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ON THE ASYMPTOTICS OF THE DIFFERENCE EQUATION WITH A PROPORTIONAL DELAY

Abstract. This paper deals with asymptotic properties of a vector difference equation with delayed argument

$$\Delta x_k = Ax_k + Bx_{\lfloor \lambda k \rfloor}, \qquad 0 < \lambda < 1, \quad k = 0, 1, 2, \dots,$$

where A, B are constant matrices and the term $\lfloor \lambda k \rfloor$ is the integer part of λk. Our aim is to emphasize some resemblances between the asymptotic behaviour of this delay difference equation and its continuous counterpart.

Keywords: asymptotics of difference equations, approximation methods for dynamical systems.

Mathematics Subject Classification: 39A11, 37M99.

1. INTRODUCTION

In this paper, we consider the vector delay differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(\lambda t), \qquad 0 < \lambda < 1, \quad t \in [0, \infty) \tag{1}$$

and its difference analogy

$$\Delta \mathbf{x}_k = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{x}_{\lambda_k}, \qquad k = 0, 1, 2, \dots, \tag{2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are nonzero $n \times n$ complex matrices, λ is a real scalar, $\mathbf{x}_k$ means $\mathbf{x}(k)$ and $\lambda_k = \lfloor \lambda k \rfloor$, $k = 0, 1, 2, \dots$ (the symbol $\lfloor\ \rfloor$ denotes the integer part). In the scientific literature equation (1) is usually referred to as the "pantograph equation". The name of this equation goes back to the paper of Ockendon and Tayler [12], in which the dynamics of electric locomotive pantograph moving along a trolley wire was studied.

Our aim is to derive the asymptotic bounds of solutions of (2) and demonstrate its resemblance to the asymptotic estimates known for the continuous case.

This paper is organized as follows: In Section 2, we mention some qualitative properties of the differential pantograph equation. Section 3 is devoted to the analysis of the difference pantograph equation. In particular, we derive the asymptotic estimate of all solutions of the studied difference equation under some assumptions imposed on matrices **A** and **B**.

2. THE PANTOGRAPH EQUATION IN THE CONTINUOUS CASE

Pantograph equation (1) and its modifications have been the subject of many qualitative and numerical investigations, because of their interesting applications, as well as specific properties of solutions. The first paper dealing with this equation was that of Kato and McLeod [6], where the authors studied the scalar equation

$$\dot{x}(t) = ax(t) + bx(\lambda t), \quad 0 < \lambda < 1, \quad t \in [0, \infty) \tag{3}$$

under various assumptions on the signs of a and b. Their results particularly imply that in the case of $a < 0$, the asymptotic estimate of all solutions x of (3) yields

$$x(t) = O(t^\gamma) \quad \text{as} \quad t \to \infty, \qquad \gamma = \frac{\log \frac{|b|}{-a}}{\log \lambda^{-1}},$$

(i.e., $|x(t)| \le Lt^\gamma$ as $t \to \infty$ for a suitable real $L > 0$). Moreover, the property

$$x(t) = o(t^\gamma) \qquad \text{as} \quad t \to \infty \tag{4}$$

(i.e., $x(t)/t^\gamma \to 0$ as $t \to \infty$) holds for the zero solution of (3) only.

In the vector case, we mention the result of Lim [9], where the following asymptotic estimate of solutions of vector differential equation (1) was derived:
Let $\mathbf{A} = \operatorname{diag}(a_{11}, \dots, a_{nn})$ with $\operatorname{Re} a_{11} \le \operatorname{Re} a_{22} \le \cdots \le \operatorname{Re} a_{nn} < 0$ and let $\mathbf{B} = (b_{ij})$. If γ is defined by

$$\gamma = \frac{\log\left(\frac{\max_{1 \le i \le n} \sum_{j=1}^{n} |b_{ij}|}{-\operatorname{Re} a_{nn}}\right)}{\log \lambda^{-1}},$$

then every solution $\mathbf{x}$ of (1) is $O(t^\gamma)$ as $t \to \infty$, i.e., $|\mathbf{x}(t)| \le Lt^\gamma$ as $t \to \infty$ for a suitable real $L > 0$, where $|\,|$ is now a suitable vector norm. This result will be compared with the asymptotic estimate derived for the difference pantograph equation in the next section.

In the remaining part of this section we demonstrate the usefulness of the knowledge of the above mentioned asymptotic behaviour of equation (3) in the study of a specific phenomenon occuring in the qualitative theory of differential equations with unbounded delays. To illustrate this, we consider the initial value problem

$$\dot{x}(t) = -0.2x(t) - x(0.99t), \quad x(0) = 1, \quad t \in [0, \infty). \tag{5}$$

The numerical analysis of problem (5) shows that the solution x of equation (5) rapidly tends (in absolute value) to the zero function ($x(t) \approx 10^{-50}$ for some t). In other words, because of insufficient accuracy of the computation this solution becomes eventually the zero function. However, this interpretation is inconsistent with the above mentioned result recalled from [6] (particularly, $x(t)/t^{160.14}$ does not tend to zero as $t \to \infty$, i.e., $x(t)$ is unbounded as $t \to \infty$).

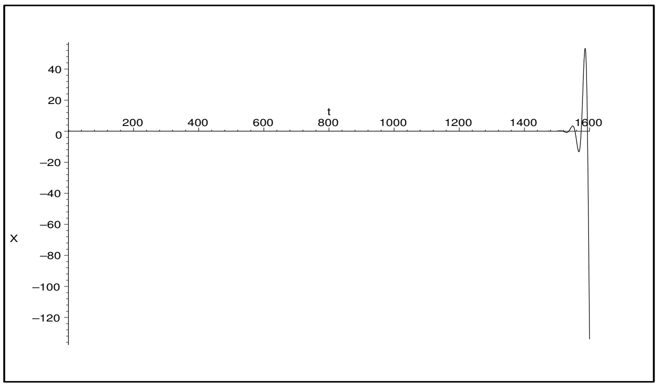

Fig. 1. Numerical solution of (5)

In fact, the solution of (5) actually tends to zero, but it is "exploding" after some critical moment t^* (see Fig. 1). This phenomenon is known in the numerical analysis as a "numerical nightmare". The problem then consists in finding the critical value t^*, as well as the relations for bounding curves. Some results concerning these problems can be found in Liu [10], and, in a more general situation, in Čermák [2], Kundrát [7], Lehninger and Liu [8], Makay and Terjéki [11].

3. THE PANTOGRAPH EQUATION IN THE DIFFERENCE CASE

In this section we are going to discuss difference pantograph equation (2), where the complex matrix $\mathbf{A} = \operatorname{diag}(a_{11}, \ldots, a_{nn})$ is diagonal. Equation (2) is a system of difference equations obtained from equation (1) via the modified Euler method (see Bellen and Zennaro [1]) with stepsize $h = 1$. Our aim is to compare the asymptotic behaviour of exact equation (1) with its discretization (2).

Recently, several papers discussing the asymptotic properties of delay difference equations have appeared. In addition to the above quoted paper [10], we especially recall papers by Iserles [4, 5] dealing with the qualitative and numerical investigation of pantograph equation (3) and Péics [13] or Györi and Pituk [3] comparing some qualitative theorems for delay differential and difference equations.

To describe the asymptotic bounds of solutions of discretized pantograph equation (2), we utilize the positive solution ρ of the system of functional inequalities

$$\sum_{j=1}^{n} |b_{ij}|\rho(\lfloor\lambda t\rfloor) \le (1-|1+a_{ii}|)\rho(t), \qquad \text{for all} \quad i=1,\dots,n, \quad t\in[0,\infty). \tag{6}$$

We assume that the diagonal elements of the matrix $\mathbf{A}$ fulfil the condition $|1+a_{ii}|<1$ $(i=1,\dots,n)$, where the symbol $|\cdot|$ stands for the absolute value. This property particularly implies that diagonal elements have negative real parts. One can see that the power function

$$\rho(t)=\begin{cases} t^{\gamma} & \text{for } \beta\ge\alpha,\\ (t+\frac{1}{1-\lambda})^{\gamma} & \text{for } \beta<\alpha,\end{cases} \tag{7}$$

where

$$\gamma=\frac{\log\beta/\alpha}{\log\lambda^{-1}}, \quad \beta=\max_{1\le i\le n}\sum_{j=1}^{n}|b_{ij}|, \quad \alpha=\min_{1\le i\le n}(1-|1+a_{ii}|), \tag{8}$$

fulfils system of inequalities (6) in the case $\beta\ge\alpha$ and $\beta<\alpha$, respectively.

In the next theorem we show that the solution of discretized pantograph equation (2) fulfils the asymptotic estimate $\mathbf{x}_k=O(k^{\gamma})$ as $k\to\infty$. Later we compare this estimate with the estimate derived in [9] for the exact pantograph equation.

Theorem 3.1. *Let $\mathbf{x}_k$ be a solution of equation* (2)*, where $\mathbf{A}=\operatorname{diag}(a_{11},\dots,a_{nn})$ is a diagonal complex matrix, $\mathbf{B}=(b_{ij})$ is a nonzero $n\times n$ complex matrix and λ is a real scalar with $0<\lambda<1$. If $|a_{ii}+1|<1$ $(i=1,\dots,n)$, then*

$$\mathbf{x}_k=O(k^{\gamma}) \qquad \text{as } k\to\infty, \tag{9}$$

where γ is given by (8)*.*

Proof. First we rewrite the system of difference equations (2) as

$$\mathbf{x}_{k+1}=\overline{\mathbf{A}}\mathbf{x}_k+\mathbf{B}\mathbf{x}_{\lambda_k}, \quad k\in\mathbb{N}^0, \qquad \overline{\mathbf{A}}=\operatorname{diag}(\overline{a}_{ii}), \tag{10}$$

where $\overline{a}_{ii}$ are complex constants such that $\overline{a}_{ii}=1+a_{ii}$, $i=1,\dots,n$. Then we substitute $\mathbf{y}_k=\mathbf{x}_k/\rho_k$ in (10), where $\rho_k=\rho(k)$ (ρ being given by (7)), to obtain

$$\rho_{k+1}\mathbf{y}_{k+1}=\overline{\mathbf{A}}\rho_k\mathbf{y}_k+\mathbf{B}\rho_{\lambda_k}\mathbf{y}_{\lambda_k}, \qquad k\in\mathbb{N}^0, \tag{11}$$

i.e., each component of (11) has to fulfil

$$\rho_{k+1}(y_i)_{k+1}=\overline{a}_{ii}\rho_k(y_i)_k+\sum_{j=1}^{n}b_{ij}\rho_{\lambda_k}(y_j)_{\lambda_k}, \qquad k\in\mathbb{N}^0,\ i=1,\dots,n. \tag{12}$$

Here the symbol $(y_i)_k$ means the value of i-th component of $\mathbf{y}$ at k (similarly for other related notations) and $\lambda_k=\lfloor\lambda k\rfloor$ – see above. Now we show that every solution $\mathbf{y}_k$ of equation (11) is bounded as $k\to\infty$. We denote $T_m:=\lambda^{-m}t_0$, $m=0,1,2,\dots,$

where $t_0 > 0$, and set $I_0 := [0, t_0]$, $I_m := [T_{m-1}, T_m]$ for all $m = 1, 2, \dots$. To express and estimate $\mathbf{y}_k$, $k \in I_{m+1} \cap \mathbb{N}^0$, in terms of $\mathbf{y}_k$, $k \in (\bigcup_{j=0}^{m} I_m) \cap \mathbb{N}^0$, we have to distinguish between two cases:

(i) Let $\overline{a}_{ii} \neq 0$ for a given $i \in \{1, \dots, n\}$. Multiplying (12) by $1/\overline{a}_{ii}{}^{k+1}$ we get

$$\frac{\rho_{k+1}(y_i)_{k+1}}{\overline{a}_{ii}{}^{k+1}} - \frac{\rho_k(y_i)_k}{\overline{a}_{ii}{}^k} = \sum_{j=1}^{n} \frac{b_{ij}}{\overline{a}_{ii}{}^{k+1}} \rho_{\lambda_k}(y_j)_{\lambda_k}, \qquad i = 1, \dots, n,$$

i.e.,

$$\Delta\left(\frac{\rho_k(y_i)_k}{\overline{a}_{ii}{}^k}\right) = \sum_{j=1}^{n} \frac{b_{ij}}{\overline{a}_{ii}{}^{k+1}} \rho_{\lambda_k}(y_j)_{\lambda_k}, \qquad i = 1, \dots, n. \tag{13}$$

Now we take any $\overline{k} \in I_{m+1} \cap \mathbb{N}^0$, $m = 0, 1, 2, \dots$. We define nonnegative integers $r_m(\overline{k}) := \lfloor \overline{k} - T_m \rfloor$ and denote $\overline{k}_m := \overline{k} - r_m(\overline{k}) - 1$. Summing equation (13) from $\overline{k}_m$ to $\overline{k} - 1$ we get

$$(y_i)_{\overline{k}} = \frac{\rho_{\overline{k}_m} \overline{a}_{ii}{}^{\overline{k}}}{\rho_{\overline{k}} \overline{a}_{ii}{}^{\overline{k}_m}} (y_i)_{\overline{k}_m} + \frac{\overline{a}_{ii}{}^{\overline{k}}}{\rho_{\overline{k}}} \sum_{s=\overline{k}_m}^{\overline{k}-1} \sum_{j=1}^{n} \frac{b_{ij}}{\overline{a}_{ii}{}^{s+1}} \rho_{\lambda_s}(y_j)_{\lambda_s}.$$

We denote $M_m := \sup \left\{ |\mathbf{y}(t)|, \quad t \in (\bigcup_{j=0}^{m} I_j) \cap \mathbb{N}^0 \right\}$. In accordance with (6), there is

$$|(y_i)_{\overline{k}}| \leq \frac{\rho_{\overline{k}_m} |\overline{a}_{ii}|^{\overline{k}}}{\rho_{\overline{k}} |\overline{a}_{ii}|^{\overline{k}_m}} M_m + \frac{|\overline{a}_{ii}|^{\overline{k}}}{\rho_{\overline{k}}} \sum_{s=\overline{k}_m}^{\overline{k}-1} \frac{(1 - |\overline{a}_{ii}|)\rho_s}{|\overline{a}_{ii}|^{s+1}} M_m.$$

Using the relation $\frac{(1-|\overline{a}_{ii}|)}{|\overline{a}_{ii}|^{s+1}} = \Delta\left(\frac{1}{|\overline{a}_{ii}|}\right)^s$ and summing by parts we get

$$|(y_i)_{\overline{k}}| \leq \frac{\rho_{\overline{k}_m} |\overline{a}_{ii}|^{\overline{k}}}{\rho_{\overline{k}} |\overline{a}_{ii}|^{\overline{k}_m}} M_m + \frac{|\overline{a}_{ii}|^{\overline{k}}}{\rho_{\overline{k}}} \sum_{s=\overline{k}_m}^{\overline{k}-1} \left(\Delta\left(\frac{1}{|\overline{a}_{ii}|}\right)^s \right) \rho_s M_m \leq \tag{14}$$

$$\leq M_m \left\{ \frac{\rho_{\overline{k}_m} |\overline{a}_{ii}|^{\overline{k}}}{\rho_{\overline{k}} |\overline{a}_{ii}|^{\overline{k}_m}} + \frac{|\overline{a}_{ii}|^{\overline{k}}}{\rho_{\overline{k}}} \left(\frac{\rho_{\overline{k}}}{|\overline{a}_{ii}|^{\overline{k}}} - \frac{\rho_{\overline{k}_m}}{|\overline{a}_{ii}|^{\overline{k}_m}} - \sum_{s=\overline{k}_m}^{\overline{k}-1} \left(\frac{1}{|\overline{a}_{ii}|}\right)^{s+1} \Delta\rho_s \right) \right\} =$$

$$= M_m \left\{ 1 - \frac{|\overline{a}_{ii}|^{\overline{k}}}{\rho_{\overline{k}}} \sum_{s=\overline{k}_m}^{\overline{k}-1} \left(\frac{1}{|\overline{a}_{ii}|}\right)^{s+1} \Delta\rho_s \right\}. \tag{15}$$

If $\gamma \geq 0$, then ρ is a nondecreasing function on I, hence $\Delta\rho$ is nonnegative on I and $|(y_i)_{\overline{k}}| \leq M_m$, $i = 1, \dots, n$. Since $\overline{k} \in I_{m+1} \cap \mathbb{N}^0$ was arbitrary, we derive $M_{m+1} \leq M_m$, i.e., M_m is bounded as $m \to \infty$. Hence the sequence y_k

is bounded. If $\gamma < 0$, then ρ is a decreasing function on I such that $\Delta\rho$ is nondecreasing on I. Then from (15) there follows

$$|(y_i)_{\overline{k}}| \le M_m \left\{ 1 + \frac{\rho_{\overline{k}_m} - \rho_{\overline{k}_m+1}}{\rho_{\overline{k}}} \sum_{s=\overline{k}_m}^{\overline{k}-1} \left(\frac{|\overline{a}_{ii}|^{\overline{k}}}{|\overline{a}_{ii}|^{s+1}} \right) \right\} \le$$
$$\le M_m \left\{ 1 + \frac{\rho_{\overline{k}_m} - \rho_{\overline{k}_m+1}}{\rho_{\overline{k}}} \delta_i \right\} \le M_m \left\{ 1 + \delta_i \frac{-\Delta\rho_{T_m-1}}{\rho_{T_{m+1}}} \right\},$$

where $\delta_i := 1/(1 - |\overline{a}_{ii}|)$, $i = 1, \dots, n$. Moreover,

$$\frac{-\Delta\rho_{T_m-1}}{\rho_{T_{m+1}}} = O(\lambda^m) \qquad \text{as } m \to \infty \tag{16}$$

by virtue of the binomial formula. Hence, $|(y_i)_{\overline{k}}| = M_m(1 + O(\lambda^m))$ as $m \to \infty$.

(ii) Let $\overline{a}_{ii} = 0$ for some $i = 1, \dots, n$. Using the same notation as above we get

$$(y_i)_{k+1} = \frac{1}{\rho_{k+1}} \sum_{j=1}^{n} b_{ij} \rho_{\lambda_k} (y_j)_{\lambda_k} \qquad \Rightarrow \qquad |(y_i)_{\overline{k}}| \le \frac{\rho_k}{\rho_{k+1}} M_m. \tag{17}$$

It is obvious that for $\gamma \ge 0$ there is $|(y_i)_{\overline{k}}| \le M_m$. For $\gamma < 0$ one can see that

$$\left(\frac{\overline{k} - 1}{\overline{k}} \right)^{\gamma} M_m \le \left(\frac{\lambda^{-m} t_0 - 1}{\lambda^{-m} t_0} \right)^{\gamma} M_m = (1 + O(\lambda^m)) M_m \qquad \text{as } m \to \infty.$$

Summarizing cases (i) and (ii), we get

$$|\mathbf{y}_{\overline{k}}| \le M_m (1 + O(\lambda^m)) \qquad \text{as } m \to \infty,$$

i.e.,

$$M_{m+1} \le M_m (1 + O(\lambda^m)) \le M_1 \prod_{j=1}^{m} \left(1 + O(\lambda^j)\right).$$

Since the product converges as $m \to \infty$, $\mathbf{y}_k$ is bounded as $k \to \infty$. □

Remark 3.1. *As we noted above, difference equation* (2) *is obtained from equation* (1) *via the Euler discretization with stepsize* $h = 1$. *The previous result can be easily extended (after some changes in the assumptions) onto the case of the Euler discretization with an arbitrary choice of stepsize.*

Remark 3.2. *The asymptotic estimate described in the previous theorem can be also extended to the case when* $\mathbf{A}$ *is diagonalizable with eigenvalues* a_{ii} *fulfilling the assumptions of Theorem 3.1. We explain this procedure just at discrete pantograph equation* (2) *with concrete data originating in* [12].

Example 3.1. *The problem of the dynamics of a current collection system near a trolley wire support leads to system of equations* (1) *with*

$$\mathbf{A} = \begin{pmatrix} -1.40 & 0 & 0.25 & -0.30 \\ 0 & -0.10 & 0 & 0.37 \\ -1.90 & 0 & 0 & -0.71 \\ -1.09 & -0.95 & 1 & -0.62 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0.59 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda = 0.65$$

(see [12]*). To apply Theorem 3.1, we first transform this system into a system with a diagonal matrix at* $\mathbf{x}(t)$*. Matrix* $\mathbf{A}$ *is diagonalizable via a regular matrix* $\mathbf{P}$ *such that* $\mathbf{A} = \mathbf{PDP}^{-1}$*, where* $\mathbf{D} = \mathrm{diag}(-0.56 + 0.78i,\ -0.56 - 0.78i,\ -0.40,\ -0.59)$*. Note that all the eigenvalues of* $\mathbf{A}$ *(forming the diagonal elements* d_{ii} *of* $\mathbf{D}$*) have negative real parts and the condition* $|a_{ii} + 1| < 1$*,* $i = 1, \ldots, 4$*, is fulfilled.*

If we denote $\mathbf{C} = \mathbf{P}^{-1}\mathbf{BP}$*, then*

$$\mathbf{C} = \begin{pmatrix} -0.30 + 0.096i & -0.17 - 0.27i & 2.0 + 0.74i & -0.61 - 0.22i \\ -0.17 + 0.27i & -0.30 - 0.096i & 2.0 - 0.74i & -0.61 + 0.22i \\ 0.19 - 0.15i & 0.19 + 0.15i & -1.6 & 0.49 \\ 1.1 - 0.84i & 1.1 + 0.84i & -9.4 & 2.8 \end{pmatrix}. \tag{18}$$

Putting $\mathbf{z}(t) = \mathbf{P}^{-1}\mathbf{x}(t)$*, equation* (1) *becomes*

$$\dot{\mathbf{z}}(t) = \mathbf{Dz}(t) + \mathbf{Cz}(0.65t), \qquad t \in [0, \infty]. \tag{19}$$

Asymptotic estimate of any exact solution $\mathbf{z}$ *of equation* (19) *derived in* [9] *(see Section 2) yields*

$$\mathbf{z}(t) = O(t^{8.3594763}). \tag{20}$$

Applying our asymptotic result to the discretization of equation (19) *in the form*

$$\mathbf{z}_{k+1} - \mathbf{z}_k = \mathbf{Dz}(t) + \mathbf{Cz}_{0.65_k}, \qquad k = 0, 1, 2, \ldots \tag{21}$$

we obtain the estimate $\mathbf{z}_k = O(k^\gamma)$ *as* $k \to \infty$ *with*

$$\gamma = 11.4278495.$$

Performing the substitution $\mathbf{x}(t) = \mathbf{Pz}(t)$ *(with a constant matrix* $\mathbf{P}$*), we can see that original pantograph equation* (1) *and its discretization* (2) *admit the above mentioned asymptotic estimates as well.*

One can see that the asymptotic estimate of the solution of equation (21) *does not correspond with estimate* (20)*. This is caused by the stepsize of the mesh* $\mathbb{N}^0$*, which is equal to one (see also Remark 3.2). It can be shown that in the case of the stepsize* h *tending to zero, the asymptotic estimate of the solution of modified equation* (21) *reaches estimate* (20)*.*

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