$

by the formula

$$\mathcal{N}^0_{c,d}(v) := (v_n - c_n) r_{n-1} r_{n-2} \ldots r_1 + \ldots + (v_2 - c_2) r_1 + (v_1 - c_1), \quad \text{for } v \in \mathcal{E}^0(R[c,d]),$$

where $r_i = d_i - c_i + 1$ for each $i \in \{1, \ldots, n\}$. Note that $\mathcal{N}^0_{c,d}$ is a one-to-one function. Indeed, if $\mathcal{N}^0_{c,d}(v) = m$ for some vertex v and integer number m, one can easily check that then the coordinates of v are defined recursively by the sequence of equations

$$v_1 = m \; mod \; r_1 + c_1,$$

$$v_i = \frac{m - ((v_{i-1} - c_{i-1}) r_{i-2} \ldots r_1 + \ldots + (v_1 - c_1))}{r_{i-1} \ldots r_1} \; mod \; r_i + c_i, \quad \text{for } i \in \{2, \ldots, n\}.$$

So, the function $(\mathcal{N}^0_{c,d})^{-1}$ inverse to $\mathcal{N}^0_{c,d}$ is given on the image of the latter.

Now we are able to define the function

$$\mathcal{N} = \mathcal{N}_{a,b} : \mathcal{E}(R[a,b]) \to \mathbf{N}.$$

Let $\sigma \in \mathcal{E}(R[a,b])$. Then $\sigma = \sigma(s,t) =]s_1, t_1[\times \ldots \times]s_n, t_n[$ for some points $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$, where

$$]s_i, t_i[= \begin{cases} (s_i, t_i), & \text{if} \quad s_i = t_i + 1, \\ \{s_i\}, & \text{if} \quad s_i = t_i \end{cases}$$

for $i \in \{1, \ldots n\}$. Put

$$\mathcal{N}_{a,b}(\sigma) := 2^{d(R[a,b])} (\mathcal{N}^0_{a,b}(s) + 1) - (\mathcal{N}^0_{s,s+1}(t) + 1).$$

Since $0 \leq 2^{d(R[a,b])} - 1 - \mathcal{N}^0_{s,s+1}(t) \leq 2^{d(R[a,b])} - 1$, $\mathcal{N}_{a,b}$ is a one-to-one function with the inverse defined, for each $m \in \mathbf{N}$ belonging to its image, by

$$(\mathcal{N}_{a,b})^{-1}(m) = \sigma(s,t),$$

where

$$s = (\mathcal{N}^0_{a,b})^{-1}(m\ div\ 2^{d(R[a,b])})$$

and

$$t = (\mathcal{N}^0_{s,s+1})^{-1}(2^{d(R[a,b])} - 1 - m\ mod\ 2^{d(R[a,b])}),$$

where $m\ div\ n$ denotes the integer part of the quotient m/n and $m\ mod\ n$ denotes the remainder of the division m/n.

So, we have defined the order $\mathcal{N} = \mathcal{N}_{a,b}$ of cells in the representable rectangle $R[a,b]$.

3.2. FINDING A SOLUTION OF THE SYSTEM OF LINEARLY DEPENDENT EQUATIONS

The goal of this section is to implement the algorithm for finding any solution of the system of linear equations:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad i = 1, \ldots, n \tag{1}$$

As the equations are linearly dependent, the determinant of the matrix $A = [a_{ij}]_{i,j=1,\ldots,n}$ is equal to 0, and one cannot apply the classical Gauss algorithm. However, it can be easily modified to be used also in this case in the following way (cf. [10], Section 9.2).

The first step is to reduce the matrix by eliminating all its zero rows and columns. Of course, the obtained matrix may not be square any more. Let row_num and col_num denote the number of its rows and columns, respectively. Now we triangulate A, i.e., carry out operations like replacement, summation and subtraction of the rows or columns, in order to obtain an equivalent system of equations with a corresponding matrix such that its left upper maximal minor is lower triangular.

We proceed with the following algorithm:

```
i := row_num;
m := min(row_num, col_num);
for j := col_num downto col_num − m + 1 do
begin
   if a_ij = 0 then
     for s := i − 1 downto 1 do
         if a_sj ≠ 0 then
           exchange i-th and s-th rows;
   if a_ij = 0 then
     for t := j − 1 downto 1 do
         if a_it ≠ 0 then
           exchange j-th and t-th columns;
   if a_ij ≠ 0 then
     for s := i − 1 downto 1 do
         replace s-th row by such a linear combination of s-th and i-th rows
         that a_sj = 0;
```

end;
$i := i - 1$;
if $row\_num < col\_num$ then
    replace first row_num columns by the last ones
else
    replace first col_num rows by the last ones;

Let us now denote the left upper maximal minor of the output matrix by B. It is obvious from the construction that B is lower triangular. Moreover, if a certain coefficient a_{ii} on its diagonal is equal to 0, all the coefficients a_{si} for $s < i$ as well as a_{it} for any t are zeros, too. Thus, finding any solution x of system of equations (1) is now very simple. For indices i such that $i > min(row\_num, col\_num)$ or $a_{ii} = 0$, we put $x_i := 0$. In fact, these coefficients of the solution can be chosen arbitrarily, as a set of columns with indices i such that $i \leq min(row\_num, col\_num)$ and $a_{ii} \neq 0$ forms a minor of maximal rank in A. The remaining coefficients of the solution are determined the same way as in the classical Gauss algorithm:

for $i := 1$ to $m := min(row\_num, col\_num)$ do
    if $a_{ii} = 0$ then
        $x_i := 0$
    else
    begin
        $temp := 0$;
        for $t := 1$ to $i - 1$ do
            $temp := temp + a_{it} * x_t$;
        $x_i := (b_i - temp)/a_{ii}$
    end;
for $i := m + 1$ to col_num do
    $x_i := 0$;

4. EXAMPLE

Consider the following one-dimensional cycle whose coboundary consists of 3 cubic cells (Fig. 1).

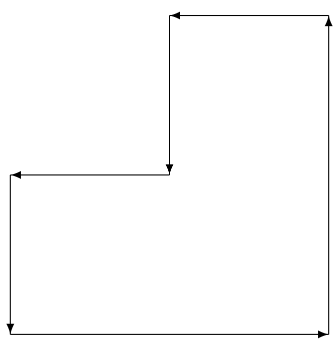

Fig. 1

The geometric construction of a coboundary of this cycle was presented in [4]. Our goal is to obtain it using the algebraic algorithm.

First of all, we enclose the cycle in its convex hull, i.e., a square which is a union of 9 zero-dimensional, 12 one-dimensional and 4 two-dimensional cells. Then we number all cells using the previously defined order. The numbers are as follows (Fig. 2).

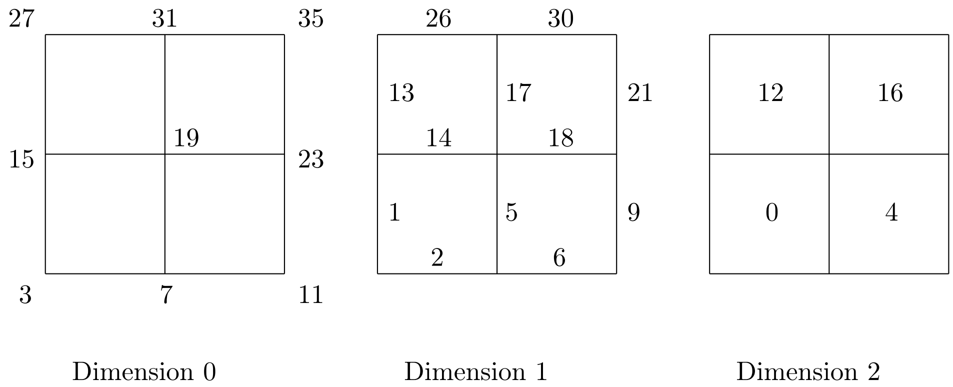

Fig. 2

Next step is to express the problem by means of system (1) of linear equations. In our example, after the reduction, the corresponding matrix A is equal to

$$
\begin{array}{c c}
 & \begin{pmatrix} 0 & 4 & 12 & 16 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 2 \\ 5 \\ 6 \\ 9 \\ 13 \\ 14 \\ 17 \\ 18 \\ 21 \\ 26 \\ 30 \end{pmatrix} &
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 \\
0 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\end{array},
$$

where the numbers in the left column and the top row denote the numbers of one-dimensional and two-dimensional cells in the convex hull of the cycle, respectively.

The vector b is equal to

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

The process of triangulation changes A into

$$\begin{array}{cc} & (\;\; 0 \quad\; 4 \quad\; 12 \quad\; 16 \;\;) \\ \begin{pmatrix} 14 \\ 18 \\ 26 \\ 30 \\ 9 \\ 13 \\ 21 \\ 17 \\ 1 \\ 2 \\ 5 \\ 6 \end{pmatrix} & \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

and b into

$$\begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, we obtain the following easily solvable system of linear equations:

$$\begin{cases} -x_0 = -1, \\ -x_4 = -1, \\ -x_{12} = 0, \\ -x_{16} = -1. \end{cases}$$

Its solution

$$\begin{cases} x_0 = 1, \\ x_4 = 1, \\ x_{12} = 0, \\ x_{16} = 1 \end{cases}$$

indicates the coefficients of the cells in a coboundary of the cycle. Therefore, it consists of 3 two-dimensional cells whose numbers are 0, 4 and 16.

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