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THE USE OF INTEGRAL INFORMATION IN THE SOLUTION OF A TWO-POINT BOUNDARY VALUE PROBLEM

Abstract. We study the worst-case ε-complexity of a two-point boundary value problem $u''(x) = f(x)\,u(x)$, $x \in [0,T]$, $u(0) = c$, $u'(T) = 0$, where $c, T \in \mathbb{R}$ ($c \neq 0$, $T > 0$) and f is a nonnegative function with r ($r \geq 0$) continuous bounded derivatives. We prove an upper bound on the complexity for linear information showing that a speed-up by two orders of magnitude can be obtained compared to standard information. We define an algorithm based on integral information and analyze its error, which provides an upper bound on the ε-complexity.

Keywords: boundary value problem, complexity, worst case setting, linear information.

Mathematics Subject Classification: 68Q25, 65L10.

1. INTRODUCTION

The goal of this paper is to study possible advantages of using linear information in the solution of the following problem. Let $f \in \mathcal{C}^r([0,T])$ ($r \geq 0$) be a nonnegative function, $c \neq 0$, $T > 0$ be real numbers. We wish to find a function $u = u_f$ such that

$$u''(x) = f(x)\,u(x)\,, \quad x \in [0,T]\,, \quad u(0) = c, \quad u'(T) = 0, \quad (c \neq 0)\,. \tag{1}$$

It is known that there exists the unique solution u_f to this problem (see [6]). We consider the worst-case setting in which the error of an algorithm is measured by its worst performance in a class of right-hand side functions.

Any method for solving (1) is based on some information on f, which usually consists of the values of f or its derivatives (standard information). Standard information for problem (1) was considered in [4]. Lemma 4.4 from [4] implies that the lower bound on the error of any algorithm using standard information is of order n^{-r}. This yields that the minimal cost of solving problem (1) up to precision ε (the ε-complexity) is $\Omega\left(\left(\frac{1}{\varepsilon}\right)^{1/r}\right)$. In a different setting, a similar problem was also considered in [7].

In this paper, we are concerned with the following question: is it possible to successfully use linear information instead of standard information for solving our problem, and how much such a change influences the ε-complexity?

The use of linear information yields a speed-up for initial-value problems (see [3]). The other motivation for studying linear information comes from the fact that deterministic algorithms based on linear information for initial-value problems lead to almost optimal algorithms in the randomized and quantum setting (see [5])

In this paper, we define an algorithm for solving (1) based on integral information, and we analyze its error. We show that by using n evaluations we can obtain the worst-case error $O\left(n^{-(r+2)}\right)$, which leads to the upper complexity bound $O\big(\left(\frac{1}{\varepsilon}\right)^{1/(r+2)}\big)$. This shows that the use of integral information allows us to improve the worst-case complexity by 2 orders of magnitude compared to standard information.

The paper is organized as follows. In Section 3 we define an algorithm ϕ^* based on linear information, and in Section 4 we prove necessary facts that are used in the error analysis. The error of ϕ^* and the ε-complexity of the problem are studied in Section 5 (Theorem 1).

2. PROBLEM FORMULATION

We consider the following problem:

Find $u = u_f$ such that

$$\begin{cases} u''(x) = f(x)\, u(x), & \\ u(0) = c, & \\ u'(T) = 0, & x \in [0, T], \end{cases} \tag{2}$$

where $T > 0$ and $c \neq 0$. We assume that, $f \in \mathcal{F}_r$ where

$$\mathcal{F}_r = \left\{ f : f \in \mathcal{C}^r([0,T]), f(x) \geq 0, \left\| f^{(i)} \right\| \leq D_i,\ i = 0, 1, \ldots, r \right\},\ r \geq 0,$$

and D_i are given positive numbers. The problem considered in this paper arises with the following optimal control problem (see [6]):

Find $u = u_f \in D = \{u : [0,T] \to \mathbb{R},\ u(0) = c,\ u \in \mathcal{C}^2([0,T])\}$ such that

$$\int_0^T \left([u'(x)]^2 + f(x)\,(u(x))^2 \right) dx \to \min \ \ (\text{in } D). \tag{3}$$

It is known (see to [6]) that (3) has the unique solution $u = u_f$ and that u is also the unique solution boundary value problem (2).

Any method for solving (2) is based on certain information on f. By *information with n evaluations* about f we mean n real numbers $N(f)$, where

$$N : \mathcal{F}_r \to \mathbb{R}^n$$

is a given operator.

Standard information $N(f)$ is defined by an arbitrary selection of n values of f or its derivatives. The class of all standard information operators will be denoted by $\mathcal{N}^{st}$. More generally, *linear information* has the form

$$N(f) = [v_1, \ldots, v_n]$$

with $v_i = L_i(f; v_1, \ldots, v_{i-1})$, where $L_i(\cdot; v_1, \ldots, v_{i-1})$ is a linear functional. The class of all linear information operators will be denoted by $\mathcal{N}^{lin}$.

Approximation to the solution u_f is given by an *algorithm* ϕ. By an algorithm we mean any mapping defined on $N(\mathcal{F}_r)$ which transforms $N(f)$ into a piecewise continuous function

$$\phi(N(f)) : [0, T] \to \mathbb{R}$$

on $[0, T]$.

The *error* of an algorithm at f is measured in the supremum norm on $[0, T]$

$$e(\phi, f) = \max_{x \in [0,T]} |u_f(x) - \phi(N(f))(x)| .$$

We shall study the *worst-case error* of ϕ in the class $\mathcal{F}_r$, defined by

$$e(\phi, \mathcal{F}_r) = \sup_{f \in \mathcal{F}_r} e(\phi, f) .$$

The cost of an algorithm is meant as a number n of functional evaluations accessed by an algorithm.

Bounds on $e(\phi, \mathcal{F}_r)$ will provide us with bounds on the ε-complexity of problem (2). The ε-complexity is defined to be the minimal cost of an algorithm with respect to all algorithms and all information operators (from the class $\mathcal{N}^{st}$ or $\mathcal{N}^{lin}$). That is, given $\varepsilon > 0$, we define the ε-complexity by

$$\mathrm{comp}^{\mathrm{st(lin)}}(\varepsilon) = \min\{n : \exists N \text{ with } n \text{ evaluations (standard or linear)}, \\ \exists \phi \text{ such that } e(\phi, \mathcal{F}_r) \le \varepsilon\}.$$

Standard information for the solution of (2) has been studied in [4]. The ε-complexity of (2) for standard information, as a corollary from Lemma 4.4 in [4], is equal to

$$\mathrm{comp}^{\mathrm{st}}(\varepsilon) = \Theta\left(\varepsilon^{-\frac{1}{r}}\right) . \tag{4}$$

In the present paper we show how linear information on f may be used to solve (2). We shall show that the use of linear information defined by integrals of f yields a speed-up over algorithms which use standard information.

3. INTEGRAL INFORMATION IN THE SOLUTION OF (2)

In this section we define algorithm ϕ^* that uses linear information to solve (2). Consider a uniform partition of $[0, T]$ with points $x_i = ih$, where $h = T/n$ and $i = 0, 1, \ldots, n$.

Denote by $u_{j,i}$ the solution of initial value problem

$$\begin{cases} u''(x) = f(x)\, u(x), \\ u(x_i) = \delta_{j,1}, \\ u'(x_i) = \delta_{j,2}, \end{cases} \tag{5}$$

where $x \in [x_i, x_{i+1}]$, $\delta_{j,k}$ is the Kronecker delta, $j = 1, 2$ and $i = 0, 1, \ldots, n-1$.

Let $s = [s_0, s'_0, s_1, s'_1, \ldots, s_n, s'_n]^T$ be the solution of a system of linear equations

$$\begin{cases} b_{i+1}s_i + a_{i+1}s'_i - s'_{i+1} = 0 \\ d_{i+1}s_i + r_{i+1}s'_i - s_{i+1} = 0 \qquad i = 0, 1, \ldots, n-1, \\ \quad s_0 = c, \ s'_n = 0 \end{cases} \tag{6}$$

where $a_{i+1} = u'_{2,i}(x_{i+1})$, $b_{i+1} = u'_{1,i}(x_{i+1})$, $r_{i+1} = u_{2,i}(x_{i+1})$ and $d_{i+1} = u_{1,i}(x_{i+1})$. The system can be written in the matrix form

$$A_n s = p \tag{7}$$

where matrix A_n is given by

$$A_n = \begin{bmatrix} a_1 & 0 & -1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ r_1 & -1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \vdots \\ 0 & b_2 & a_2 & 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & d_2 & r_2 & -1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & b_n & a_n & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & d_n & r_n & -1 \end{bmatrix} \tag{8}$$

and $p = [-b_1 c, -d_1 c, 0, \ldots, 0]^T$. Then the solution u_f of (2) can be written as

$$u_f(x) = s_i u_{1,i}(x) + s'_i u_{2,i}(x), \tag{9}$$

for $x \in [x_i, x_{i+1}]$ and $i = 0, 1, \ldots, n-1$ (see [6, pp. 493–497 and 485]). The matrix A_n of this system of equations is nonsingular, since (2) has a unique solution.

The algorithm ϕ^* will be based on the computation of certain approximations to s_i and s'_i. To compute approximately s_i and s'_i, we first approximate the unknown numbers a_{i+1}, b_{i+1}, d_{i+1} and r_{i+1} in (6) by $\tilde{a}_{i+1}$, $\tilde{b}_{i+1}$, $\tilde{d}_{i+1}$ and $\tilde{r}_{i+1}$. Note that

$$u'(x_{i+1}) = u'(x_i) + \int_{x_i}^{x_{i+1}} f(t)\, u(t)\, dt, \tag{10}$$

$$u(x_{i+1}) = u(x_i) + u'(x_i)h + \int_{x_i}^{x_{i+1}} \int_{x_i}^{t} f(s)\,u(s)\,dsdt. \tag{11}$$

Define an approximation $l_{j,i}(x)$ of $u_{j,i}(x)$ by Taylor's expansion of $u_{j,i}$ at x_i,

$$l_{j,i}(x) = \sum_{k=0}^{r+1} \frac{u_{j,i}^{(k)}(x_i)}{k!}(x - x_i)^k. \tag{12}$$

Using (10) and (11), for $i = 0, 1, \ldots, n-1$, we define

$$\tilde{b}_{i+1} = u'_{1,i}(x_i) + \int_{x_i}^{x_{i+1}} f(x)\,l_{1,i}(x)\,dx,$$

$$\tilde{d}_{i+1} = u_{1,i}(x_i) + u'_{1,i}(x_i)h + \int_{x_i}^{x_{i+1}} \int_{x_i}^{t} f(s)\,l_{1,i}(s)\,dsdt.$$

Replacing $u_{1,i}$ with $u_{2,i}$ and $l_{1,i}$ with $l_{2,i}$, we obtain similar formulas for $\tilde{a}_{i+1}$ and $\tilde{r}_{i+1}$:

$$\tilde{a}_{i+1} = u'_{2,i}(x_i) + \int_{x_i}^{x_{i+1}} f(x)\,l_{2,i}(x)\,dx,$$

$$\tilde{r}_{i+1} = u_{2,i}(x_i) + u'_{2,i}(x_i)h + \int_{x_i}^{x_{i+1}} \int_{x_i}^{t} f(s)\,l_{2,i}(s)\,dsdt.$$

Let us denote by $\tilde{A}_n$ and $\tilde{p}$ a new matrix and a new right-hand side vector, respectively, with coefficients $a_{i+1},\ b_{i+1},\ d_{i+1},\ r_{i+1}$ replaced by $\tilde{a}_{i+1},\ \tilde{b}_{i+1},\ \tilde{d}_{i+1},\ \tilde{r}_{i+1}$. We shall later show that $\tilde{A}_n$ is nonsingular. By solving the linear system

$$\tilde{A}_n\tilde{s} = \tilde{p} \tag{13}$$

we obtain a new solution vector $\tilde{s} = [\tilde{s}_0, \tilde{s}'_0, \tilde{s}_1, \tilde{s}'_1, \ldots, \tilde{s}_n, \tilde{s}'_n]^T$.

We now define an algorithm ϕ^*. We do it by replacing $u_{j,i}$, s_i and s'_i in (9) with $l_{j,i}$, $\tilde{s}_i$ and $\tilde{s}'_i$, respectively. The approximate solution $\tilde{u}_f(x)$ is defined as

$$\tilde{u}_f(x) = \tilde{s}_i l_{1,i}(x) + \tilde{s}'_i l_{2,i}(x). \tag{14}$$

Using (5) to express the derivatives of $u_{j,i}$, we see that the approximations $\tilde{a}_{i+1}, \tilde{b}_{i+1}, \tilde{r}_{i+1}, \tilde{d}_{i+1}$ are based on the values of f and its derivatives up to the order $r-1$ at some points. Additionally, they use the values of integrals of the form

$$\int_{x_i}^{x_{i+1}} f(t)\,dt,\ \int_{x_i}^{x_{i+1}} \int_{x_i}^{t} f(s)\,dsdt,\ \int_{x_i}^{x_{i+1}} f(t)(t - x_i)^k\,dt,\ \int_{x_i}^{x_{i+1}} \int_{x_i}^{t} f(s)(s - x_i)^k\,dsdt.$$

The algorithm is thus based on linear information. We shall analyze the error of algorithm ϕ^* in the next sections.

4. PRELIMINARY RESULTS

We shall in the sequel show that the matrix $\tilde{A}_n$ in (13) is nonsingular. We first establish some bounds concerning the matrix A_n.

Fact 1.

$$\|A_n\|_\infty = O(1).$$

Proof. Let us first note that

$$\begin{aligned}
a_{i+1} &= u'_{2,i}(x_{i+1}) = \\
&= u'_{2,i}(x_i) + \frac{u''_{2,i}(x_i)}{1!}(x_{i+1}-x_i) + \frac{u'''_{2,i}(x_i)}{2!}(x_{i+1}-x_i)^2 + \ldots = \\
&= 1 + \frac{f(x_i)\cdot u_{2,i}(x_i)}{1}(x_{i+1}-x_i) + C_a(h,i,f)\cdot h^2 = \\
&= 1 + O\left(h^2\right),
\end{aligned}$$

where

$$|C_a(h,i,f)| \le C_a$$

and constant C_a depends only on parameters of the class $\mathcal{F}_r$. By $C_b(h,i,f)$, $C_d(h,i,f)$ and $C_r(h,i,f)$, we denote similar constants appearing in the expansions of b_{i+1}, d_{i+1} and r_{i+1}, respectively. Additionally, by C_b, C_d and C_r we denote bounds on these constants which only depend on parameters of the class $\mathcal{F}_r$. Now, similarly as for coefficient a_{i+1}, we find

$$\begin{aligned}
b_{i+1} &= f(x_i)(x_{i+1}-x_i) + O\left(h^2\right), \\
d_{i+1} &= 1 + O\left(h^2\right), \\
r_{i+1} &= (x_{i+1}-x_i) + O\left(h^3\right),
\end{aligned}$$

where constants in the "O" notation depend only on the parameters of the class $\mathcal{F}_r$. This yields that

$$\|A_n\|_\infty = \max_{i=1,\ldots,n}\{|b_i| + |a_i| + 1; |d_i| + |r_i| + 1\} = O(1),$$

where the constant depends only on parameters of the class $\mathcal{F}_r$. This completes the proof of the fact. □

Let us consider the system

$$A_n z = l \tag{15}$$

where the matrix A_n is given by (8),

$$z = [w_1, y_1, w_2, y_2, \ldots, w_n, y_n]^T$$

and

$$l = [l_1, l_2, l_3, \ldots, l_{2n-1}, l_{2n}]^T.$$

Define sequences $\{p_i\}$ and $\{k_i\}$ by

$$\begin{aligned} p_n &= 0, \\ p_{i-1} &= (b_i + p_i d_i) \,/\, (a_i + p_i r_i) \end{aligned} \tag{16}$$

and

$$\begin{aligned} k_{n+1} &= 0, \\ k_i &= (l_{2i-1} + p_i l_{2i} + k_{i+1}) \,/\, (a_i + p_i r_i) \end{aligned} \tag{17}$$

for $i = n, n-1, \ldots, 1$. The following fact holds:

Fact 2. *The solution of $A_n z = l$ can be expressed as*

$$\begin{aligned} y_0 &= 0, \\ y_i &= (d_i - p_{i-1} r_i)\, y_{i-1} + r_i k_i - l_{2i}, \\ w_i &= k_i - p_{i-1} y_{i-1} \end{aligned}$$

for $i = 1, 2, \ldots, n$.

Proof. We proceed by induction. Let $i = n$. Taking the last two equations of the system $A_n z = l$:

$$\begin{cases} b_n y_{n-1} + a_n w_n = l_{2n-1} \\ d_n y_{n-1} + r_n w_n - y_n = l_{2n} \end{cases}$$

we obtain

$$w_n = \frac{l_{2n-1}}{a_n} - \frac{b_n}{a_n} y_{n-1} = k_n - p_{n-1} y_{n-1},$$

where

$$\begin{aligned} k_n &= \frac{l_{2n-1} + p_n l_{2n} + k_{n+1}}{a_n + p_n r_n}, \\ p_{n-1} &= \frac{b_n + p_n d_n}{a_n + p_n r_n}. \end{aligned}$$

Moreover,

$$\begin{aligned} y_n &= d_n y_{n-1} + r_n w_n - l_{2n} = \\ &= d_n y_{n-1} + r_n (k_n - p_{n-1} y_{n-1}) - l_{2n} = \\ &= (d_n - r_n p_{n-1})\, y_{n-1} + r_n k_n - l_{2n}. \end{aligned}$$

where k_n and p_n are defined by (16) and (17), respectively.
By induction, let us assume that

$$w_{i+1} = k_{i+1} - p_i y_i \tag{18}$$

for fixed $i \in \{n-1, \ldots, 2\}$. Taking equations

$$\begin{cases} b_i y_{i-1} + a_i w_i - w_{i+1} = l_{2i-1} \\ d_i y_{i-1} + r_i w_i - y_i = l_{2i} \end{cases}$$

and using (18), we obtain

$$\begin{aligned} w_i &= \frac{l_{2i-1} + l_{2i}p_i + k_{i+1}}{a_i + p_i r_i} - \frac{b_i + d_i p_i}{a_i + r_i p_i} y_{i-1} = \\ &= k_i - p_{i-1} y_{i-1} \end{aligned}$$

and

$$\begin{aligned} y_i &= d_i y_{i-1} + r_i w_i - l_{2i} = \\ &= d_i y_{i-1} + r_i \left(k_i - p_{i-1} y_{i-1}\right) - l_{2i} = \\ &= \left(d_i - r_i p_{i-1}\right) y_{i-1} + r_i k_i - l_{2i}. \end{aligned}$$

Finally, by induction, let us assume that

$$w_2 = k_2 - p_1 y_1. \tag{19}$$

From the first pair of equations of (15)

$$\begin{cases} a_1 w_1 - w_2 = l_1 \\ r_1 w_1 - y_1 \ = l_2 \end{cases}$$

using (19) we obtain

$$\begin{aligned} w_1 &= \frac{l_1 + l_2 p_1 + k_2}{a_1 + p_1 r_1} - p_0 y_0 = \\ &= k_1 - p_0 y_0 \end{aligned}$$

and

$$\begin{aligned} y_1 &= r_1 w_1 - l_2 = \\ &= d_1 y_0 + r_1 \left(k_1 - p_0 y_0\right) - l_2 = \\ &= \left(d_1 - p_0 r_1\right) y_0 + r_1 k_1 - l_2, \end{aligned}$$

where $y_0 = 0$. □

Now we are ready to prove an upper bound on $\left\|A_n^{-1}\right\|_\infty$. The following fact holds:

Fact 3. *For matrix A_n defined by* (8), *there is*

$$\left\|A_n^{-1}\right\|_\infty = O\left(n\right). \tag{20}$$

Proof. Let us write the matrix A_n in the form

$$A_n = F_n + G_n,$$

where

$$F_n = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ h & -1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \vdots \\ 0 & f(x_1)h & 1 & 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & h & -1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & f(x_{n-1})h & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & h & -1 \end{bmatrix}$$

and

$$G_n = h^2 \begin{bmatrix} C_a(h,i,f) & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ hC_r(h,i,f) & 0 & 0 & 0 & \cdots & \cdots & \cdots & \vdots \\ 0 & hC_b(h,i,f) & C_a(h,i,f) & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & C_d(h,i,f) & hC_r(h,i,f) & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & hC_b(h,i,f) & C_a(h,i,f) & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & C_d(h,i,f) & hC_r(h,i,f) & 0 \end{bmatrix}.$$

Note that

$$\|F_n\|_\infty = O(1)$$

and

$$\|G_n\|_\infty = O\left(h^2\right), \tag{21}$$

where constant in the "O" notation depends only on the parameters of the class $\mathcal{F}_r$.
Note first that F_n is nonsingular.
We now show that $\left\|F_n^{-1}\right\|_\infty = O(n)$. To find F_n^{-1}, we solve $2n$ equations

$$F_n \cdot f_j = e_j$$

for $j = 1, 2, \ldots, 2n$ where

$$F_n^{-1} = [f_1, f_2, \ldots, f_{2n}],$$
$$f_j = \left[w_1^{(j)}, y_1^{(j)}, \ldots, w_n^{(j)}, y_n^{(j)}\right]^T$$

and

$$e_j = \left[0, \ldots, 0, \underset{(j)}{1}, 0, \ldots, 0\right]^T.$$

We consider two cases in which we prove that all elements of the matrix F_n^{-1} are bounded by a constant. Let us first note that the sequence $\{p_i\}$ given by (16) with $b_i = f(x_{i-1})\,h$, $d_i = a_i = 1$ and $r_i = h$ is nonnegative and bounded by a constant independent of i and h; denote it by C_p. Indeed, it is easy to note that $p_i \geq 0$ for $i = 1, 2, \ldots, n$ and $\frac{1}{1+hp_i} < 1$. Moreover, from the formula for p_{i-1}, we can write

$$p_{i-1} \leq D_0 h + p_i.$$

Taking the sum over i and remembering that $p_n = 0$, we obtain the inequality:

$$\sum_{i=1}^{n} p_{i-1} \leq nD_0 h + \sum_{i=1}^{n} p_i,$$
$$p_0 \leq D_0 T := C_p.$$

Case 1.
Fix t, $t = 1, 2, \ldots, n$, and take $j = 2t$. Let us put

$$e_{2t} = [l_1, l_2, \ldots, l_{2t}, \ldots, l_{2n}]^T = \\ = \left[0, \ldots, 0, \underset{(2t)}{1}, 0, \ldots, 0\right]^T.$$

The sequence $\{k_i\}$ defined by (17) now takes the form:

$$k_{n+1} = k_n = \ldots = k_{t+1} = 0,$$
$$k_t = \frac{p_t}{1 + p_t h}$$

and

$$k_i = \frac{k_{i+1}}{1 + p_i h}$$

for $i = t-1, \ldots, 1$. Thus the sequence $\{k_i\}$ is bounded by a constant C_p for any $i = 1, 2, \ldots, n$. From Fact 2, we derive:

$$\begin{array}{ll} w_i = k_i - p_{i-1} y_{i-1} & \text{for} \quad i = 1, 2, \ldots, t, \\ w_i = -p_{i-1} y_{i-1} & \text{for} \quad i = t+1, \ldots, n, \end{array} \tag{22}$$

and

$$\begin{array}{ll} y_0 = 0, & \\ y_i = (1 - p_{i-1} h)\, y_{i-1} + h k_i & \text{for} \quad i = 1, \ldots, t-1, \\ y_t = (1 - p_{t-1} h)\, y_{t-1} + h k_t - 1, & \\ y_i = (1 - p_{i-1} h)\, y_{i-1} & \text{for} \quad i = t+1, \ldots n. \end{array}$$

Since p_i and k_i are bounded by C_p, the inequality $\left(1 + \frac{a}{n}\right)^n \leq e^a$ yields that

$$\begin{array}{ll} |w_i| \leq C_p \left(1 + e^{C_p T}\right) & \text{for} \quad i = 1, 2, \ldots t, \\ |w_i| \leq C_p e^{C_p T} & \text{for} \quad i = t+1, \ldots, n, \end{array}$$

and

$$\begin{array}{lll} |y_i| \le (1 + C_p h)^i - 1 \le e^{C_p T} - 1 & \text{for} & i = 1, 2, \ldots, t-1, \\ |y_t| \le (1 + C_p h)^t \le e^{C_p T}, & & \\ |y_i| \le (1 + C_p h)^i \le e^{C_p T} & \text{for} & i = t+1, \ldots, n. \end{array}$$

Case 2.

Fix t, $t = 1, 2, \ldots, n$, and take $j = 2t - 1$. In the same way as in Case 1, we obtain the sequence $\{k_i\}$ in the following form

$$k_{n+1} = k_n = \ldots = k_{t+1} = 0,$$
$$k_t = \frac{1}{1 + p_t h}$$

and

$$k_i = \frac{k_{i+1}}{1 + p_i h}$$

for $i = t-1, \ldots, 1$. Analogously as in Case 1, by using Fact 2, we obtain a formula for w_i identical as in (22). Additionally we obtain

$$\begin{array}{lll} y_i = (1 - p_{i-1} h)\, y_{i-1} + h k_i & \text{for} & i = 1, \ldots, t, \\ y_i = (1 - p_{i-1} h)\, y_{i-1} & \text{for} & i = t+1, \ldots n, \end{array}$$

and $y_0 = 0$. In a similar way as in Case 1, we obtain

$$|y_i| \le \frac{(1 + C_p h)^i - 1}{C_p} \le \frac{e^{C_p T} - 1}{C_p}$$

for $i = 1, 2, \ldots, t$, and

$$\begin{aligned} |y_i| &\le \frac{1}{C_p} \left[(1 + C_p h)^i - (1 + C_p h)^{i-t} \right] = \\ &= \frac{(1 + C_p h)^i}{C_p} \left(1 - \frac{1}{(1 + C_p h)^t} \right) \le \\ &\le \frac{1}{C_p} e^{C_p T} \end{aligned}$$

for $i = t+1, \ldots, n$. Using that in (22), we also obtain

$$\begin{array}{lll} |w_i| \le C_p \left(1 + \frac{1}{C_p} \left(e^{C_p T} - 1 \right) \right) = C_p + e^{C_p T} - 1 & \text{for} & i = 1, 2, \ldots t, \\ |w_i| \le e^{C_p T} & \text{for} & i = t+1, \ldots, n. \end{array}$$

These two cases lead to a conclusion that all elements of the matrix F_n^{-1} are bounded by a constant, which yields

$$\left\| F_n^{-1} \right\|_\infty = O(n). \tag{23}$$

Now consider the matrix A_n^{-1}. Note that

$$A_n^{-1} = (F_n + G_n)^{-1} = \\ = \left(I + F_n^{-1} G_n\right)^{-1} F_n^{-1}.$$

For any matrix K with $\|K\|_\infty < 1$, there holds

$$\left\|(I + K)^{-1}\right\|_\infty \leq \frac{1}{1 - \|K\|_\infty}. \tag{24}$$

Hence,

$$\begin{aligned} \left\|A_n^{-1}\right\|_\infty &= \left\|\left(I + F_n^{-1} G_n\right)^{-1} F_n^{-1}\right\|_\infty \leq \\ &\leq \left\|\left(I + F_n^{-1} G_n\right)^{-1}\right\|_\infty \left\|F_n^{-1}\right\|_\infty \leq \\ &\leq \frac{\left\|F_n^{-1}\right\|_\infty}{1 - \left\|F_n^{-1}\right\|_\infty \|G_n\|_\infty}. \end{aligned} \tag{25}$$

Using now (21) and (23) in (25), we obtain

$$\left\|A_n^{-1}\right\|_\infty \leq Cn,$$

where the constant C is independent of n. This ends the proof of Fact 3. □

5. ERROR OF THE ALGORITHM ϕ^*

To prove an upper bound on the error of ϕ^*, we need some results concerning the quality of approximating $a_{i+1}, b_{i+1}, d_{i+1}$ and r_{i+1} in (6) with $\tilde{a}_{i+1}, \tilde{b}_{i+1}, \tilde{d}_{i+1}$ and $\tilde{r}_{i+1}$.

Lemma 1. *For $f \in \mathcal{F}_r$, there holds*

$$\varepsilon_n = \max_{i=1,\ldots,n} \max\left\{|a_i - \tilde{a}_i|, \left|b_i - \tilde{b}_i\right|, \left|d_i - \tilde{d}_i\right|, |r_i - \tilde{r}_i|\right\} = O\left(n^{-(r+3)}\right) \tag{26}$$

as $n \to +\infty$.

Proof. Let us first note that for $x \in [x_i, x_{i+1}]$ and $j = 1, 2$ the following inequality holds:

$$\sup_{x \in [x_i, x_{i+1}]} |u_{j,i}(x) - l_{j,i}(x)| = \sup_{x \in [x_i, x_{i+1}]} \left|\frac{u_{j,i}^{(r+2)}(\xi_x)}{(r+2)!}(x - x_i)^{r+2}\right| \leq C_j \cdot h^{r+2}. \tag{27}$$

Hence

$$\begin{aligned} \max_{i=1,\ldots,n} |a_{i+1} - \tilde{a}_{i+1}| &= \max_{i=1,\ldots,n} \left|\int_{x_i}^{x_{i+1}} f(x)\left(u_{2,i}(x) - l_{2,i}(x)\right) dx\right| \leq \\ &\leq D_0 \cdot h \cdot \sup_{x \in [x_i, x_{i+1}]} |u_{2,i}(x) - l_{2,i}(x)| \leq \\ &\leq D_0 \cdot h \cdot C_2 \cdot h^{r+2} = O\left(h^{r+3}\right) \end{aligned} \tag{28}$$

and

$$\max_{i=1,\ldots,n}|r_{i+1}-\tilde{r}_{i+1}| = \max_{i=1,\ldots,n}\left|\int\limits_{x_i}^{x_{i+1}}\int\limits_{x_i}^{t} f(s)\left(u_{2,i}(s)-l_{2,i}(s)\right)dsdt\right| \leq$$
$$\leq \frac{1}{2}h^2\cdot D_0\cdot \sup_{x\in[x_i,x_{i+1}]}|u_{2,i}(x)-l_{2,i}(x)| \leq \tag{29}$$
$$\leq \frac{1}{2}h^2\cdot D_0\cdot C_2\cdot h^{r+2} = O\left(h^{r+4}\right)$$

In a similar way, we obtain

$$\max_{i=1,\ldots,n}\left|b_{i+1}-\tilde{b}_{i+1}\right| = O\left(h^{r+3}\right), \tag{30}$$
$$\max_{i=1,\ldots,n}\left|d_{i+1}-\tilde{d}_{i+1}\right| = O\left(h^{r+4}\right). \tag{31}$$

Using (28) $-$ (31), we get the desired upper bound on ε_n. □

We now show that system (13) has a unique solution.

Fact 4. *Matrix $\tilde{A}_n$ in* (13) *is nonsingular.*

Proof. From Fact 3. there follows that

$$\left\|A_n^{-1}\right\|_\infty = O(n),$$

as $n\to\infty$. Since

$$\tilde{A}_n = A_n\left(I + A_n^{-1}\left(\tilde{A}_n - A_n\right)\right)$$

and $\left\|\tilde{A}_n - A_n\right\|_\infty = O\left(h^{r+3}\right)$ (see (26)), we infer that $\tilde{A}_n$ is nonsingular for a sufficiently large n. □

Lemma 2. *System* (13) *with coefficients $\tilde{a}_{i+1},\tilde{b}_{i+1},\tilde{d}_{i+1}$ and $\tilde{r}_{i+1}$ has the unique solution $\tilde{s}$ such that*

$$\|s-\tilde{s}\|_\infty = O\left(n^{-(r+2)}\right). \tag{32}$$

Proof. Note first that there exists a constant S_1 independent of n (see [6]) such that

$$\sup_{x\in[0,T]}|u_f(x)| \leq S_1. \tag{33}$$

Since

$$u_f'(T) = u_f'(x) + \int\limits_x^T f(t)\,u_f(t)\,dt$$

for $x \in [0,T]$, then

$$\sup_{x\in[0,T]} \left|u_f'(x)\right| \le \sup_{x\in[0,T]} \left|\int_x^T f(t)\, u_f(t)\, dt\right| \le T \cdot D_0 \cdot S_1 := S_2. \tag{34}$$

Moreover, from Lemma 1, we derive

$$\|\tilde{p} - p\|_\infty = |c| \max\left\{\left|\tilde{b}_1 - b_1\right|, \left|\tilde{d}_1 - d_1\right|\right\} \le |c|\, \varepsilon_n, \tag{35}$$

where c is a number given in (2). From Fact 4 we know that matrix $\tilde{A}_n$ is nonsingular. Now using (10), (11), (24) and (35) we obtain

$$\begin{aligned} \|s - \tilde{s}\|_\infty &\le \frac{\left\|A_n^{-1}\right\|_\infty}{1 - \left\|A_n^{-1}\right\|_\infty \left\|\tilde{A}_n - A_n\right\|_\infty} \left(\|s\|_\infty \left\|\tilde{A}_n - A_n\right\|_\infty + \|\tilde{p} - p\|_\infty\right) = \\ &= O\left(\left\|A_n^{-1}\right\|_\infty \varepsilon_n\right) = O\left(h^{r+2}\right), \end{aligned} \tag{36}$$

which proves (32). □

Based on Lemma 2, we now prove a theorem that gives an upper bound on the error of ϕ^*.

Theorem 1. *Let f be a function from class $\mathcal{F}_r$. There exists a constant K_1, depending only on parameters of the class $\mathcal{F}_r$, such that for a sufficiently small h there holds*

$$e(\phi^*, \mathcal{F}_r) \le K_1 \cdot h^{r+2}.$$

Proof. We shall find an upper bound on

$$\sup_{x\in[0,T]} |u_f(x) - \tilde{u}_f(x)| = \max_{i=0,\ldots,n-1} E_i, \tag{37}$$

where $E_i = \sup_{x\in[x_i,x_{i+1}]} |u_f(x) - \tilde{u}_f(x)|$. Note that

$$E_i = \sup_{x\in[x_i,x_{i+1}]} |s_i u_{1,i}(x) + s_i' u_{2,i}(x) - \tilde{s}_i l_{1,i}(x) - \tilde{s}_i' l_{2,i}(x)| \le H_i + M_i, \tag{38}$$

where

$$H_i = \sup_{x\in[x_i,x_{i+1}]} |s_i u_{1,i}(x) - \tilde{s}_i l_{1,i}(x)|, \tag{39}$$

$$M_i = \sup_{x\in[x_i,x_{i+1}]} |s_i' u_{2,i}(x) - \tilde{s}_i' l_{2,i}(x)|. \tag{40}$$

Note first that there exist constants R_1 and R_2 depending only on $D_0, \ldots, D_r$ and T (see [6, pp. 420–422]) such that for each i

$$\sup_{x\in[x_i,x_{i+1}]} |u_{1,i}(x)| \le R_1$$

and

$$\sup_{x\in[x_i,x_{i+1}]} |u_{2,i}(x)| \le R_2.$$

Combining this with (27), we obtain the bound

$$\begin{aligned} H_i = & \sup_{x\in[x_i,x_{i+1}]} |s_i u_{1,i}(x) - \tilde{s}_i u_{1,i}(x) + \tilde{s}_i u_{1,i}(x) - \tilde{s}_i l_{1,i}(x)| \le \\ & \le |s_i - \tilde{s}_i| \cdot \sup_{x\in[x_i,x_{i+1}]} |u_{1,i}(x)| + |\tilde{s}_i| \cdot \sup_{x\in[x_i,x_{i+1}]} |u_{1,i}(x) - l_{1,i}(x)| \le \\ & \le |s_i - \tilde{s}_i| \cdot R_1 + (|s_i - \tilde{s}_i| + |s_i|) \cdot C_1 h^{r+2}. \end{aligned}$$

In a similar way, we obtain

$$M_i \le |s_i' - \tilde{s}_i'| \cdot R_2 + (|s_i' - \tilde{s}_i'| + |s_i'|) \cdot C_2 h^{r+2}. \tag{41}$$

Using (33), (34) and Lemma 2, we consequently obtain

$$\begin{aligned} \max_{i=0,\dots,n-1} E_i \le & \max_{i=0,\dots,n-1} H_i + \max_{i=0,\dots,n-1} M_i \le \\ & \le \max_{i=0,\dots,n-1} |s_i - \tilde{s}_i| \cdot \left(R_1 + C_1 h^{r+2}\right) + \max_{i=0,\dots,n-1} |u_f(x_i)| \cdot C_1 h^{r+2} + \\ & \quad + \max_{i=0,\dots,n-1} |s_i' - \tilde{s}_i'| \cdot \left(R_2 + C_2 h^{r+2}\right) + \max_{i=0,\dots,n-1} \left|u_f'(x_i)\right| \cdot C_2 h^{r+2} \le \\ & \le C h^{r+2} \left(R_1 + C_1 h^{r+2}\right) + S_1 C_1 h^{r+2} + \\ & \quad + C h^{r+2} \left(R_2 + C_2 h^{r+2}\right) + S_1 C_2 h^{r+2} \le \\ & \le h^{r+2} \left(C R_1 + C R_2 + S_1 C_1 + S_2 C_2\right) + \left(C C_1 + C C_2\right) h^{r+4} \le \\ & \le K_1 h^{r+2}, \end{aligned} \tag{42}$$

where constant K_1 is independent of n. This ends the proof of the theorem. □

The above result leads to the following upper bound on the complexity of (2).

Corollary 2. *For problem* (2) *there exists a positive constant K such that*

$$comp^{lin}(\varepsilon) \le K \left(\frac{1}{\varepsilon}\right)^{1/(r+2)}. \tag{43}$$

Comparing this with the complexity of this problem based on standard information described by (4), we see that the use of integrals of f yields an improvement over standard information. In this case the ε-complexity turns out to be of order $(1/\varepsilon)^{1/(r+2)}$; therefore, we get a speed-up by 2 in the denominator of the exponent, in comparison with an algorithm based on standard information.

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