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FURTHER PROPERTIES OF THE RATIONAL
RECURSIVE SEQUENCE \( x_{n+1} = \frac{ax_{n-1}}{b+cx_nx_{n-1}} \)

Abstract. In this paper we consider the difference equation

\[ x_{n+1} = \frac{ax_{n-1}}{b+cx_nx_{n-1}}, \quad n = 0, 1, \ldots \]  (E)

with positive parameters \( a \) and \( c \), negative parameter \( b \) and nonnegative initial conditions. We investigate the asymptotic behavior of solutions of equation (E).

Keywords: difference equation, explicit formula, positive solutions, asymptotic stability.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

In this paper we consider the following rational difference equation

\[ x_{n+1} = \frac{ax_{n-1}}{b+cx_nx_{n-1}}, \quad n = 0, 1, \ldots \]  (E)

where \( b \) is a negative real number and \( a \) and \( c \) are positive real numbers and the initial conditions \( x_{-1}, x_0 \) are nonnegative real numbers such that at least one of them is positive. Eq. (E) in the case of positive \( b \) was considered in [1]. We use the explicit formula for solutions of Eq. (E) in investigating their behavior.

There has been a lot of work concerning the asymptotic behavior of solutions of rational difference equations. Second order rational difference equations were investigated, for example, in [1–13]. This paper is motivated by the short notes [4], where the author studied the rational difference equation

\[ x_{n+1} = \frac{x_{n-1}}{-1 + x_nx_{n-1}}, \quad n = 0, 1, \ldots \]
2. MAIN RESULTS

Let \( p = \frac{b}{q}, q = \frac{s}{\pi} \). Then Eq. (E) can be rewritten as

\[
x_{n+1} = \frac{x_{n-1}}{p + qx_n x_{n-1}}, \quad n = 0, 1, \ldots
\]

(E1)

The change of variables \( x_n = \frac{1}{\sqrt{q}} y_n \) reduces the above equation to

\[
y_{n+1} = \frac{y_{n-1}}{p + y_n y_{n-1}}, \quad n = 0, 1, \ldots
\]  

(E2)

where \( p \) is a negative real number, the initial conditions \( y_{-1}, y_0 \) are nonnegative real numbers such that at least one of them is positive. We will also assume \( y_0 y_{-1} \neq c^n (1-p) \) for \( n = 1, 2, \ldots, p \neq -1 \) and \( y_0 y_{-1} \neq 1 \) for \( p = -1 \) (which ensures that the denominator in Eq. (E2) is not equal to zero). Hereafter, we focus our attention on \( y \) or \( y \) numbers such that at least one of them is positive. We will also assume \( \alpha \) equilibrium

Let

\[
\{y_n\} = \{0, 0, 0, \frac{y_{n-1}}{p}, 0, \frac{y_{n-1}}{p^2}, \ldots\}\text{ or } \{y_n\} = \{y_{-1}, 0, \frac{y_{n-1}}{p}, 0, \frac{y_{n-1}}{p^2}, \ldots\}.
\]

Obviously, if \( p = -1 \), these solutions are 4-periodic.

Definition 1. ([8]) For every pair of initial conditions \((x_{-1}, x_0) \in I \times I\), the difference equation

\[
x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots
\]

(E3)

has the unique solution \( \{x_n\}_{n=1}^\infty \), which is called a recursive sequence. An equilibrium point of (E3) is a point \( \alpha \in I \) with \( f(\alpha, \alpha) = \alpha \); it is also called a trivial solution of Eq. (E3).

Definition 2. ([13]) Let \( \alpha \) be an equilibrium point of Eq.(E3):

(i) \( \alpha \) is stable if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any initial conditions \((x_{-1}, x_0) \in I \times I\) with \( |x_{-1} - \alpha| + |x_0 - \alpha| < \delta \), the inequality \( |x_n - \alpha| < \varepsilon \) holds for \( n = 1, 2, \ldots \);

(ii) \( \alpha \) is a local attractor if there exists \( \gamma > 0 \) such that \( x_n \to \alpha \) holds for any initial conditions \((x_{-1}, x_0) \in I \times I\) with \( |x_{-1} - \alpha| + |x_0 - \alpha| < \gamma \);

(iii) \( \alpha \) is locally asymptotically stable if it is stable and is a local attractor;

(iv) \( \alpha \) is a repeller if there exists \( \gamma > 0 \) such that for each \((x_{-1}, x_0) \in I \times I\) with \( |x_{-1} - \alpha| + |x_0 - \alpha| < \gamma \), there exists \( N \) such that \( |x_N - \alpha| \geq \gamma \).

Assume \( \alpha \) is an equilibrium point of Eq. (E3). Let \( r = -\frac{\partial f(\alpha, \alpha)}{\partial x_n} \), \( s = -\frac{\partial f(\alpha, \alpha)}{\partial x_{n-1}} \). Then the linearized equation associated with Eq. (E3) about the equilibrium \( \alpha \) is

\[
z_{n+1} + rz_n + sz_{n-1} = 0.
\]

(E4)
Further properties of the rational recursive sequence $x_{n+1} = \frac{ax_n - b}{cx_n - d}$

Theorem A ([7])(Linearized stability theorem).

(i) If $|r| < 1 + s$ and $s < 1$, then $\alpha$ is locally asymptotically stable.

(ii) If $|r| < |1 + s|$ and $|s| > 1$ then $\alpha$ is a repeller.

The equilibria of Eq. (E2) are the solutions of the equation

$$\bar{y} = \frac{\bar{y}}{p + \bar{y}^2}.$$ 

So, equilibrium points of Eq. (E2) are $\bar{y} = 0$ and $\bar{y} = \pm \sqrt{1 - p}$. The local asymptotic behavior of the zero equilibrium of Eq. (E2) is characterized by the following result.

Theorem 1. The following statements are true:

(i) if $p \in (-\infty, -1)$, then $\bar{y} = 0$ is locally asymptotically stable;
(ii) if $p \in (-1, 0)$, then $\bar{y} = 0$ is a repeller.

Proof. For Eq. (E2), there is

$$\frac{\partial f}{\partial y_n} = -\frac{y_n^{n-1}}{(p + y_n y_{n-1})^2},$$

$$\frac{\partial f}{\partial y_{n-1}} = \frac{p}{(p + y_n y_{n-1})^2}.$$ 

Therefore, for $\bar{y} = 0$ we get $r = 0$, $s = -\frac{1}{p}$ and the linearized equation associated with Eq. (E2) about the equilibrium $\bar{y} = 0$ is

$$z_{n+1} - \frac{1}{p} z_{n-1} = 0.$$

(i) The result follows from Theorem A(i) and the following relations

$$|r| - (1 + s) = -1 + \frac{1}{p} < 0,$$

and

$$s = -\frac{1}{p} < 1.$$

(ii) The result follows from Theorem A(ii) and the following relations

$$|r| - |1 + s| = -\left|\frac{p - 1}{p}\right| = \frac{1-p}{p} < 0$$

and

$$-\frac{1}{p} > 1.$$

This completes the proof. \qed
It is easy to see that the method used in the proof of Theorem 1 in [1] can be used in our case too. Thus the following formula holds for all solutions of Eq. (E2) with positive initial conditions $y_1, y_0$ such that $y_0 y_1 \neq \frac{p^n(1-p)}{p^n-1}$ for $n = 1, 2, \ldots, p \neq -1$ and $y_0 y_1 \neq 1$ for $p = -1$.

If all parameters and initial conditions in Eq. (E) are positive, then all solutions of Eq. (E) are positive, too. It is not true in the case of negative $b$. In the next theorem we give sufficient conditions for every solution of Eq. (E2) to be positive.

**Theorem 2.** Assume that $p \in (-1, 0)$. Let \{\(y_n\)\} be a solution of Eq. (E2) with positive initial conditions $y_1, y_0$ such that $y_0 y_1 \neq \frac{p^n(1-p)}{p^n-1}$ for $n = 1, 2, \ldots$. If $y_0 y_1 > -p$ then \{\(y_n\)\} is positive.

**Proof.** Let \{\(y_n\)\} be a solution of Eq. (E2). From (1), for the subsequence \{\(y_{2n-1}\)\} there follows

\[
y_{2n-1} = \frac{y_{n-1} \prod_{k=0}^{n-1} [p^{2i} + y_0 y_{1-i} \sum_{k=0}^{2i-1} p^k]}{\prod_{k=0}^n [p^{2i+1} + y_0 y_{1-i} \sum_{k=0}^{2i+1} p^k]}.
\]

Obviously, for $p \in (-1, 0)$,

\[p^{2i} + y_0 y_{1-i} \sum_{k=0}^{2i-1} p^k > 0\]

for all $i = 0, 1, \ldots$. On the other hand, if $y_0 y_{1-i} > -p$, then

\[p^{2i+1} + y_0 y_{1-i} \sum_{k=0}^{2i} p^k > 0\]  \hspace{1cm} (2)

for all $i = 0, 1, \ldots$. Therefore, all terms of the sequence \{\(y_{2n-1}\)\} are positive. For $n$ even the proof is similar.

**Remark 1.** If $y_0 y_{1-i} = 1 - p$ then from (E2) we get $y_{n+1} = \frac{y_{n-1}}{p+y_0 y_{n-1}} = y_{n-1}$. Hence \{\(y_{2n}\)\} = \{\(y_0, y_0, y_0, \ldots\)\} and \{\(y_{2n-1}\)\} = \{\(y_{1}, y_{-1}, y_{-1}, \ldots\)\}.
Further properties of the rational recursive sequence \( x_{n+1} = \frac{ax_n - 1}{bx_n + px_{n-1}} \)

**Theorem 3.** Assume that \( p \in (-1, 0) \). Let \( \{y_n\} \) be a solution of Eq. (E2) with positive initial conditions \( y_{-1}, y_0 \) such that \( y_0 y_{-1} \neq \frac{p^{n-1} - \mu}{p^n - 1} \) for \( n = 1, 2, \ldots \). If \(-p < y_0 y_{-1} < 1 - p\) then the subsequence \( \{y_{2n}\} \) is decreasing and subsequence \( \{y_{2n-1}\} \) is increasing.

**Proof.** Let \( \{y_n\} \) be a solution of Eq. (E2). From (1), for the subsequence \( \{y_{2n}\} \) there follows

\[
y_{2n} = \frac{\prod_{i=0}^{n-1} [p^{2i+1} + y_0 y_{-1} \sum_{k=0}^{2i} p^k]}{\prod_{i=0}^{n-1} [p^{2i+2} + y_0 y_{-1} \sum_{k=0}^{2i+1} p^k]}
\]

Thus for \( n \geq 1 \)

\[
\frac{y_{2n+2}}{y_{2n}} = \frac{\prod_{i=0}^{n} [p^{2i+1} + y_0 y_{-1} \sum_{k=0}^{2i} p^k] \prod_{i=0}^{n-1} [p^{2i+2} + y_0 y_{-1} \sum_{k=0}^{2i+1} p^k]}{\prod_{i=0}^{n-1} [p^{2i+2} + y_0 y_{-1} \sum_{k=0}^{2i} p^k] \prod_{i=0}^{n} [p^{2i+1} + y_0 y_{-1} \sum_{k=0}^{2i} p^k]}
\]

\[
= \frac{p^{2n+1} + y_0 y_{-1} \sum_{k=0}^{2n} p^k}{p^{2n+2} + y_0 y_{-1} \sum_{k=0}^{2n+1} p^k}
\]

Since \( y_0 y_{-1} < 1 - p \), there is

\[
y_0 y_{-1} p^{2n+1} > p^{2n+1} - p^{2n+2}.
\]

Hence

\[
y_0 y_{-1} (\sum_{k=0}^{2n+1} p^k - \sum_{k=0}^{2n} p^k) > p^{2n+1} - p^{2n+2},
\]

and therefore

\[
p^{2n+1} + y_0 y_{-1} \sum_{k=0}^{2n} p^k < p^{2n+2} + y_0 y_{-1} \sum_{k=0}^{2n+1} p^k.
\]

From the above inequality, by (2) and (3) it follows that the subsequence \( \{y_{2n}\} \) is decreasing. Similarly we prove that the subsequence \( \{y_{2n-1}\} \) is increasing. This completes the proof. \( \Box \)

**Theorem 4.** Assume that \( p \leq -2 \). Let \( \{y_n\} \) be a solution of Eq. (E2) with positive initial conditions \( y_{-1}, y_0 \in (0, 1) \). Then the subsequences \( \{y_{4n-1}\} \) and \( \{y_{4n}\} \) are both positive and decreasing, while subsequences \( \{y_{4n+1}\} \) and \( \{y_{4n+2}\} \) are both negative and increasing.
Proof. Let \(y_1, y_0 \in (0, 1)\). Then \(y_1, y_2 \in (0, 1)\) and \(y_3, y_4 \in (-1, 0)\). By induction we can prove that \(\{y_{4n-1}\}, \{y_{4n}\} \subset (0, 1)\) and \(\{y_{4n+1}\}, \{y_{4n+2}\} \subset (-1, 0), n = 0, 1, \ldots\). Since, by (1),

\[
\frac{y_{4n+4}}{y_{4n}} = \frac{(p^{4n+4} + y_0y-1 - p^{4n+1})}{(p^{4n+2} + y_0y-1 - p^{4n+3})} < 1,
\]

we have

\[
y_{4n+4} < y_{4n}, \quad n = 0, 1, \ldots
\]

Similarly we can see that \(y_{4n+3} < y_{4n-1}\), and \(y_{4n+5} > y_{4n+1}, y_{4n+6} > y_{4n+2}\) for \(n = 0, 1, \ldots\) and the result follows.

3. NUMERICAL RESULTS

**Example 1.** Let \(y_1 = \frac{3}{4}, y_0 = 1\) be the initial conditions of Eq. (E2) with \(p = -\frac{1}{2}\). Then, by Theorem 2, the solution is positive.

Table 1 sets forth the values of \(y_n\) for selected small \(n\)’s.

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<th>(y)</th>
<th>(n)</th>
<th>(y)</th>
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**Example 2.** Let \(p = -2/3, y(1) = 4/3, y(0) = 1\). Thus the condition \(-p < y(0)y(-1) < 1 - p\) holds and by Theorem 3, the subsequence \(\{y_{2n}\}\) is decreasing and subsequence \(\{y_{2n-1}\}\) is increasing.

Table 2 sets forth the values of \(y_n\) for selected small \(n\)’s.

**Example 3.** Let \(p = -11, y(-1) = 0.2, y(0) = 0.5\). Then, by Theorem 4, the subsequences \(\{y_{2n-1}\}\) and \(\{y_{2n}\}\) are both positive and decreasing, while the subsequences \(\{y_{4n+1}\}\) and \(\{y_{4n+2}\}\) are both negative and increasing.
Further properties of the rational recursive sequence $x_{n+1} = \frac{ax_{n-1}}{bx_{n}x_{n-1}+c}$

Table 3 sets forth the values of $y_n$ for selected small $n$’s.

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</table>

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Received: September 23, 2005.