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CONSTRUCTION OF AN INTEGRAL MANIFOLD FOR LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

Abstract. In this paper we establish sufficient conditions for the existence of an asymptotic integral manifold of solutions of a linear system of differential-difference equations with a small parameter. This integral manifold is described by a linear system of differential equations without deviating argument.

Keywords: system with deviating argument, integral manifold of solutions, fundamental matrix, exponential dichotomy.

Mathematics Subject Classification: 34K06.

1. INTRODUCTION AND PRELIMINARIES

The theory of linear differential equations with deviating argument is well established. There are numerous important papers on the subject. One of the classical works here is [2]. For a recent account of the theory, we refer the reader to [1] and the references given there.

We consider the linear system of differential equations with deviating argument

\[
\begin{align*}
\frac{dX(t)}{dt} &= A(t)X(t) + \mu \sum_{k=1}^{n} (A_k(t)X(t + \tau_k(t)) + B_k(t)Y(t + \tau_k(t))) \\
\frac{dY(t)}{dt} &= B(t)Y(t) + \mu \sum_{k=1}^{n} (C_k(t)X(t + \tau_k(t)) + D_k(t)Y(t + \tau_k(t))),(1)
\end{align*}
\]

where \( t \geq 0, \mu \) is a small parameter, \( \dim X(t) = p, \dim Y(t) = q, \)

\[
|\tau_k(t)| \leq \tau \quad (k = 1, \ldots, n; \ t \geq 0). \quad (2)
\]
We assume that the matrices in System (1) are bounded:

\[
\|A(t)\| \leq \alpha_0, \quad \sum_{k=1}^{n} \|A_k(t)\| \leq \alpha,  \\
\sum_{k=1}^{n} \|B_k(t)\| \leq \alpha, \quad \sum_{k=1}^{n} \|C_k(t)\| \leq \alpha, \quad \sum_{k=1}^{n} \|D_k(t)\| \leq \alpha,  
\]

(3)

If the parameter \(\mu = 0\), System (1) decouples into two independent subsystems. Let the system

\[
\frac{dX(t)}{dt} = A(t)X(t)
\]

have a fundamental matrix of solutions \(P(t, s)\) normalized at \(t = s\) and satisfy the condition

\[
\|P(t, s)\| \leq c_1 e^{\|t-s\|} \quad (c_1 \geq 1, \ \varepsilon > 0)  
\]

(4)

Let the system

\[
\frac{dY(t)}{dt} = B(t)X(t)
\]

have a fundamental matrix of solutions \(Q(t, s)\) normalized at \(t = s\) and satisfy the condition

\[
\|Q(t, s)\| \leq c_2 e^{\lambda \|t-s\|} \quad (c_2 \geq 1, \ \lambda > \varepsilon)  
\]

(5)

Thus, for \(\mu = 0\), System (1) possesses an exponential dichotomy with an exponent \(\sigma\), where \(-\lambda < \sigma < -\varepsilon\) (see [3]).

We will construct an integral manifold of solutions of System (1) in the form of ( [3,4])

\[
\frac{dX(t)}{dt} = H(t, \mu)X(t), \quad Y(t) = K(t, \mu)X(t).  
\]

(6)

Let a fundamental matrix of solutions of System (6) be denoted by \(N(t, s, \mu)\), it follows that

\[
X(t) = N(t, s, \mu)X(s).  
\]

(7)

From this we obtain

\[
X(t + \tau_k(t)) = N(t + \tau_k(t), t, \mu)X(t),  \\
Y(t + \tau_k(t)) = K(t + \tau_k(t), \mu)N(t + \tau_k(t), t, \mu)X(t).  
\]

(8)

If the solutions of System (6) satisfy (1), then

\[
H(t, \mu) = A(t) + \mu \sum_{k=1}^{n} (A_k(t)N(t + \tau_k(t), t, \mu) + \\
+ B_k(t)K(t + \tau_k(t), \mu)N(t + \tau_k(t), t, \mu))  
\]

(9)
Construction of an Integral Manifold for Linear Differential-Difference Equations

\[ \frac{\partial K(t, \mu)}{\partial t} + K(t, \mu)(A(t) + \mu \sum_{k=1}^{n} (A_k(t)N(t + \tau_k(t), t, \mu) + B_k(t)K(t + \tau_k(t), \mu)N(t + \tau_k(t), t, \mu))) = B(t)K(t, \mu) + \mu \sum_{k=1}^{n} (C_k(t)N(t + \tau_k(t), t, \mu) + D_k(t)K(t + \tau_k(t), \mu)N(t + \tau_k(t), t, \mu)) \]  

(10)

We now proceed to considering the auxiliary matrix differential equation

\[ \frac{\partial K(t, \mu)}{\partial t} = B(t)K(t, \mu) - K(t, \mu)A(t) + F(t), \]  

(11)

where \( \|F(t)\| \leq b \) for \( t \geq 0 \). It is easy to check that the matrix

\[ K(t, \mu) = \int_{0}^{t} Q(t, s)F(t)P(s, t)ds \]  

(12)

is a solution of (11). In addition, under conditions (4) and (5), we obtain

\[ \|K(t, \mu)\| = c_1c_2 \int_{0}^{t} e^{-\lambda(t-s)} \sup \|F(t)\| e^{(t-s)} \leq \frac{c_1c_2}{\lambda - \dot{\epsilon}} \sup \|F(t)\|, \quad t \geq 0. \]  

(13)

Application of (11) and (12) enables us to write System (10) in the form

\[ K(t, \mu) = \mu \int_{0}^{t} Q(t, s) \sum_{k=1}^{n} (C_k(s) + D_k(s)K(s + \tau_k(s), \mu))K(s, t, \mu) - K(s, \mu)A_k(s) - K(s, \mu)B_k(s)K(s + \tau_k(s), \mu))N(s + \tau_k(s), t, \mu)P(s, t)ds. \]  

(14)

Our purpose here is to give a proof that an integral manifold of solutions of System (1) exists in form (6).

2. SUCCESSIVE APPROXIMATIONS

System (9),(14) of matrix equations defines the matrices \( H(t, \mu), \ K(t, \mu) \) and it can be solved by the method of successive approximations. We start this process by letting \( H_0(t, \mu) = 0, \ K_0(t, \mu) = 0 \) and

\[ H_{j+1}(t, \mu) = A(t) + \mu \sum_{k=1}^{n} (A_k(t)N_j(t + \tau_k(t), t, \mu) + B_k(t)K_j(t + \tau_k(t), \mu)N_j(t + \tau_k(t), t, \mu)), \]

\[ K_{j+1}(t, \mu) = \mu \int_{0}^{t} Q(t, s) \sum_{k=1}^{n} (C_k(s) + D_k(s)K_j(s + \tau_k(s), \mu)) - K_j(s, \mu)A_k(s) - K_j(s, \mu)B_k(s)K_j(s + \tau_k(s), \mu))N_j(s + \tau_k(s), t, \mu)P(s, t)ds, \]  

(15)

for \( j=0,1,2,\ldots \)
Let $N_j(t, s, \mu)$ be a fundamental matrix of solutions of the system

$$\frac{dX(t)}{dt} = H_j(t, \mu)X(t) \quad (j = 0, 1, 2, \ldots).$$

(16)

Supposing that the inequalities

$$\|H_j(t, \mu)\| \leq h_j, \quad \|K_j(t, \mu)\| \leq k_j,$$

take place, we find the estimates

$$\|N_j(t, s, \mu)\| \leq e^{h_j|t-s|}, \quad \|N_j(t + \tau_k(t), t, \mu)\| \leq e^{\tau h_j}.$$

And owing to System (15) it follows that

$$h_{j+1} \leq \alpha_0 + |\mu| \alpha (1 + k_j)e^{\tau h_j},$$
$$k_{j+1} \leq |\mu| \alpha \beta (1 + k_j)^2 e^{\tau h_j},$$

(17)

where

$$\beta = \frac{c_1c_2}{\lambda - \varepsilon}.$$ 

In order for the sequences $\{h_j\}$ and $\{k_j\}$ to be bounded from below, i.e.

$$h_j \geq h > 0, \quad k_j \geq k > 0, \quad j = 0, 1, 2, \ldots$$

it is necessary and sufficient that the system of nonlinear equations

$$\alpha_0 + |\mu| \alpha (1 + k) e^{\tau h} = h, \quad |\mu| \alpha \beta (1 + k)^2 e^{\tau h} = k$$

(18)

has a positive solution.

We need to find the largest value $\mu = \mu_0$ for which System (18) has multiple solutions. In this connection $h$ and $k$ achieve the maximum. We obtain

$$\mu_0 = \frac{2(\tau + \sqrt{\tau^2 + 4\beta^2})}{\alpha(\tau + 2\beta + \sqrt{\tau^2 + 4\beta^2})} \exp \left\{-\tau (\alpha + \frac{2}{\tau + 2\beta + \sqrt{\tau^2 + 4\beta^2}})\right\}$$

(19)

$$h = \alpha_0 + \frac{k}{\beta(1 + k)}, \quad k = \frac{2\beta}{\tau + \sqrt{\tau^2 + 4\beta^2}}.$$ 

It follows that for $|\mu| \leq \mu_0$ the sequences of matrices $H_j(t, \mu)$ and $K_j(t, \mu)$ ($j=0,1,2,\ldots$) are well defined and bounded in norm for $t \geq 0$.

3. CONVERGENCE

Let us turn to prove that the sequences of matrices $H_j(t, \mu)$ and $K_j(t, \mu)$ ($j = 0, 1, 2, \ldots$) converge uniformly in $t$, for $t \geq 0$.

Let us introduce the notation

$$\|H_j(t, \mu) - H_{j-1}(t, \mu)\| \leq u_j, \quad \|K_j(t, \mu) - K_{j-1}(t, \mu)\| \leq v_j \quad (j = 1, 2, 3, \ldots).$$
Then from System (15), for $\mu = \mu_0$ there follows:

$$
\begin{align*}
 u_{j+1} &\leq \ell(1+k)\tau u_j + lv_j, \\
v_{j+1} &\leq \ell \beta \tau (1+k)^2 u_j + 2\ell \beta (1+k)v_j,
\end{align*}
$$

(20)

where

$$
\ell \equiv \frac{1}{2\beta(1+k)+\tau}.
$$

The matrix of coefficients of the expression on the right hand side of (20)

$$
R(\mu) = \begin{pmatrix}
\ell \tau (1+k) & \ell \\
\ell \beta \tau (1+k)^2 & 2\ell \beta
\end{pmatrix}
$$

has the largest eigenvalue (in terms of the absolute value)

$$
\rho_{\text{max}} = \frac{2\beta + \tau + \sqrt{\tau^2 + 4\beta^2}}{2(2\beta + \sqrt{\tau^2 + 4\beta^2})(\tau + \sqrt{\tau^2 + 4\beta^2})} \equiv 1.
$$

For $|\mu| < \mu_0$, the absolute values of eigenvalues of the matrix $R(\mu)$ are less than 1 and, therefore, the terms of the series

$$
H(t, \mu) = \sum_{j=0}^{\infty} (H_{j+1}(t, \mu) - H_j(t, \mu)),
$$

(21)

$$
K(t, \mu) = \sum_{j=0}^{\infty} (K_{j+1}(t, \mu) - K_j(t, \mu)).
$$

(22)

are bounded from above by the terms of the decreasing geometric progression. It follows that series (21) and (22) converge uniformly for $|\mu| \leq \mu_1 < \mu_0$.

We can now formulate our main results.

**Theorem 3.1.** Let System (1) of differential-difference equations be dichotomic for $\mu = 0$ and $t \geq 0$, caused by inequalities (4) and (5). If conditions (2) and (3) hold, then System (1) has an integral manifold of form (6), where the matrices $H$ and $K$ depend analytically on $\mu$, provided $|\mu| < \mu_0$, where $\mu_0$ is given by (19).

The construction of integral manifolds in form (6) for systems of differential-difference equations can be used in the investigation of qualitative properties of this systems.

**Example 3.2.** For the delay system

$$
\begin{align*}
\frac{dx(t)}{dt} &= \mu \cos tx(t) + \mu y(t - \tau), \\
\frac{dy(t)}{dt} &= -y(t) + \mu bx(t - \tau),
\end{align*}
$$

(23)
we are interested in the construction of an integral manifold of solutions in form (6)

\[
\frac{dx(t)}{dt} = h(t, \mu)x(t), \quad y(t) = k(t, \mu)x(t).
\]

The functions \(k\) and \(h\) are defined by the system

\[
\begin{align*}
  h(t, \mu) &= \mu \cos t + \mu ak(t - \tau, \mu) \exp \int_{t}^{t-\tau} h(r, \mu)dr, \\
  k(t, \mu) &= \mu \int_{0}^{t} \left( e^{-(t-s)}(b - k(s, \mu)) \cos s - ak(s, \mu)k(s - \tau, \mu) \right) \exp \int_{s}^{s-\tau} h(r, \mu)dr)ds.
\end{align*}
\]

Solving System (24) by the method of successive approximations, we find

\[
h(t, \mu) = \mu \cos t + \mu^{2}ab + O(\mu^{3}).
\]

The zero solution of the differential equation

\[
\frac{dx(t)}{dt} = (\mu \cos t + \mu^{2}ab + O(\mu^{3}))x(t)
\]

and also the zero solution of System (23) are asymptotically stable for a sufficiently small values of \(\mu > 0\) and \(ab < 0\), and unstable if \(ab > 0\).

REFERENCES


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