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THE REACTANCE WAVE DIFFRACTION PROBLEM BY A STRIP IN A SCALE OF BESSEL POTENTIAL SPACES

Abstract. We consider a boundary-transmission problem for the Helmholtz equation, in a Bessel potential space setting, which arises within the context of wave diffraction theory. The boundary under consideration consists of a strip, and certain reactance conditions are assumed on it. Operator theoretical methods are used to deal with the problem and, as a consequence, several convolution type operators are constructed and *associated* to the problem. At the end, the well-posedness of the problem is shown for a range of regularity orders of the Bessel potential spaces, and for a set of possible reactance numbers (dependent on the wave number).

Keywords: Helmholtz equation, boundary-transmission problem, wave diffraction, convolution type operator, Wiener–Hopf operator, Fredholm property, factorization.

Mathematics Subject Classification: Primary: 35J05; Secondary: 35J25, 47A53, 47B35, 47F05, 45E10, 78A45.

1. INTRODUCTION

We will consider a plane wave diffraction problem with reactance conditions [5, 6, 16, 17] on a finite strip [8, 9, 10]. The analysis will be taken from an operator-theoretical viewpoint.

The problem is initially presented in the form of a boundary-transmission value problem. All the conditions will be then assembled in a single equation characterized by an operator acting between Bessel potential spaces. Although this first operator is not given in explicit form, it turns out that the construction of specific operator relations lead us to related explicit convolution operators. As about these convolution type operators, a new chain of operator relations based on factorization procedures leads us to the invertibility of all of them in a smoothness parameters range of the Bessel potential spaces in use. Consequently, the well-posedness of the wave diffraction problem is obtained, including the continuous dependence on the data in a scale of spaces.

As mentioned, the theory will be developed in the framework of Bessel potential spaces. A Bessel potential space can be defined as the linear space of distributions, $\phi = r_{\mathbb{R}^n\to\Omega}\varphi$, that are obtained by restricting to $\Omega \subset \mathbb{R}^n$ the elements of the space

$$H^s(\mathbb{R}^n) = \left\{\varphi \in \mathcal{D}'(\mathbb{R}^n) \colon \ \|\varphi\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathcal{F}\varphi\|_{L^2(\mathbb{R}^n)} < +\infty\right\},$$

where $s \in \mathbb{R}$ and

$$(\mathcal{F}\varphi)(\xi) = \int\limits_{\mathbb{R}^n} e^{i\xi\cdot x}\varphi(x)\,dx\,, \qquad \xi \in \mathbb{R}^n\,. \tag{1.1}$$

Moreover, the $H^s(\Omega)$ space endowed with the norm

$$\|\phi\|_{H^s(\Omega)} = \inf\left\{\|\varphi\|_{H^s(\mathbb{R}^n)} \colon \ \varphi \in H^s(\mathbb{R}^n), r_{\mathbb{R}^n\to\Omega}\varphi = \phi\right\}$$

becomes a Banach space. For $\mathcal{I} \subseteq \mathbb{R}_+$, we will by $\widetilde{H}^s(\mathcal{I})$ denote the closed subspace of $H^s(\mathbb{R})$ defined by the distributions with support contained in $\overline{\mathcal{I}}$. Moreover, in the special case of $s = 0$, we will use the more common notation of $L^2_+(\mathbb{R})$ and $L^2(\mathbb{R}_+)$ for representing the Lebesgue spaces $\widetilde{H}^0(\mathbb{R}_+)$ and $H^0(\mathbb{R}_+)$, respectively.

2. FORMULATION OF THE WAVE DIFFRACTION PROBLEM

We will consider the problem of wave diffraction by a finite strip with reactance conditions. The finite strip is denoted here by $\Sigma =]0, a[$ where the dependence on one variable was dropped due to perpendicular wave incidence (which leads us from strips to intervals); see Figure 1.

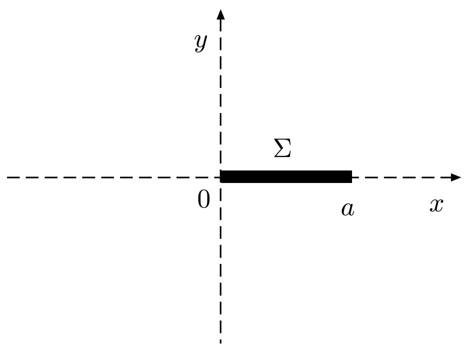

Fig. 1. The geometry of the problem

From the mathematical point of view, the problem can be formulated in a Bessel potential space setting as the following boundary-transmission problem for the Helmholtz equation: Find $\varphi \in L^2(\mathbb{R}^2)$, with $\varphi_{|\mathbb{R}^2_\pm} \in H^s(\mathbb{R}^2_\pm)$, $s > 1/2$, such that

$$\left(\Delta + k^2\right)\varphi = 0 \qquad \text{in} \quad \mathbb{R}^2_\pm, \tag{2.1}$$

$$\begin{cases} \varphi_0^+ - \varphi_0^- = h_1 \\ \varphi_1^+ - \varphi_1^- + q\varphi_0^+ = h_2 \end{cases} \qquad \text{on} \quad \Sigma, \tag{2.2}$$

$$\begin{cases} \varphi_0^+ - \varphi_0^- = 0 \\ \varphi_1^+ - \varphi_1^- = 0 \end{cases} \qquad \text{on} \quad \mathbb{R}\setminus\overline{\Sigma}, \tag{2.3}$$

where $\mathbb{R}^2_\pm$ represents the upper/lower half-plane, $k \in \mathbb{C}$ (with $\operatorname{Im} k > 0$) stands for the wave number, $\varphi_0^\pm = \varphi_{|y=\pm 0}$, $\varphi_1^\pm = (\partial\varphi/\partial y)_{|y=\pm 0}$, $q \in \mathbb{C}$ is the reactance number and the elements $h_1 \in r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-1/2}(\Sigma)$, $h_2 \in r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-3/2}(\Sigma)$ are arbitrarily given (since the dependence on the data is to be studied for well-posedness). The Bessel potential spaces of order $s > 1/2$, $s - 1/2$ and $s - 3/2$ are naturally involved due to the (generalized) *Trace Theorem* [4]. Note also that since for $j = 0, 1$ the traces $\varphi_j^\pm$ belong to the Bessel potential spaces on the real line $H^{s-1/2-j}(\mathbb{R})$, it is natural to use the operator $r_{\mathbb{R}\to\Sigma}$ in view of the above characterization of the data h_1 and h_2 (which appear in the $\Sigma \subset \mathbb{R}$ boundary).

For the case when Σ is a half-line, the corresponding problem has previously been considered by many authors as a Sommerfeld type problem. In [17, §5], such a half-line problem was also regarded as a certain class of general screen problems that were analyzed upon the boundary conditions considered.

The reason to consider the data in the restricted tilde spaces $r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-1/2}(\Sigma)$ and $r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-3/2}(\Sigma)$ is a consequence of the overlapping of the information in (2.2) and (2.3). Such realizations of the data are known as *compatibility conditions* [19] and appear in several different kinds of wave diffraction problems. In fact, the first compatibility condition follows directly from the first equations in (2.2) and (2.3), whilst the second one follows from the second equations in (2.2) and (2.3) and by noting that we have the continuous embedding $H^{s-1/2}(\Sigma) \hookrightarrow r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-3/2}(\Sigma)$.

From the operator-theoretical point of view, the problem (2.1)–(2.3) can be described with use of a single operator

$$L^{(s)}\colon\ D^s(L^{(s)}) \to \left(r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-1/2}(\Sigma)\right) \times \left(r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-3/2}(\Sigma)\right) \tag{2.4}$$

defined as $L^{(s)}\varphi = (h_1, h_2)^T$ if $D(L^{(s)})$ is provided to be the subspace of $H^s(\mathbb{R}^2_+) \times H^s(\mathbb{R}^2_-)$ whose functions satisfy Helmholtz equation (2.1) and transmission condition (2.3). The operator $L^{(s)}$ is then said to be associated with the reactance problem. In what follows, we will analyze whether $L^{(s)}$ is a bounded and invertible operator. This will guarantee the well-posedness of the problem, including the continuous dependence upon the data.

3. CORRESPONDING EQUATIONS WITH CONVOLUTION TYPE OPERATORS ON TILDE SPACES

In this section we shall explore the structure behind the operator $L^{(s)}$ defined in (2.4). This will be done within the framework of convolution type operators. To this end, we will first consider some operator extension procedures of the following type.

Definition 1. *Let us consider two operators $W_1\colon\ X_1 \to Y_1$ and $W_2\colon\ X_2 \to Y_2$, acting between Banach spaces.*

(i) W_1 *and* W_2 *are said to be* algebraically equivalent after extension *if there exist additional Banach spaces* Z_1 *and* Z_2 *and invertible linear operators* $E : Y_2 \times Z_2 \to Y_1 \times Z_1$ *and* $F : X_1 \times Z_1 \to X_2 \times Z_2$ *such that*

$$\begin{bmatrix} W_1 & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} W_2 & 0 \\ 0 & I_{Z_2} \end{bmatrix} F. \tag{3.1}$$

(ii) *If, in addition to* (i)*, the invertible and linear operators E and F in (3.1) are bounded, then we will say that* W_1 *and* W_2 *are* topologically equivalent after extension *operators (or simply say that* W_1 *and* W_2 *are* equivalent after extension *operators [1]).*

(iii) W_1 *and* W_2 *are said to be* equivalent *operators in the particular case when* $W_1 = EW_2F$*, for some bounded invertible linear operators* $E\colon\ Y_2 \to Y_1$ *and* $F\colon\ X_1 \to X_2$*.*

We remark that the equivalence after extension notion (ii) is equivalent to the notion of *matricial coupling* – which is well-known to be very important in solving certain classes of integral equations, and is also important in (linear algebra) *matrix completion problems* (cf. e.g. [1]). In the present work we will use above notion (i) in the proof of Theorem 3, notion (ii) in Theorems 2, 3 and 7, and notion (iii) in the proof of Theorem 3 and in Corollary 6.

Let

$$t(\xi) = \left(\xi^2 - k^2\right)^{1/2}, \qquad \xi \in \mathbb{R} \tag{3.2}$$

denote the branch of the square root that tends to $+\infty$ as $\xi \to +\infty$ with branch cuts along $\pm k \pm i\eta$, $\eta \geq 0$. Then the following result on the structure of the operator $L^{(s)}$ and related convolution type operators holds true.

Theorem 2. *Let $s - 1/2 \notin \mathbb{N}$. The operator $L^{(s)}$ is equivalent after extension to the convolution type operator*

$$\widetilde{\mathcal{W}}_{\Phi,\Sigma} = r_{\mathbb{R}\to\Sigma}\mathcal{F}^{-1}\Phi \cdot \mathcal{F}\colon\ \widetilde{H}^{s-3/2}(\Sigma) \to r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-3/2}(\Sigma), \tag{3.3}$$

where $\mathcal{F}$ denotes the Fourier transformation (cf. (1.1) *for $n = 1$), and*

$$\Phi = 1 - qt^{-1}/2, \tag{3.4}$$

with q being the reactance number in (2.2)*, and t^{-1} the inverse of the function t defined in* (3.2) *(and which depends on the wave number k).*

Proof. Note first the well-known fact [11, 17] that a function $\varphi \in L^2(\mathbb{R}^2)$, with $\varphi_{|\mathbb{R}^2_\pm} \in H^s(\mathbb{R}^2_\pm)$, satisfies Helmholtz equation (2.1) if and only if it can be expressed as

$$\varphi(x,y) = \mathcal{F}^{-1}_{\xi\mapsto x} e^{-t(\xi)y} \mathcal{F}_{x\mapsto\xi} \varphi_0^+(x)\chi_{\mathbb{R}_+}(y) + \mathcal{F}^{-1}_{\xi\mapsto x} e^{t(\xi)y} \mathcal{F}_{x\mapsto\xi} \varphi_0^-(x)\chi_{\mathbb{R}_-}(y) \tag{3.5}$$

for $(x,y) \in \mathbb{R}^2$, where $\mathcal{F}_{x\mapsto\xi}\varphi(x,y) = \int_{\mathbb{R}} \varphi(x,y)e^{i\xi x}dx$, and $\chi_{\mathbb{R}_+}$ and $\chi_{\mathbb{R}_-}$ denote the characteristic functions of the positive and negative half-line, respectively.

Define the space

$$Z^s = \Big\{ (\phi,\psi) \in \big[H^{s-1/2}(\mathbb{R})\big]^2 : \\ \phi - \psi \in \widetilde{H}^{s-1/2}(\Sigma), \quad \mathcal{F}^{-1} t \cdot \mathcal{F}(\phi+\psi) \in \widetilde{H}^{s-3/2}(\Sigma) \Big\}.$$

Then (under the present conditions) the trace operator $T_0\colon D(L^{(s)}) \to Z^s$ defined by

$$T_0\varphi = \varphi_0 := \begin{bmatrix} \varphi_0^+ \\ \varphi_0^- \end{bmatrix}$$

is an invertible operator. In fact, such a trace operator is continuously invertible with the inverse operator $K\colon\ \varphi_0 \mapsto \varphi$ defined by representation formula (3.5). Moreover, with the help of the operators T_0 and K, the operator $L^{(s)}$ can be rewritten in the form of an operator matrix composition depending on $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$ (which can be checked by direct computation):

$$L^{(s)} = \begin{bmatrix} 0 & r_{\mathbb{R}\to\Sigma} \\ I_{r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-1/2}(\Sigma)} & \frac{q}{2} r_{\mathbb{R}\to\Sigma} \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{W}}_{\Phi,\Sigma} & 0 \\ 0 & I_{\widetilde{H}^{s-1/2}(\Sigma)} \end{bmatrix} \mathcal{W}_{\Phi_1,\mathbb{R}} T_0, \tag{3.6}$$

where $\mathcal{W}_{\Phi_1,\mathbb{R}}$ is the convolution operator on the whole line

$$\mathcal{W}_{\Phi_1,\mathbb{R}} = \mathcal{F}^{-1}\Phi_1 \cdot \mathcal{F}\colon\ Z^s \to \widetilde{H}^{s-3/2}(\Sigma) \times \widetilde{H}^{s-1/2}(\Sigma),$$

with

$$\Phi_1 = \begin{bmatrix} -t & -t \\ 1 & -1 \end{bmatrix}.$$

Now it can easily be verified that the matrix operator

$$\begin{bmatrix} 0 & r_{\mathbb{R}\to\Sigma} \\ I_{r_{\mathbb{R}\to\Sigma}}\widetilde{H}^{s-3/2}(\Sigma) & \frac{q}{2} r_{\mathbb{R}\to\Sigma} \end{bmatrix},$$

which maps $r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-3/2}(\Sigma) \times \widetilde{H}^{s-1/2}(\Sigma)$ into $r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-1/2}(\Sigma) \times r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-3/2}(\Sigma)$, is a bounded invertible operator with the inverse

$$\begin{bmatrix} -\frac{q}{2} I_{r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-1/2}(\Sigma)} & I_{r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-3/2}(\Sigma)} \\ l_{0_{|r_{\mathbb{R}\to\Sigma}\widetilde{H}}}^{s-1/2}(\Sigma) & 0 \end{bmatrix}$$

(where l_0 denotes the zero extension operator). Additionally, $\mathcal{W}_{\Phi_1,\mathbb{R}}T_0$ is continuously invertible with the inverse operator

$$K\mathcal{W}_{\Phi_1,\mathbb{R}}^{-1} = K\mathcal{W}_{\Phi_1^{-1},\mathbb{R}} \colon\ \widetilde{H}^{s-3/2}(\Sigma) \times \widetilde{H}^{s-1/2}(\Sigma) \to D(L^{(s)}).$$

Consequently, (3.6) represents a factorization of the operator $L^{(s)}$ in the form of an equivalence after extension relation between $L^{(s)}$ and the convolution type operator on a finite interval, $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, defined in (3.3). □

Due to the basic properties of the Fourier transformation, the operator $\mathcal{F}^{-1}\Phi \cdot \mathcal{F}$ in (3.3) has the form of a *convolution operator* (on the real line). Such kind of operators, when composed with restriction operators like $r_{\mathbb{R}\to\Sigma}$ and defined on domains having elements supported on $\overline{\Sigma}$, are known as *convolution type operators (on finite intervals)* in the operator theory literature, cf., e.g., [3, 9].

From now on we will always be considering $s - 1/2 \notin \mathbb{N}$ (besides $s > 1/2$).

4. NEW RELATED OPERATORS LEADING TO THE FREDHOLM PROPERTY

Our final goal is to study the invertibility of the operator $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$ defined in (3.3). To this end, in the first step, we will make use of other convolution type operators in such a process. We do this by first considering an auxiliary bounded and invertible convolution type operator, and then extending the operator $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$ with use of the announced auxiliary operator, which will allow us to work in the L^2-space setting. All this will lead to the Fredholm property at the end of the present section.

We recall that a bounded linear operator $W\colon\ X \to Y$ is called a *Fredholm operator* if $\ker W$ has a finite dimension and $\operatorname{im} W$ has a finite codimension (in Y). In particular, Fredholm operators have closed range and are generalized invertible. Additionally, for Fredholm operators W the *Fredholm index* is defined by

$$\operatorname{ind} W = \dim\ker W - \dim\operatorname{coker} W.$$

4.1. AUXILIARY OPERATORS

We start by considering the auxiliary convolution type operator

$$\mathcal{W}_{t^{-1},\Sigma} = r_{\mathbb{R}\to\Sigma}\mathcal{F}^{-1}t^{-1} \cdot \mathcal{F}\colon\ \widetilde{H}^{s-3/2}(\Sigma) \to H^{s-1/2}(\Sigma) \tag{4.1}$$

that will help us to identify some additional properties of $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$.

Theorem 3. *The convolution type operator* $\mathcal{W}_{t^{-1},\Sigma}$*, defined in* (4.1)*, is equivalent after extension to the Wiener–Hopf operator*

$$\mathcal{W}_{\Upsilon,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon \cdot \mathcal{F}\colon\ [L^2_+(\mathbb{R})]^2 \to [L^2(\mathbb{R}_+)]^2, \tag{4.2}$$

with

$$\Upsilon = \zeta^{s-1}\begin{bmatrix}\zeta^{-1/2}\,\tau_{-a} & 0\\ 1 & \zeta^{1/2}\,\tau_a\end{bmatrix}, \tag{4.3}$$

where $\zeta = \lambda_-/\lambda_+$ *and* $\lambda_\pm(\xi) = \xi \pm k$, $\tau_b(\xi) = \exp[i\xi b]$, *for* $\xi \in \mathbb{R}$.

This means that there are bounded and invertible linear operators E_1 *and* F_1 *and Banach spaces* X *and* Y *such that*

$$\begin{bmatrix} \mathcal{W}_{t^{-1},\Sigma} & 0 \\ 0 & I_X \end{bmatrix} = E_1 \begin{bmatrix} \mathcal{W}_{\Upsilon,\mathbb{R}_+} & 0 \\ 0 & I_Y \end{bmatrix} F_1. \tag{4.4}$$

Proof. It is known from [14, Theorem 2.1] (see also [7, 15] for some generalizations) that $\mathcal{W}_{t^{-1},\Sigma}$ is algebraically equivalent after extension to the new Wiener–Hopf operator

$$\mathcal{W}_{\Upsilon_1,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon_1\cdot\mathcal{F}\colon\ \widetilde{H}^{s-3/2}(\mathbb{R}_+)\times\widetilde{H}^{s-1/2}(\mathbb{R}_+) \to H^{s-3/2}(\mathbb{R}_+)\times H^{s-1/2}(\mathbb{R}_+),$$

with a Fourier symbol

$$\Upsilon_1 = \begin{bmatrix} \tau_{-a} & 0 \\ t^{-1} & \tau_a \end{bmatrix}.$$

Thus, identity (3.1) in Definition 1(i) holds with W_1 and W_2 being replaced by $\mathcal{W}_{t^{-1},\Sigma}$ and $\mathcal{W}_{\Upsilon_1,\mathbb{R}_+}$ (and for some linear invertible – not necessarily bounded – operators E and F).

Next we show that the Wiener–Hopf operator $\mathcal{W}_{\Upsilon_1,\mathbb{R}_+}$ is equivalent to $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$. Here, the operator equivalence in question is constructed in an explicit way and can be directly obtained by computing the following operator composition

$$\mathcal{W}_{\Upsilon_1,\mathbb{R}_+} = E_2\mathcal{W}_{\Upsilon,\mathbb{R}_+}F_2, \tag{4.5}$$

where E_2 and F_2 are defined by

$$E_2 = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\begin{bmatrix} \lambda_-^{-s+3/2} & 0 \\ 0 & \lambda_-^{-s+1/2} \end{bmatrix}\cdot\mathcal{F}l_0\colon$$
$$[L^2(\mathbb{R}_+)]^2 \to H^{s-3/2}(\mathbb{R}_+)\times H^{s-1/2}(\mathbb{R}_+)$$

and

$$F_2 = l_0 r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\begin{bmatrix} \lambda_+^{s-3/2} & 0 \\ 0 & \lambda_+^{s-1/2} \end{bmatrix}\cdot\mathcal{F}\colon\ \widetilde{H}^{s-3/2}(\mathbb{R}_+)\times\widetilde{H}^{s-1/2}(\mathbb{R}_+) \to \left[L_+^2(\mathbb{R})\right]^2.$$

In fact, the bounded operators E_2 and F_2 are invertible with

$$E_2^{-1} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\begin{bmatrix} \lambda_-^{s-3/2} & 0 \\ 0 & \lambda_-^{s-1/2} \end{bmatrix}\cdot\mathcal{F}l_0\colon$$
$$H^{s-3/2}(\mathbb{R}_+)\times H^{s-1/2}(\mathbb{R}_+) \to [L^2(\mathbb{R}_+)]^2 \tag{4.6}$$

and

$$F_2^{-1} = l_0 r_{\mathbb{R}\to\mathbb{R}_+} \mathcal{F}^{-1} \begin{bmatrix} \lambda_+^{-s+3/2} & 0 \\ 0 & \lambda_+^{-s+1/2} \end{bmatrix} \cdot \mathcal{F} : \\ [L_+^2(\mathbb{R})]^2 \to \widetilde{H}^{s-3/2}(\mathbb{R}_+) \times \widetilde{H}^{s-1/2}(\mathbb{R}_+) \quad (4.7)$$

(see [21, §2.10.3]). In view of the structure of the Fourier symbols of E_2 and F_2 [21], it follows that the right hand-side of (4.5) can be rewritten in the form of an unique Wiener–Hopf operator with Υ_1 as its Fourier symbol.

We now study the Fredholm property of the Wiener–Hopf operator $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$. The Fourier symbol Υ (see (4.3)) of the Wiener–Hopf operator $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$ belongs to the C^*-algebra of the semi-almost periodic (SAP) two by two matrix functions on the real line (see [3, 20]). This means that Υ belongs to the smallest closed subalgebra of $[L^\infty(\mathbb{R})]^{2\times 2}$ which contains the (classical) algebra of (two by two) *almost periodic elements* and the (two by two) continuous matrices with possible jumps at infinity. Additionally, the element in the second row and first column of Υ (that is, the lifted Fourier symbol of $\mathcal{W}_{t^{-1},\Sigma}$) is ζ^{s-1}, and the determinant of Υ is equal to $\zeta^{2(s-1)}$. Thus, using the well-known criteria for the Fredholm property of such operators (see, e.g., [2]), we conclude that $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$ is a Fredholm operator with Fredholm index $-2[s-1]$, where $[\theta]$ denotes the integer part of the real number decomposed in the form $\theta = [\theta] + \{\theta\}$ with $-1/2 < \{\theta\} < 1/2$ (for $\theta - 1/2 \notin \mathbb{Z}$).

Now, since $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$ is equivalent to $\mathcal{W}_{\Upsilon_1,\mathbb{R}_+}$ and algebraically equivalent after extension to $\mathcal{W}_{t^{-1},\Sigma}$ (through operator identity (4.5)), and by noting the structure of identity (4.5), it follows that also the operators $\mathcal{W}_{\Upsilon_1,\mathbb{R}_+}$ and $\mathcal{W}_{t^{-1},\Sigma}$ are Fredholm operators with Fredholm index $-2[s-1]$. Moreover, from the operator identities provided by both the equivalence relation and the algebraic equivalence after extension relation, we conclude that the corresponding defect spaces of all the three operators $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$, $\mathcal{W}_{\Upsilon_1,\mathbb{R}_+}$ and $\mathcal{W}_{t^{-1},\Sigma}$ have the same dimensions. From this, and since by [1, Theorem 3] Fredholm operators in Banach spaces are equivalent after extension if and only if their corresponding defect spaces have equal dimensions, we arrive at the last statement of Theorem 3. □

We will now be concerned with the factorization of the Fourier symbol Υ. The factorization will be done in such a way that the influence of the oscillating behavior (at infinity) of the elements in Υ will be absent in the new main factor and, additionally, it will allow the identification of the inverse of the corresponding Wiener–Hopf operator. In view of these goals, we propose the following factorization for Υ:

$$\Upsilon = \begin{bmatrix} \tau_{-a}\rho & -1 \\ 1 & 0 \end{bmatrix} \times \\ \times \begin{bmatrix} \zeta^{s-1} & \zeta^{s-1}\tau_a\left(\zeta^{s-1/2} - \rho\right) \\ \zeta^{s-1}\tau_{-a}\left(\rho - \zeta^{s-3/2}\right) & \zeta^{s-1}\rho\left(\zeta^{s-1/2} + \zeta^{s-3/2} - \rho\right) \end{bmatrix} \times \begin{bmatrix} 1 & \tau_a\rho \\ 0 & 1 \end{bmatrix} =$$

$$= \left(\begin{bmatrix} \tau_{-a}\rho & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \tau_{-a}\left(\rho - \zeta^{s-3/2}\right) & 1 \end{bmatrix} \begin{bmatrix} \lambda_-^{s-1} & 0 \\ 0 & \lambda_-^{s-1} \end{bmatrix} \right) \times$$
$$\times \left(\begin{bmatrix} \lambda_+^{-s+1} & 0 \\ 0 & \lambda_+^{-s+1} \end{bmatrix} \begin{bmatrix} 1 & \tau_a\left(\zeta^{s-1/2} - \rho\right) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tau_a\rho \\ 0 & 1 \end{bmatrix} \right)$$
$$=: \Upsilon_- \Upsilon_+ . \tag{4.8}$$

Here we are using the normalized sine function

$$\rho(\xi) = \frac{2}{\pi} \int_0^{\xi} \frac{\sin y}{y}\, dy$$

which reveals the following useful behavior at infinity

$$\rho(\xi) = \mathrm{sign}\xi + \mathcal{O}\left(|\xi|^{-1}\right) .$$

It is also important to observe that

$$\tau_{\pm a}\rho \in H_{\pm}^{\infty} , \tag{4.9}$$

i.e., $\tau_{\pm a}\rho$ are functions bounded and holomorphic in the upper/lower half-planes.

Theorem 4. *For $[s-1] = 0$ (i.e., $1/2 < s < 3/2$), the Wiener–Hopf operator $\mathcal{W}_{\Upsilon,\mathbb{R}_+}$ is invertible with its inverse being given by the formula*

$$\mathcal{W}_{\Upsilon,\mathbb{R}_+}^{-1} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon_+^{-1} \cdot \mathcal{F} l_0 r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon_-^{-1} \cdot \mathcal{F} . \tag{4.10}$$

Proof. The result is a direct consequence of the structure of Υ_- and Υ_+ (particularly because of (4.9)) and of the corresponding factorization (4.8), involving also the (generalized) L^2–factorization [18] of

$$\begin{bmatrix} \zeta^{s-1} & 0 \\ 0 & \zeta^{s-1} \end{bmatrix} = \begin{bmatrix} \lambda_-^{s-1} & 0 \\ 0 & \lambda_-^{s-1} \end{bmatrix} \begin{bmatrix} \lambda_+^{-s+1} & 0 \\ 0 & \lambda_+^{-s+1} \end{bmatrix} \tag{4.11}$$

due to $-1/2 < s - 1 < 1/2$. □

Corollary 5. *If $[s-1] = 0$, then the convolution type operator $\mathcal{W}_{t^{-1},\Sigma}$, defined in* (4.1)*, is bounded and invertible with its inverse being*

$$\mathcal{W}_{t^{-1},\Sigma}^{-1} = B_{11} ,$$

where B_{11} is the operator in the first block (with respect to the natural space decomposition) of the operator matrix

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = F_1^{-1} \begin{bmatrix} r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon_+^{-1} \cdot \mathcal{F} l_0 r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Upsilon_-^{-1} \cdot \mathcal{F} & 0 \\ 0 & I_Y \end{bmatrix} E_1^{-1} ,$$

and E_1 and F_1 are the same as in Theorem 3.

Proof. The result follows directly from Theorems 3 and 4 in conjunction with (4.4) and (4.10). □

4.2. FREDHOLM OPERATORS FOR THE WAVE DIFFRACTION PROBLEM

Taking advantage of the composition of the operators $\mathcal{W}_{t^{-1},\Sigma}$ and $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, we now arrive at the following result.

Corollary 6. *Let $[s-1]=0$. The operator $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$ is equivalent to*

$$\mathcal{W}_{\Phi,\Sigma,s-1/2} = r_{\mathbb{R}\to\Sigma}\mathcal{F}^{-1}\Phi\cdot\mathcal{F}l\colon\ H^{s-1/2}(\Sigma)\to H^{s-1/2}(\Sigma), \tag{4.12}$$

where $l\colon\ H^{s-1/2}(\Sigma)\to H^{s-1/2}(\mathbb{R})$ is an extension operator (whose particular choice does not affect the definition of $\mathcal{W}_{\Phi,\Sigma,s-1/2}$).

Proof. From Theorem 3 and the special form of Φ (see (3.4)), the following equivalent equations follow:

$$\begin{aligned}&\widetilde{\mathcal{W}}_{\Phi,\Sigma}f = g\\ &\mathcal{W}_{t^{-1},\Sigma}l_0\widetilde{\mathcal{W}}_{\Phi,\Sigma}f = \mathcal{W}_{t^{-1},\Sigma}l_0 g\\ &r_{\mathbb{R}\to\Sigma}\mathcal{F}^{-1}\Phi\cdot\mathcal{F}l\mathcal{W}_{t^{-1},\Sigma}f = \mathcal{W}_{t^{-1},\Sigma}l_0 g\end{aligned} \tag{4.13}$$

for $f\in\widetilde{H}^{s-3/2}(\Sigma)$ and $g\in r_{\mathbb{R}\to\Sigma}\widetilde{H}^{s-3/2}(\Sigma)$, where l is an operator of extension whose particular form does not affect the left hand-side of (4.13). In fact, equation (4.13), which involves the action of the operator $\mathcal{W}_{\Phi,\Sigma,s-1/2}$, can be written in the form

$$\varphi(\xi) - \frac{q}{2}\int_0^a \mathcal{F}^{-1}t^{-1}(\xi-x)\,\varphi(x)\,dx = \psi(\xi)\,, \qquad \xi\in\Sigma.$$

It is clear that the above equation is dependent on $\varphi\in H^{s-1/2}(\Sigma)$ and independent of the remaining part of the extension $l\varphi = l\mathcal{W}_{t^{-1},\Sigma}f\in H^{s-1/2}(\mathbb{R})$. □

Now, instead of studying $\mathcal{W}_{\Phi,\Sigma,s-1/2}$ directly, we consider the following image and domain extension of $\mathcal{W}_{\Phi,\Sigma,s-1/2}$

$$\mathcal{W}_{\Phi,\Sigma,0} = r_{\mathbb{R}\to\Sigma}\mathcal{F}^{-1}\Phi\cdot\mathcal{F}l_0\colon\ L^2(\Sigma)\to L^2(\Sigma), \tag{4.14}$$

which is a linear and bounded operator. From the Fredholm theory of Wiener–Hopf operators with continuous Fourier symbols (see e.g. [12]), it follows that for $[s-1]=0$ we have $\dim\operatorname{coker}\mathcal{W}_{\Phi,\Sigma,s-1/2} = \dim\operatorname{coker}\mathcal{W}_{\Phi,\Sigma,0}$ and $\dim\ker\mathcal{W}_{\Phi,\Sigma,s-1/2} = \dim\ker\mathcal{W}_{\Phi,\Sigma,0}$. This is also a consequence of the structure of Φ (which can be presented in terms of operators as the *identity plus additional smoothing*) and of the space embedding $H^{s-1/2}(\Sigma)\hookrightarrow L^2(\Sigma)$. Moreover, if we have the knowledge of $\mathcal{W}^{-1}_{\Phi,\Sigma,0}$ (the inverse of $\mathcal{W}_{\Phi,\Sigma,0}$), then a representation of the inverse of $\mathcal{W}_{\Phi,\Sigma,s-1/2}$ can be derived from $\mathcal{W}^{-1}_{\Phi,\Sigma,0}$ by use of the corresponding space restrictions.

Let us denote by $\mathcal{S}_k$ the set of all complex reactance numbers q which turn the continuous Fourier symbol Φ away from zero (cf. Fig. 2 for the wave number $k=5+i/2$ and the reactance number $q=10-10i\in\mathcal{S}_{5+i/2}$).

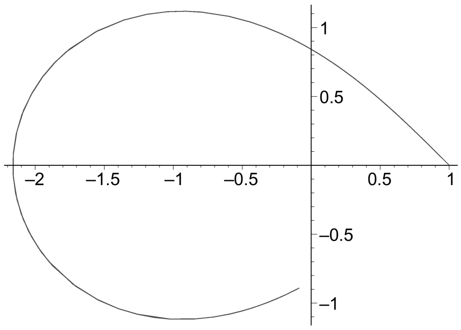

Fig. 2. The graph of Φ for $k = 5 + i/2$ and $q = 10 - 10i$

Theorem 7. *Let $q \in \mathcal{S}_k$. The convolution type operator $\mathcal{W}_{\Phi,\Sigma,0}$ is equivalent after extension to the following Fredholm operator with vanishing Fredholm index:*

$$\mathcal{W}_{\Psi,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Psi \cdot \mathcal{F}l_0 \colon \left[L^2(\mathbb{R}_+)\right]^2 \to [L^2(\mathbb{R}_+)]^2,$$

where

$$\Psi = \begin{bmatrix} \tau_{-a} & 0 \\ \Phi & \tau_a \end{bmatrix}$$

and $\tau_a(\xi) = \exp[i\xi a]$ *for* $\xi \in \mathbb{R}$.

Proof. Since $\Phi(\pm\infty) = 1$ and $\Phi(\xi) \neq 0$, for all $\xi \in \mathbb{R}$, the operator $\mathcal{W}_{\Psi,\mathbb{R}_+}$ has the Fredholm property (see, for example, [2, Theorem 4.1]). Further, $\mathcal{W}_{\Psi,\mathbb{R}_+}$ has zero Fredholm index since the continuous function on the real line Φ has no jumps at infinity and traces out a curve in the complex plane that leaves zero outside of its interior delimited domain (cf., e.g., the Fredholm index formula (2.14) in [13, Theorem 2.10]). The statement of the theorem now follows by arguing similarly as in the proof of Theorem 3. □

From Theorem 7 and the equivalence relations between the operators $L^{(s)}$, $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, $\mathcal{W}_{\Phi,\Sigma,s-1/2}$, and $\mathcal{W}_{\Phi,\Sigma,0}$ the following corollary follows directly now.

Corollary 8. *If $q \in \mathcal{S}_k$ and $1/2 < s < 3/2$, then the operators $L^{(s)}$, $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, $\mathcal{W}_{\Phi,\Sigma,s-1/2}$, and $\mathcal{W}_{\Phi,\Sigma,0}$ are Fredholm operators with zero Fredholm index.*

5. ANALYSIS OF THE FOURIER SYMBOL Ψ AND INVERTIBILITY OF RELATED OPERATORS

We are now in a position to prove the invertibility of all our main convolution type operators arising from the wave diffraction problem. In doing this, we need a new operator factorization scheme provided with the help of an auxiliary invertible Wiener–Hopf operator.

Lemma 9. *If $q \in \mathcal{S}_k$, then the Wiener–Hopf operator*

$$\mathcal{W}_{\Phi,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Phi\cdot\mathcal{F}l_0\colon\ L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)$$

is invertible with the inverse $\mathcal{W}^{-1}_{\Phi,\mathbb{R}_+} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Phi_+^{-1}\cdot\mathcal{F}l_0 r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\Phi_-^{-1}\cdot\mathcal{F}l_0$, *where*

$$\Phi_\pm = \exp\left\{\frac{1}{2}(I\pm S_\mathbb{R})\log\left(1-\frac{q}{2}t^{-1}\right)\right\} \tag{5.1}$$

and $S_\mathbb{R}$ is the Cauchy integral operator on $\mathbb{R}$:

$$\left(S_\mathbb{R} f\right)(\xi) = \frac{1}{\pi i}\int\limits_{\mathbb{R}}\frac{f(x)}{x-\xi}\,dx$$

(where the integral is understood in the principal value sense).

Proof. Note that for $q \in \mathcal{S}_k$ the Fourier symbol $\Phi = 1 - q\,t^{-1}/2$ is a non-vanishing continuous function on the real line with the same non-zero limits at $\pm\infty$. Thus, by virtue of the well-known Fredholm criterium for Wiener–Hopf operators with continuous Fourier symbols (see, e.g., [3, Theorem 2.15]), it follows that $\mathcal{W}_{\Phi,\mathbb{R}_+}$ is a Fredholm operator. Further, we observe that as ξ moves from $-\infty$ to $+\infty$, the point $\Phi(\xi)$ traces out a continuous oriented closed curve in $\mathbb{C}\setminus\{0\}$ starting from $\Phi(-\infty)=1$ till $\Phi(0)$ and then coming back again by the same way to $\Phi(+\infty)=1$ (cf. the example of Fig. 2). Therefore, the graph of Φ has zero windings around the origin (recall also the definition of the set $\mathcal{S}_k$), and it follows that $\mathcal{W}_{\Phi,\mathbb{R}_+}$ has a zero Fredholm index.

On the other hand, since $\mathcal{W}_{\Phi,\mathbb{R}_+}$ is a scalar Wiener–Hopf operator with a non-zero Fourier symbol, *Coburn Theorem* (see [3, Theorem 2.5]) can be applied to derive that $\ker\mathcal{W}_{\Phi,\mathbb{R}_+} = \{0\}$ or the range of $\mathcal{W}_{\Phi,\mathbb{R}_+}$ is dense and then closed in $L^2(\mathbb{R}_+)$. Consequently, $\mathcal{W}_{\Phi,\mathbb{R}_+}$ is invertible. By the factorization theory of continuous functions (see [18, Chapter III, §5]) we obtain representation (5.1) for the construction of the inverse operator. □

Theorem 10. *Let $q \in \mathcal{S}_k$. The Wiener–Hopf operator $\mathcal{W}_{\Psi,\mathbb{R}_+}\colon\ \left[L^2(\mathbb{R}_+)\right]^2 \to \left[L^2(\mathbb{R}_+)\right]^2$ is a bounded and invertible operator.*

Proof. The operator $\mathcal{W}_{\Psi,\mathbb{R}_+}$ is bounded because Ψ is an essentially bounded function.

We now prove the invertibility of $\mathcal{W}_{\Psi,\mathbb{R}_+}$. We start by factorizing $\mathcal{W}_{\Psi,\mathbb{R}_+}$ into the form

$$\mathcal{W}_{\Psi,\mathbb{R}_+} = \begin{bmatrix} I & r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}l_0\mathcal{W}^{-1}_{\Phi,\mathbb{R}_+} \\ 0 & I\end{bmatrix}\begin{bmatrix}\mathcal{C} & 0\\ 0 & \mathcal{W}_{\Phi,\mathbb{R}_+}\end{bmatrix}\times$$
$$\times\begin{bmatrix} 0 & -I \\ I & \mathcal{W}^{-1}_{\Phi,\mathbb{R}_+} r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}l_0\end{bmatrix} \tag{5.2}$$

where

$$\mathcal{C} = r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}l_0\,\mathcal{W}^{-1}_{\Phi,\mathbb{R}_+}r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}l_0\colon\; L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+).$$

We recall that the existence of the inverse of $\mathcal{W}_{\Phi,\mathbb{R}_+}$ is guaranteed by Lemma 9, since $q\in\mathcal{S}_k$.

From (5.2) and Lemma 9 it can be seen that $\mathcal{W}_{\Psi,\mathbb{R}_+}$ is invertible if and only if $\mathcal{C}$ is invertible. By (5.2) and Theorem 7, we conclude that $\mathcal{C}$ is a Fredholm operator with a vanishing Fredholm index. Thus, to derive the invertibility of $\mathcal{W}_{\Psi,\mathbb{R}_+}$ it is enough to show that $\mathcal{C}$ is an injective operator, that is, $\langle\mathcal{C}\varphi,\varphi\rangle_{L^2(\mathbb{R}_+)}=0$ implies $\varphi=0$. Now for $\varphi\in L^2(\mathbb{R}_+)$,

$$\langle\mathcal{C}\varphi,\varphi\rangle_{L^2(\mathbb{R}_+)} = \left\langle\mathcal{W}^{-1}_{\Phi,\mathbb{R}_+}r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}l_0\varphi, r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}l_0\varphi\right\rangle_{L^2(\mathbb{R}_+)}.$$

Thus it is enough to show that

$$\left\langle\mathcal{W}^{-1}_{\Phi,\mathbb{R}_+}\phi,\phi\right\rangle_{L^2(\mathbb{R}_+)}=0 \qquad \text{implies } \phi=0 \tag{5.3}$$

since the right a-shift operator $r_{\mathbb{R}\to\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F}l_0\colon\; L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)$ is obviously injective.

Let $\psi=\mathcal{W}^{-1}_{\Phi,\mathbb{R}_+}\phi$. Then (5.3) is equivalent to

$$\left\langle\psi,\mathcal{W}_{\Phi,\mathbb{R}_+}\psi\right\rangle_{L^2(\mathbb{R}_+)}=0 \qquad \text{implies } \psi=0. \tag{5.4}$$

Since $\Phi=1-q\,t^{-1}/2$ and

$$\left\langle\psi,\mathcal{W}_{\Phi,\mathbb{R}_+}\psi\right\rangle_{L^2(\mathbb{R}_+)} = \langle\mathcal{F}l_0\psi,\Phi\cdot\mathcal{F}l_0\psi\rangle_{L^2(\mathbb{R})},$$

it follows that (5.4) is true. □

From Theorem 10 and the constructed equivalence relations between the operators $L^{(s)}$, $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, $\mathcal{W}_{\Phi,\Sigma,s-1/2}$, and $\mathcal{W}_{\Phi,\Sigma,0}$, we immediately obtain the following assertion.

Corollary 11. *If $q\in\mathcal{S}_k$ and $1/2<s<3/2$, then the operators $L^{(s)}$, $\widetilde{\mathcal{W}}_{\Phi,\Sigma}$, $\mathcal{W}_{\Phi,\Sigma,s-1/2}$, and $\mathcal{W}_{\Phi,\Sigma,0}$ are all invertible.*

Bearing in mind Section 2, this corollary directly yields the following result on the well-posedness of the wave diffraction problem within a range of regularity orders of the Bessel potential spaces, and for a set of possible reactance numbers (dependent on the wave number).

Corollary 12. *If $q\in\mathcal{S}_k$ and $1/2<s<3/2$, then there is a unique solution $\varphi\in L^2(\mathbb{R}^2)$, with $\varphi_{|\mathbb{R}^2_\pm}\in H^s(\mathbb{R}^2_\pm)$, to the reactance diffraction problem* (2.1)–(2.3). *This solution is continuously dependent on the given data.*

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