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# $J$-CONVEXITY CONSTANTS

**Abstract.** We introduce the $J$-convexity constants on Banach spaces and give some properties of the constants. We give the relations between the $J$-convexity constants and the $n$-th von Neumann-Jordan constants. Using the quantitative indices we estimate the value of $J$-convexity constants in Orlicz spaces.

**Keywords:** new quantitative index, $J$-convexity constants, $n$-th von Neumann-Jordan constants, Orlicz spaces.

**Mathematics Subject Classification:** 46B20, 46E30.

## 1. INTRODUCTION

Much of the significance of the concept of superreflexivity of a Banach space $X$ is due to its numerous equivalent characterizations, see, e. g., Beauzamy [1, Part 4]. One of these characterizations is $J(n,\varepsilon)$-convexity. We restate the definition from [2] as follows.

**Definition 1.1.** *Given $n$ and $0 < \varepsilon < 1$, we say that a Banach spaces $X$ is $J(n,\varepsilon)$-convex if for all elements $z_1,\ldots,z_n \in U_X = \{x \in X : \|x\| \leq 1\}$ there is*

$$\inf_{1\leq k\leq n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| < n(1-\varepsilon).$$

**Definition 1.2.** *We define the $J$-convexity constants, for $n \geq 2$, by*

$$J(n,X) = \sup\left\{ \inf_{1\leq k\leq n} \left\| \sum_{n=1}^{k} z_h - \sum_{n=k+1}^{n} z_h \right\| : z_1,\ldots,z_n \in U_X \right\}, \tag{1}$$

*and*

$$J_n(X) = \inf\Big\{\varepsilon : 0 < \varepsilon < 1, \textit{and there exists } z_1,\ldots,z_n \in U_X \textit{such that} \\ \inf_{1\leq k\leq n} \left\| \sum_{n=1}^{k} z_h - \sum_{n=k+1}^{n} z_h \right\| \geq n(1-\varepsilon)\Big\}. \tag{2}$$

It is known that a Banach space is superreflexive if and only if it is $J(n,\varepsilon)$-convex for some $n$ and $\varepsilon > 0$ ([3] and [4]). It is evident that:

(i) $J(n,X) \le n$ and $0 \le J_n(X) < 1$ for $n \ge 2$.
(ii) $X$ is superreflexive if and only if $J_n(X) > 0$ for some $n$ or, equivalently, $J(n,X) < n$ for some $n$.
(iii) $J(n,X) = n(1-\varepsilon)$ if and only if $J_n(X) = \varepsilon$.
(iv) $J(n,X) = n$ if and only if $J_n(X) = 0$.
(v) For a Banach spaces $X$ the following conditions are equivalent:
  1) $X$ is not superreflexive,
  2) $J_n(X) = 0$ for all $n \in N$,
  3) $J(n,X) = n$ for all $n \in N$.

Let

$$\Phi(u) = \int_0^{|u|} \phi(t)dt, \quad \Psi(v) = \int_0^{|v|} \psi(s)ds$$

be a pair of complementary $N$-functions, where the right derivative $\phi$ of $\Phi$ is right-continuous and nondecreasing, $\phi(t) > 0$ whenever $t > 0$, $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$; the right derivative $\psi$ of $\Psi$ satisfies the same conditions as $\phi$. Assume that $(G,\Sigma,m)$ is a (Lebesgue) measure space and $L^0(G,\Sigma,m)$ is the space of $\Sigma-$measurable functions defined on $G$. The Orlicz space is defined as

$$L^\Phi(G) = \{x \in L^0 : x \text{ is measurable in } G, \rho_\Phi(\lambda x)dt < \infty \text{ for some } \lambda > 0\},$$

where $\rho_\Phi(x) = \int_G \Phi(x(t))dt$. The Luxemburg norm (gauge norm) and the Orlicz norm in $L^\Phi(G)$ are defined, respectively, by

$$\|x\|_{(\Phi)} = \inf\{c > 0 : \rho_\Phi\left(\frac{x}{c}\right) \le 1\}$$

and

$$\|x\|_\Phi = \inf_{k>0} \frac{1}{k}[1 + \rho_\Phi(kx)].$$

As usual, we denote $L^{(\Phi)} = (L^\Phi, \|\cdot\|_{(\Phi)}), L^\Phi = (L^\Phi, \|\cdot\|_\Phi)$ for short.

An $N$-function $\Phi(u)$ is said to satisfy the $\triangle_2$-condition for small $u$ (for all $u$ or for large $u$), which is written as $\Phi \in \triangle_2(0) (\Phi \in \triangle_2$ or $\Phi \in \triangle_2(\infty))$, if there exist $u_0 > 0$ and $c > 0$ such that $\Phi(2u) \le c\Phi(u)$ for $0 \le u \le u_0$ (for all $u \ge 0$ or for $u \ge u_0$). An $N$-function $\Phi(u)$ satisfies the $\nabla_2$-condition for small $u$ (for all $u \ge 0$ or for large $u$), which is written as $\Phi \in \nabla_2(0)$ ($\Phi \in \nabla_2$ or $\Phi \in \nabla_2(\infty)$), if its complementary $N$-function (see [6] or [8]) $\Psi \in \triangle_2(0)$ ($\Psi \in \triangle_2$ or $\Psi \in \triangle_2(\infty)$). The basic facts on Orlicz spaces can be found in [8].

New quantitative indices for an $N$-function $\Phi$ are defined by

$$\alpha_\Phi(n) = \liminf_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)}, \qquad \beta_\Phi(n) = \limsup_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)},$$

$$\alpha^0_\Phi(n) = \liminf_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)}, \qquad \beta^0_\Phi(n) = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)},$$

$$\overline{\alpha}_\Phi(n) = \inf_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)}, \qquad \overline{\beta}_\Phi(n) = \sup_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(nu)}.$$

For $n = 2$, we denote these constants by $\alpha_\Phi, \beta_\Phi, \alpha^0_\Phi, \beta^0_\Phi, \overline{\alpha}_\Phi$ and $\overline{\beta}_\Phi$(see [8]). Clearly, $\frac{1}{n} \le \overline{\alpha}_\Phi(n) \le \min\{\alpha_\Phi(n), \alpha^0_\Phi(n)\}$, $\max\{\beta_\Phi(n), \beta^0_\Phi(n)\} \le \overline{\beta}_\Phi(n) \le 1$.

**Proposition 1.1.** ([7]) *Let $\Phi$ be an $N$-function. Then:*

(i) $\Phi \notin \triangle_2(\infty) \Longleftrightarrow \beta_\Phi(n) = 1$;  $\Phi \notin \nabla_2(\infty) \Longleftrightarrow \alpha_\Phi(n) = \frac{1}{n}$.
(ii) $\Phi \notin \triangle_2(0) \Longleftrightarrow \beta^0_\Phi(n) = 1$;  $\Phi \notin \nabla_2(0) \Longleftrightarrow \alpha^0_\Phi(n) = \frac{1}{n}$.
(iii) $\Phi \notin \triangle_2 \Longleftrightarrow \overline{\beta}_\Phi(n) = 1$;  $\Phi \notin \nabla_2 \Longleftrightarrow \overline{\alpha}_\Phi(n) = \frac{1}{n}$.

The following results concern these new indices.

**Proposition 1.2.** ([7]) *Let $\Phi$ and $\Psi$ be a pair of complementary $N$-functions and $n \ge 2$. Then:*

$$n\alpha_\Phi(n)\beta_\Psi(n) = 1 = n\alpha_\Psi(n)\beta_\Phi(n), \tag{3}$$

$$n\alpha^0_\Phi(n)\beta^0_\Psi(n) = 1 = n\alpha^0_\Psi(n)\beta^0_\Phi(n), \tag{4}$$

$$n\overline{\alpha}_\Phi(n)\overline{\beta}_\Psi(n) = 1 = n\overline{\alpha}_\Psi(n)\overline{\beta}_\Phi(n). \tag{5}$$

## 2. THE RELATIONS BETWEEN $J$-CONVEXITY CONSTANTS AND VON NEUMANN-JORDAN CONSTANTS

In order to discuss the $J$-convexity constants, we need the $n$-th von Neumann-Jordan constants defined as follows.

**Definition 2.1.** *We define the* $n$-th von Neumann-Jordan constants, *for $n \ge 2$, by*

$$C^{(n)}_{NJ}(X) = \sup\left\{ \frac{\sum_{k=1}^{n} \|\sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h\|}{n \sum_{i=1}^{n} \|z_i\|^2} : z_i \in X, \sum_{i=1}^{n} \|z_i\|^2 \neq 0 \right\}.$$

When $n = 2$, $C^{(2)}_{NJ}(X)$ is the von Neumann-Jordan constants of a Banach space $X$ (see [8]).

**Theorem 2.1.**

(i) *For any Banach space $X$, there holds*

$$J(n,X) \leq \sqrt{nC_{NJ}^{(n)}(X)}. \tag{6}$$

(ii) $J(n,X) < n$ *if and only if* $C_{NJ}^{(n)}(X) < n$.

*Proof.* (i) Let $x_1, x_2, \ldots, x_n \in U(X)$. Then

$$\begin{aligned} n \min_{1\leq i\leq n} \left\| \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right\|^2 &\leq \sum_{k=1}^{n} \left\| \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right\|^2 \leq \\ &\leq C_{NJ}^{(n)}(X) n \sum_{i=1}^{n} \|x_i\|^2 \leq \\ &\leq n^2 C_{NJ}^{(n)}(X). \end{aligned}$$

(ii) By (i), the sufficiency is clear. Now we prove the necessity.

$$\begin{aligned} C_{NJ}^{(n)}(X) &= \sup \left\{ \frac{\sum\limits_{1<k\leq n} \| \sum\limits_{i=1}^{k} x_i - \sum\limits_{i=k+1}^{n} x_i \|}{n \sum\limits_{i=1}^{n} \|x_i\|^2} : \{x_i\}_1^n \subset X \text{and } \sum_{i=1}^{n} \|x_i\|^2 \neq 0 \right\} = \\ &= \sup \left\{ \frac{\sum\limits_{1\leq k\leq n} \| \sum\limits_{i=1}^{k} x_i - \sum\limits_{i=k+1}^{n} x_i \|}{n (\sum\limits_{i=1}^{n-1} \|x_i\|^2 + 1)} : \{x_i\}_1^n \subset X, \|x_i\| \leq \|x_n\| = 1 \right\}. \end{aligned}$$

Since $J(n,X) < n$, there exists $0 < \delta < 1$ such that

$$\sup \left\{ \inf_{1\leq k\leq n} \left\| \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \right\| : \{x_i\}_1^n \subset U_X \right\} < n - \delta. \tag{7}$$

Suppose $1 = \|x_n\| \geq \|x_i\| > 1 - \frac{\delta}{2(n-1)} (i = 1, 2, \ldots, n-1)$. By (7), there is

$$\inf_{1\leq k\leq n} \| \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \| < n - \delta.$$

Without loss of generality, we assume that

$$\|x_1 - x_2 - \ldots - x_n\| = \inf_{1\leq k\leq n} \| \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i \| < n - \delta.$$

Hence

$$\frac{\sum\limits_{1\le k\le n}\|\sum\limits_{i=1}^{k}x_i-\sum\limits_{i=k+1}^{n}x_i\|^2}{n(\sum\limits_{i=1}^{n-1}\|x_i\|^2+1)}=\frac{\|x_1-x_2-\ldots-x_n\|^2+\sum\limits_{2\le k\le n}\|\sum\limits_{i=1}^{k}x_i-\sum\limits_{i=k+1}^{n}x_i\|^2}{n(\sum\limits_{i=1}^{n-1}\|x_i\|^2+1)}\le$$

$$\le\frac{(n-\delta)^2}{n[(n-1)\cdot(1-\frac{\delta}{2(n-1)})^2+1]}+\frac{\sum\limits_{2\le k\le n}(\sum\limits_{i=1}^{n-1}\|x_i\|+1)^2}{n(\sum\limits_{i=1}^{n-1}\|x_i\|^2+1)}\le$$

$$\le\frac{(n-\delta)^2}{n[(n-1)-\delta+\frac{\delta^2}{4(n-1)}+1]}+\frac{\sum\limits_{2\le k\le n}n(\sum\limits_{i=1}^{n-1}\|x_i\|^2+1)}{n(\sum\limits_{i=1}^{n-1}\|x_i\|^2+1)}=$$

$$=\frac{(n-\delta)^2}{n[n-\delta+\frac{\delta^2}{4(n-1)}]}+n-1<\frac{n-\delta}{n}+n-1=n-\frac{\delta}{n}.$$

If there exists $1\le i\le n-1$ such that $\|x_i\|\le 1-\frac{\delta}{2(n-1)}$, we may, without loss of generality, assume that $\|x_1\|\le 1-\frac{\delta}{2(n-1)}$. Then

$$\frac{\sum\limits_{1\le k\le n}\|\sum\limits_{i=1}^{k}x_i-\sum\limits_{i=k+1}^{n}x_i\|^2}{n(\sum\limits_{i=1}^{n-1}\|x_i\|^2+1)}\le\frac{n(\|x_1\|+\|x_2\|+\ldots+\|x_{n-1}\|+1)^2}{n(\sum\limits_{i=1}^{n-1}\|x_i\|^2+1)}=$$

$$=n-\frac{n(\sum\limits_{i=1}^{n-1}\|x_i\|^2+1)-(\|x_1\|+\|x_2\|+\ldots+\|x_{n-1}\|+1)^2}{(\sum\limits_{i=1}^{n-1}\|x_i\|^2+1)}=$$

$$=n-\{n(\sum_{i=1}^{n-1}\|x_i\|^2+1)-[\|x_1\|^2+2\|x_1\|(\|x_2\|+\ldots+\|x_{n-1}\|+1)+$$

$$+(\|x_2\|+\ldots+\|x_{n-1}\|+1)^2]\}/(\sum_{i=1}^{n-1}\|x_i\|^2+1)=$$

$$=n-\{(1-\|x_1\|)^2+(\|x_1\|-\|x_2\|)^2+\ldots+(\|x_1\|-\|x_{n-1}\|)^2+$$

$$+(n-1)(\sum_{i=2}^{n-1}\|x_i\|^2+1)-(\|x_2\|+\ldots+\|x_{n-1}\|+1)^2\}/(\sum_{i=1}^{n-1}\|x_i\|^2+1)\le$$

$$\le n-\frac{(1-\|x_1\|)^2}{\sum\limits_{i=1}^{n-1}\|x_i\|^2+1}\le n-\frac{[1-(1-\frac{\delta}{2(n-1)})]^2}{n}=n-\frac{\delta^2}{4n(n-1)^2}.$$

Therefore,

$$C_{NJ}^{(n)}(X) \leq \max\left\{n - \frac{\delta}{n}, n - \frac{\delta^2}{4n(n-1)^2}\right\} < n.$$

□

## 3. LOWER BOUNDS FOR $J$-CONVEXITY CONSTANTS IN ORLICZ SPACES

In this section, we will give lower bounds of $J$-convexity for three Orlicz spaces.

**Theorem 3.1.** *Let $\Phi$ be an $N$-function and $n \geq 2$. Then:*

$$\max\left[\frac{1}{\alpha_\Phi(n)}, n\beta_\Phi(n)\right] \leq J(n, L^{(\Phi)}[0,1]), \tag{8}$$

$$\max\left[\frac{1}{\alpha_\Phi^0(n)}, n\beta_\Phi^0(n)\right] \leq J(n, l^{(\Phi)}), \tag{9}$$

$$\max\left[\frac{1}{\overline{\alpha}_\Phi(n)}, n\overline{\beta}_\Phi(n)\right] \leq J(n, L^{(\Phi)}[0,\infty)). \tag{10}$$

*Proof.* We only prove (8). The proofs of inequalities (9) and (10) are similar. By the definition of $\alpha_\Phi(n)$, there exists $0 < u_k \nearrow \infty$ such that

$$\lim_{k\to\infty} \frac{\Phi^{-1}(u_k)}{\Phi^{-1}(nu_k)} = \alpha_\Phi(n).$$

So for $\varepsilon > 0$, there exists $k_0 \geq 1$ such that for any $k \geq k_0$, there is

$$\frac{\Phi^{-1}(u_k)}{\Phi^{-1}(nu_k)} < \alpha_\Phi(n) + \varepsilon. \tag{11}$$

Put $u_0 = u_{k_0} > 1$. Let $G_i, 1 \leq i \leq n$ be non-overlapping subsets of $[0,1]$, and $m(G_i) = \frac{1}{nu_0}, 1 \leq i \leq n$. Define $x_i(t) = \Phi^{-1}(nu_0)\chi_{G_i}(t), 1 \leq i \leq n$. Then $\|x_i\|_{(\Phi)} = 1$, and for any $1 \leq k \leq n$, there holds

$$\|\sum_{i=1}^{k} x_i - \sum_{k+1}^{n} x_i\|_{(\Phi)} = \Phi^{-1}(nu_0)\|\chi_{G_1\cup G_2\cup\ldots\cup G_n}\|_{(\Phi)} =$$

$$= \frac{\Phi^{-1}(nu_0)}{\Phi^{-1}(u_0)} > \frac{1}{\alpha_\Phi(n) + \varepsilon}.$$

Therefore

$$J(n, L^{(\Phi)}[0,1]) > \frac{1}{\alpha_\Phi(n) + \varepsilon},$$

which proves (8), because $\varepsilon$ is arbitrary.

Now we prove that $n\beta_\Phi(n) \le J(n, L^{(\Phi)}[0,1])$. By the definition of $\beta_\Phi(n)$, for any given $\varepsilon > 0$ we choose a $v_0 > 1$ such that $\frac{\Phi^{-1}(v_0)}{\Phi^{-1}(nv_0)} > \beta_\Phi(n) - \frac{\varepsilon}{n}$. We divide $[0, \frac{1}{v_0}]$ into $n$ non-overlapping intervals $A_1, A_2, \ldots, A_n$ of the same length. Define:

$$
\begin{aligned}
x_1(t) &= \Phi^{-1}(v_0)\left(\chi_{A_1} + \chi_{A_2} + \chi_{A_3} + \ldots + \chi_{A_n}\right),\\
x_2(t) &= \Phi^{-1}(v_0)\left(-\chi_{A_1} + \chi_{A_2} + \chi_{A_3} + \ldots + \chi_{A_n}\right),\\
x_3(t) &= \Phi^{-1}(v_0)\left((-\chi_{A_1} - \chi_{A_2} + \chi_{A_3} + \ldots + \chi_{A_n})\right),\\
&\ldots\ldots\\
x_n(t) &= \Phi^{-1}(v_0)\left(-\chi_{A_1} - \chi_{A_2} - \chi_{A_3} - \ldots - \chi_{A_{n-1}} + \chi_{A_n}\right).
\end{aligned}
$$

Obviously, $\|x_i\|_{(\Phi)} = 1 (i = 1, 2, \ldots, n)$. For any $1 \le k \le n$,

$$
\|\sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i\|_{(\Phi)} \ge \|\Phi^{-1}(v_0) n \chi_{A_k}\|_{(\Phi)} = \frac{n\Phi^{-1}(v_0)}{\Phi^{-1}(nv_0)} \ge n\beta_\Phi(n) - \varepsilon.
$$

The latter implies that $J(n, L^{(\Phi)}[0,1]) \ge n\beta_\Phi(2^{n-1}) - \varepsilon$. This proves that $J_n(L^{(\Phi)}[0,1]) \ge n\beta_\Phi(n)$, because $\varepsilon$ is arbitrary. □

**Corollary 3.1.**

(i) *If* $\Phi \notin \triangle_2(\infty) \bigcap \nabla_2(\infty)$, *then* $J(n, L^{(\Phi)}[0,1]) = n$.
(ii) *If* $\Phi \notin \triangle_2(0) \bigcap \nabla_2(0)$, *then* $J(n, l^{(\Phi)}) = n$.
(iii) *If* $\Phi \notin \triangle_2 \bigcap \nabla_2$, *then* $J(n, L^{(\Phi)}[0,\infty)) = n$.

*Proof.* We only prove (i). If $\Phi \notin \triangle_2(\infty)$, then $\beta_\Phi(n) = 1$ by Proposition 1.1. By (8), there is $J(n, L^{(\Phi)}[0,1]) = n$. If $\Phi \notin \nabla_2(\infty)$, then $\alpha_\Phi(n) = \frac{1}{n}$. By (8), the same equality holds true. □

**Corollary 3.2.** *Let* $1 < p < \infty, L^p \in \{L^p[0,1], L^p[0,\infty), l^p\}$. *Then for* $n \ge 2$,

$$
\max\left\{n^{\frac{1}{p}}, n^{1-\frac{1}{p}}\right\} \le J(n, L^p),
$$
$$
\max\left\{n^{\frac{2}{p}-1}, n^{1-\frac{2}{p}}\right\} \le C_{NJ}^{(n)}(L^p).
$$

*Proof.* We put $\Phi_p(u) = |u|^p$. Then $L^{(\Phi)} = L^p$ and $\|\cdot\|_{(\Phi)} = \|\cdot\|_p$. The result is easy to verify. □

Similarly, for the Orlicz spaces with the Orlicz norm, the following theorem holds.

**Theorem 3.2.** *Let* $\Phi$ *be an* $N$*-function,* $n \ge 2$ *and* $\Psi$ *be a complementary* $N$*-function for* $\Phi$. *Then:*

$$
\max\left[n\beta_\Psi(n), \frac{1}{\alpha_\Psi(n)}\right] \le J(n, L^\Phi[0,1]), \tag{12}
$$

$$
\max\left[n\beta_\Psi^0(n), \frac{1}{\alpha_\Psi^0(n)}\right] \le J(n, l^\Phi), \tag{13}
$$

$$
\max\left[n\overline{\beta}_\Psi(n), \frac{1}{\overline{\alpha}_\Psi(n)}\right] \le J(n, L^\Phi[0,\infty)). \tag{14}
$$

**Remark.** (i) By Proposition 1.2, there is

$$\max\left\{\frac{1}{\alpha_\Phi(n)}, n\beta_\Phi(n)\right\} = \max\left\{n\beta_\Psi(n), \frac{1}{\alpha_\Psi(n)}\right\}.$$

That is to say that $J(n, L^{(\Phi)}[0,1])$ and $J(n, L^\Phi[0,1])$ have the same lower bounds etc.

(ii) If $X_\Phi$ denotes one of the Orlicz spaces in Theorem 3.1 and Theorem 3.2, then

$$\sqrt{n} \leq J(n, X_\Phi).$$

In fact, assume that $X_\Phi = L^{(\Phi)}$, then by Theorem 3.1, there holds

$$\sqrt{n} \leq \max\left\{\frac{1}{\alpha_\Phi(n)}, n\beta_\Phi(n)\right\} \leq J(n, X_\Phi).$$

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