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APPLICATION OF GREEN'S OPERATOR TO QUADRATIC VARIATIONAL PROBLEMS

Abstract. We use Green's function of a suitable boundary value problem to convert the variational problem with quadratic functional and linear constraints to the equivalent unconstrained extremal problem in some subspace of the space $\mathbf{L}_2$ of quadratically summable functions. We get the neccessary and sufficient criterion for unique solvability of the variational problem in terms of the spectrum of some integral Hilbert–Schmidt operator in $\mathbf{L}_2$ with symmetric kernel. The numerical technique is proposed to estimate this criterion. The results are demonstrated on examples: 1) a variational problem with deviating argument, and 2) the problem of the critical force for the vertical pillar with additional support point (the qualities of the pillar may vary discontinuously along the pillar's axis).

Keywords: quadratic variational problem, Sobolev space, boundary value problem, Hilbert space, Green's operator, Fredholm integral operator, spectrum.

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In applications we frequently have quadratic variational problems of the following type:

$$\begin{aligned} &\mathcal{I}(x) \overset{\text{def}}{=} \int_a^b \frac{1}{2}\left[(\psi x^{(n)})(t)\right]^2 + \frac{1}{2}\sum_{j=1}^{m}(T_{1j}x)(t)(T_{2j}x)(t) + (T_0x)(t)\,dt \to \inf, \\ &\ell^i x = \alpha^i, \quad i = 1,\dots,N, \quad N \geq n, \end{aligned} \tag{1}$$

where ℓ^i are linearly independent linear functionals, $\psi\colon\ \mathbf{L}_2 \to \mathbf{L}_2$ is a self-adjoint invertible operator, T_{ij} are operators acting to the space $\mathbf{L}_2$. The highest order derivative appears squared in one term of the integrand only, namely, $\frac{1}{2}\left[(\psi x^{(n)})(t)\right]^2$; other products may contain n-th derivative in at most one of the multipliers.

In such a case, it is natural to consider problem (1) in the Sobolev space $\mathbf{H}^n$ of functions $x\colon\ [a, b] \to \mathbb{R}$. The operators $T_{1j}\colon\ \mathbf{H}^{n-1} \to \mathbf{L}_2$ and $T_{2j}\colon\ \mathbf{H}^n \to \mathbf{L}_2$,

$i = 1, \ldots, m$, $T_0\colon\ \mathbf{H}^n \to \mathbf{L}_1$ are assumed to be continuous linear operators. Since the operators T_{1j} are defined on the space $\mathbf{H}^{n-1}$, expressions $(T_{1j}x)(t)$ do not contain the highest order derivative. We exploit this restriction in Lemma 1 below and its consequences.

For our purposes, it is most convenient to represent $\mathbf{H}^n$ $(n > 1)$ as the space of functions $x : [a, b] \to \mathbb{R}$ such that

$$x(t) = \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!} x^{(i)}(a) + \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} x^{(n)}(s)\, ds \tag{2}$$

and $x^{(n)} \in \mathbf{L}_2$ (see [6]). So the correspondence

$$x \leftrightarrow \Big(x(a), x'(a), \ldots, x^{(n-1)}(a);\ x^{(n)}(\cdot)\Big) \tag{3}$$

is the isomorphism $\mathbf{H}^n \simeq \mathbb{R}^n \times \mathbf{L}_2$, which is isometric if we put $\|x\| = \sum\limits_{i=0}^{n-1} \big|x^{(i)}(a)\big| + \big\|x^{(n)}\big\|_{\mathbf{L}_2}$. One can check that this norm is equivalent to the usually considered one, generated by the scalar product $(y, z) = \sum\limits_{i=0}^{n} \int_a^b y^{(i)}(t) z^{(i)}(t)\, dt$.

We use the notation $\mathbf{H}^0 = \mathbf{L}_2$.

To study and solve problem (1), we convert it to some unconstrained extremal problem in a subspace of $\mathbf{L}_2$.

1. ONE PROPERTY OF GREEN'S OPERATOR

Using a part of linear constraints as boundary value conditions, we construct a problem

$$\begin{aligned} &x^{(n)} = z, \quad x \in \mathbf{L}_2, \\ &\ell^i x = \alpha^i, \quad i = 1, \ldots, n. \end{aligned} \tag{4}$$

Hereinafter we assume that this problem has a unique solution $x \in \mathbf{H}^n$ for every $(z, \alpha) \in \mathbf{L}_2 \times \mathbb{R}^n$. Then, according to the Banach inverse operator theorem, x continuously depends on (z, α), and we have

$$x = \mathbf{G}z + X\alpha,$$

where $\mathbf{G}\colon\ \mathbf{L}_2 \to \mathbf{H}^n$ and $X\colon\ \mathbb{R}^n \to \mathbf{H}^n$ are continuous linear operators. $\mathbf{G}$ is Green's operator for problem (4). It is integral one [3, p. 79] with the kernel that we denote as $G(t, s)$.

We shall prove that $\mathbf{G}$ is the integral Hilbert–Schmidt operator, i.e., it has a quadratically summable kernel.

Lemma 1. *Let* $T\colon\ \mathbf{H}^{n-1} \to \mathbf{L}_2$ *be a continuous linear operator and* $\mathbf{G}\colon \mathbf{L}_2 \to \mathbf{H}^n$ *Green's operator for problem* (4)*. Then* $T\mathbf{G}\colon\ \mathbf{L}_2 \to \mathbf{L}_2$ *is an integral Hilbert–Schmidt operator with the kernel*

$$(t,\,s) \mapsto (Tg_s)(t), \tag{5}$$

where $g_s(t) = G(t,\,s)$*.*

Remark 1. *The analogous lemma for* $T\colon\ \mathbf{L}_2 \to \mathbf{L}_2$ *and arbitrary integral Hilbert–Schmidt operator* $\mathbf{G}$ *is proved in* [5]*. Using adjoints, one can demonstrate that, under the assumptions of* [5]*, the operator* $\mathbf{G}T\colon\ spL \to \mathbf{L}_2$ *is an integral Hilbert–Schmidt one too; its kernel is the function* $(t,\,s) \mapsto (T^*g^t)(s)$*, where* $g^t(s) = G(t,\,s)$*.*

Proof. If $x = \mathbf{G}z$, then x satisfies the boundary value problem

$$\begin{aligned} x^{(n)} &= z, \quad x \in \mathbf{L}_2,\\ \ell^i x &= 0, \quad i = 1,\,\dots,\,n. \end{aligned} \tag{6}$$

Due to isomorphism (3) and Riesz's theorem, we may rewrite the boundary value conditions of problem (6) in the form

$$\Psi\,\mathrm{col}\Big(x(a),\dots,x^{(n-1)}(a)\Big) + \int_a^b \mathrm{col}\Big(\phi^0(s),\dots,\phi^{n-1}(s)\Big)x^{(n)}(s)\,ds = 0,$$

where $\Psi = (\psi^i_j)^{n-1}_{i,\,j=0}$ is a matrix of reals, $\phi^0,\,\dots,\,\phi^{n-1},\,x^{(n)} \in \mathbf{L}_2$. One may argue that for unique solvability of problem (6) it is necessary that $\det\Psi \neq 0$. Accordingly, if we change notation, the system of boundary value conditions looks as follows:

$$x^{(i)}(a) = \int_a^b \phi^i(s)z(s)\,ds, \quad i = 0,\dots,n-1. \tag{7}$$

Now apply isomorphism (3) considered for the space $\mathbf{H}^{n-1}$:

$$(Tx)(t) = x(a)\tau_0(t) + \dots + x^{(n-2)}(a)\tau_{n-2}(t) + T_{n-1}x^{(n-1)}, \tag{8}$$

where $\tau_0,\dots,\tau_{n-2} \in \mathbf{L}_2$, $T_{n-1}\colon\ \mathbf{L}_2 \to \mathbf{L}_2$ is some continuous linear operator. Let $\tau_{n-1} = T_{n-1}\mathbf{1}$, where $\mathbf{1}(t) \equiv 1$, and $(\mathbf{C}z)(t) = \int_a^t z(s)\,ds$. Then, according to (7),

$$(T\mathbf{G}z)(t) = \int_a^b \sum_{i=0}^{n-1} \tau_i(t)\phi^i(s)z(s)\,ds + (T_{n-1}\mathbf{C}z)(t). \tag{9}$$

As mentioned in Remark 1, the operator $T_{n-1}\mathbf{C}$ has also a quadratically summable kernel.

To show equality (5), let first $T\colon \mathbf{H}^{n-1} \to \mathbf{L}_2$ be the embedding. If we replace n by $n-1$ in equality (2) and compare it with (9), we get

$$(\mathbf{G}z)(t) = (T\mathbf{G}z)(t) =$$
$$= \int_a^b \sum_{i=0}^{n-1} \psi_i(t)\phi^i(s)z(s)\,ds + \int_a^t \frac{(t-s)^{n-2}}{(n-2)!} \int_a^s z(\theta)\,d\theta$$
$$= \int_a^b \sum_{i=0}^{n-1} \psi_i(t)\phi^i(s)z(s)\,ds + \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} z(s)\,ds,$$

where $\psi_i(t) \overset{\text{def}}{=} \frac{(t-a)^i}{i!}$, $i = 0, \ldots, n-1$.

This argument is valid for $n > 1$; if $n = 1$, then the last formula obviously holds too.

Note that the functions

$$g_s(t) = \sum_{i=0}^{n-1} \psi_i(t)\phi^i(s) + u_s(t),$$
$$u_s(t) \overset{\text{def}}{=} \begin{cases} \frac{(t-s)^{n-1}}{(n-1)!}, & \text{if } t > s, \\ 0, & \text{if } t \le s, \end{cases}$$

belong to the space $\mathbf{H}^{n-1}$.

Now consider the general case of the operator $T\colon \mathbf{H}^{n-1} \to \mathbf{L}_2$ expressed by (8). For $\psi_k(t) = \frac{(t-a)^k}{k!}$ we have $\psi_k^{(i)}(a) = \delta_k^i$, where δ_k^i is the Kronecker delta symbol, $\psi_k^{(n-1)} \equiv \delta_k^{n-1}$; so, due to (8),

$$T\psi_k = \tau_k, \quad k = 0, \ldots, n-2,$$
$$T\psi_{n-1} = T_{n-1}\mathbf{1} = \tau_{n-1}.$$

For $u_s(\cdot)$, equalities $u_s^{(i)}(a) = 0$ and

$$u_s^{(n-1)}(t) = c_s(t) \overset{\text{def}}{=} \begin{cases} 1, & t > s, \\ 0, & t \le s, \end{cases}$$

are valid; then from (8) we get

$$(Tu_s)(t) = (T_{n-1}c_s)(t).$$

But the right-hand side function is just the kernel of the operator $T_{n-1}\mathbf{C}$ (see Remark 1).

Thus, due to (9), the function

$$(Tg_s)(t) = \sum_{i=0}^{n-1} (T\psi_k)(t)\phi^i(s) + (Tu_s)(t) = \sum_{i=0}^{n-1} \tau_k(t)\phi^i(s) + (T_{n-1}c_s)(t)$$

is the kernel of the operator $T\mathbf{G}$. □

2. FIRST STEP OF CONVERSION

It will be done following the general idea represented in [4, 3] and some later works.

The ***model boundary value problem***

$$\psi x^{(n)} = z,$$
$$\ell^i x = \alpha^i, \quad i = 1, \dots, n,$$

has the unique solution $x \in \mathbf{D}$,

$$x = \mathbf{G}\psi^{-1} z + X\alpha^{[n]}, \tag{10}$$

where $\alpha^{[n]} \overset{\text{def}}{=} \operatorname{col}\big(\alpha_1, \dots, \alpha^n\big)$.

So there is one-to-one correspondence

$$\mathbf{D}_\alpha^{[n]} \overset{\text{def}}{=} \big\{x \in \mathbf{D}\colon\ \ell^i x = \alpha^i, \quad i = 1, \dots, n\big\} \leftrightarrow \mathbf{L}_2.$$

Substitution (10) converts problem (1) to the following equivalent extremal problem in the space $\mathbf{L}_2$:

$$\begin{gathered}\mathcal{J}(z) \overset{\text{def}}{=} \mathcal{I}(\mathbf{G}\psi^{-1} z + X\alpha^{[n]}) - \mathcal{I}(X\alpha^{[n]}) \to \inf,\\ \langle l^i,\, z\rangle = \beta^i \overset{\text{def}}{=} \alpha^i - \ell^i X\alpha^{[n]}, \quad i = n+1, n+2 \dots, N.\end{gathered} \tag{11}$$

Here $l^i \in \mathbf{L}_2$ are such that $\ell^i \mathbf{G}\psi^{-1} z = \big\langle l^i, z\big\rangle$, where $\langle \cdot,\cdot \rangle$ denotes the usual scalar product in the space $\mathbf{L}_2$.

Calculating $\mathcal{J}(z)$, we get

$$\mathcal{J}(z) = \frac{1}{2}\langle z - \mathbf{K}z,\, z\rangle - \langle \theta,\, z\rangle\,,$$

where

$$\begin{aligned}\mathbf{K} &= -\psi^{-1} \sum_{j=1}^{m} \big[(T_{2j}\mathbf{G})^* T_{1j}\mathbf{G} + (T_{1j}\mathbf{G})^* T_{2j}\mathbf{G}\big]\psi^{-1},\\ \theta &= \frac{1}{2}\psi^{-1} \sum_{j=1}^{m} \big[(T_{2j}\mathbf{G})^* T_{1j} + (T_{1j}\mathbf{G})^* T_{2j}\big] X\alpha^{[n]} + \psi^{-1}\big(T_0\mathbf{G}\big)^* \mathbf{1}.\end{aligned} \tag{12}$$

Theorem 1. *The operator* $\mathbf{K}\colon\ \mathbf{L}_2 \to \mathbf{L}_2$ *is a self-adjoint integral Hilbert–Schmidt one.*

Proof. According to Lemma 1, each operator $T_{1i}\mathbf{G} : \mathbf{L}_2 \to \mathbf{L}_2$ is an integral Hilbert–Schmidt one. So the adjoint $(T_{1i}\mathbf{G})^*$ is too. Then we apply Remark 1 four times. □

3. SECOND STEP OF CONVERSION

Let Z_0 be the linear hull of the vectors l^i, $i = n+1, \dots, N$, and Z_1 its orthogonal complement. Denote by $\mathbf{P}_k$ the orthogonal projector onto Z_k. So $\mathbf{P}_0$ is the integral operator with the degenerate kernel $P(t,\, s) = \sum_{i=n+1}^{N} \sum_{j=n+1}^{N} \gamma_{ij}\, l^i(t)\, l^j(s)$, where $(\gamma_{ij})_{i,j=n+1}^{N}$ is the inverse of the Gramian matrix $\big(\langle l^i,\, l^j \rangle\big)_{i,j=n+1}^{N}$; $\mathbf{P}_1 = I - \mathbf{P}_0$ is self-adjoint Fredholm operator.

Consider the problem

$$\begin{aligned} \mathbf{P}_1 z = z_1, &\quad z_1 \in Z_1, \\ \langle l^i,\, z \rangle = \beta^i, &\quad i = n+1,\, n+2\, \dots,\, N. \end{aligned} \tag{13}$$

Since $\mathbf{L}_2 \simeq Z_1 \times \mathbb{R}^{N-n}$, this problem looks like a boundary value problem. The theory of *abstract boundary value problems* is developed in the book [3].

The solution of problem (13) exists uniquely:

$$z = z_1 + z_0, \quad z_0 \overset{\text{def}}{=} \sum_{i=n+1}^{N} \sum_{j=n+1}^{N} \gamma_{ij} \beta^j l^i \in Z_0$$

(we see that Green's operator of such an abstract boundary value problem is just embedding). This substitution converts problem (11) to the following extremal problem without constraints in the Hilbert space Z_1:

$$\mathcal{F}(z_1) \overset{\text{def}}{=} \frac{1}{2} \langle (I - \mathbf{P}_1 \mathbf{K}) z_1,\, z_1 \rangle - \langle \mathbf{P}_1 \theta - \mathbf{P}_1 (I - \mathbf{K}) z_0,\, z_1 \rangle \to \min\,. \tag{14}$$

This problem is equivalent to (1) in the following sense. The constraint region of points $x \in \mathbf{H}^n$ for problem (1) is in one-to-one correspondence with the subspace Z_1; the values of functionals differ by a constant; so minimum points of the problems correspond too.

The following theorem is quite clear.

Theorem 2 ([3, 1]). *Problem* (14) *has a minimum point* $\hat{z}_1 \in Z_1$ *if and only if* $\hat{z}_1$ *satisfies the equation*

$$(I - \mathbf{P}_1 \mathbf{K}) z_1 = \mathbf{P}_1 \theta - \mathbf{P}_1 (I - \mathbf{K}) z_0 \tag{15}$$

and the operator $I - \mathbf{P}_1 \mathbf{K}\colon\ Z_1 \to Z_1$ *is positively definite.*

In such a case, the uniqueness of a minimum point is equivalent to each of the following conditions:

a) *equation* (15) *has a unique solution,*

b) *the operator* $I - \mathbf{P}_1 \mathbf{K}$ *is strictly positively definite on the subspace* Z_1.

Let $r_+(A)$ be the largest positive eigenvalue of the operator A, if it exists, and 0 otherwise.

Theorem 3. *For a self-adjoint completely continuous operator A in a Hilbert space, the following statements are equivalent:*

a) *for any $k \in [0, 1]$, the operator $I - kA$ is invertible;*

b) *the operator $I - A$ is strictly positively definite;*

c) $r_+(A) < 1$. *If $r_+(A) = 1$, then $I - A$ is a positively definite operator.*

Proof. Proof is based on the spectral theorem of Hilbert and Schmidt (see [1, theorem 3.3]). □

Note that the operator $\mathbf{P}_1\mathbf{K}\colon\ Z_1 \to Z_1$ has the same nonzero spectrum that the operator $\mathbf{P}_1\mathbf{K}\mathbf{P}_1\colon\ \mathbf{L}_2 \to \mathbf{L}_2$ has. And the latter one is an integral Hilbert–Schmidt operator with symmetric kernel. So we have to calculate $r_+(\mathbf{P}_1\mathbf{K}\mathbf{P}_1)$.

4. A NUMERICAL METHOD

If the spectral radius $r(A)$ is an eigenvalue, then $r_+(A) = r(A) > 0$. To find $r(A)$, the following numerical technique may be used.

Let A be a self-ajoint completely continuous operator in the Hilbert space. Choose a starting function $y_0 \in \mathbf{L}_2$ and let $y_i = Ay_{i-1}$ for $i = 1, 2, \ldots$. Let E be the orthogonal complement to all eigenfunctions corresponding to the eigenvalues λ_j with $|\lambda_j| = r(A)$.

Theorem 4 ([1, Theorem 3.8]). *If $y_0 \notin E$, then the sequence of numbers $\frac{\|y_i\|}{\|y_{i-1}\|}$ tends to $r(A)$.*

Remark 2. *The sequence has the rate of convergence of a geometric progression with the ratio $\frac{\|A_1\|}{r(A)}$, where A_1 is the restriction of the operator A to the subspace E. In fact, the proof of the theorem is valid for any self-adjoint operator A such that either $r(A)$ or $-r(A)$ is an eigenvalue and $\|A_1\| < r(A)$.*

We may use the computer symbolic algebra to perform these iterations. Usually, calculating, we may determine the sign of the unique eigenvalue λ_1, such that $|\lambda_1| = r(A)$.

If $\lambda_1 > 0$, then $r_+(A) = r(A)$.

If $\lambda_1 < 0$ but $r_+(A) > 0$, then $r_+(A) = r(A + \lambda I) - \lambda$, where $\lambda > 0$ is sufficiently large; $r(A + \lambda I)$ is calculated with the described technique.

5. PARAMETRIC PROBLEMS

Now suppose that the objective functional in the problem (1) containes a positive parameter p:

$$\mathcal{I}_p(x) = \int_a^b \frac{1}{2}[(\psi x^{(n)})(t)]^2 + \frac{p}{2}\sum_{j=1}^{m}(T_{1j}x)(t)(T_{2j}x)(t) + (T_0x)(t)\,dt \to \inf, \qquad \ell^i x = \alpha^i, \quad i = 1, \ldots, N, \tag{16}$$

The unique solvability of the variational problem depends on the quadratic part of the functional $\mathcal{I}_p$ only. So we assume that $T_0 = 0$ and all $\alpha^i = 0$.

Problem (14) in the subspace Z_1 takes the form

$$\mathcal{F}_p(z_1) \stackrel{\text{def}}{=} \frac{1}{2}\,\langle (I - p\mathbf{P}_1\mathbf{K})z_1,\, z_1\rangle \to \min, \tag{17}$$

where the operator $\mathbf{K}$ is given by equality (12).

The ***critical value*** p_{cr} of the parameter is defined as follows: p_{cr} is the largest value such that for all $p \in (0, p_{\text{cr}})$ the considered extremal problems have unique minimum points, specifically, $z_1 = 0$ and $x = 0$. The following assertions are valid for both problems (17) and (16).

Theorem 5. *We have*

$$p_{cr} = \frac{1}{r_+(\mathbf{P}_1\mathbf{K}\mathbf{P}_1)}, \tag{18}$$

where we suppose that $p_{cr} = +\infty$ in the case of $r_+(\mathbf{P}_1\mathbf{K}\mathbf{P}_1) = 0$.

If $p = p_{cr}$, $T_0 = 0$, and $\alpha^i = 0$ for $i = 1, \ldots, N$, then there is a nontrivial linear variety of minimum points.

If $p > p_{cr}$, then no minimum point exists.

Proof. Equality (18) is a direct consequence of Theorems 2 and 3.

Let $\lambda \stackrel{\text{def}}{=} r_+(\mathbf{P}_1\mathbf{K}\mathbf{P}_1)$ be different from zero and u be the corresponding eigenvector of the operator $\mathbf{P}_1\mathbf{K}\mathbf{P}_1$; then $u \in Z_1$ and $\mathcal{F}_p(u) = (1 - p\lambda)\,\|u\|^2$.

Let $T_0 = 0$ and $\alpha^i = 0$, $i = 1, \ldots, N$. If $p = p_{\text{cr}}$, then $p\lambda = 1$ and for every $k \in \mathbb{R}$ the equality $\mathcal{F}_p(ku) = (1-p\lambda)k^2\,\|u\|^2 = 0$ holds. Thus, all the points ku are minimum points for problem (17). Accordingly, problem (16) has minimum points $k\mathbf{G}u$.

If $p > p_{\text{cr}}$, then $\mathcal{F}_p(u) = (1 - p\lambda)\,\|u\|^2 < 0$. Problem (17) and, consequently, problem (16) have no minimum. □

6. EXAMPLE OF THE PROBLEM WITH DEVIATING ARGUMENT

We consider the problem

$$\begin{aligned} &\frac{1}{2}\int_0^b \ddot{x}^2(t) - p(t)x(h(t))x(g(t))\,dt \to \inf, \\ &x(t) = \phi(t), \quad \text{if } t \notin [0,\, b], \\ &x(0) = \alpha^1, \quad x(b) = \alpha^2. \end{aligned} \tag{19}$$

Here $p \in \mathbf{L}_2$, $p(t) \geq 0$, h and g are measurable functions, such that both the functions

$$\phi^h(t) = \begin{cases} 0, & \text{if } \ t \in [0,\, b], \quad h(t) \in [0,\, b], \\ \phi(h(t)), & \text{if } \ t \in [0,\, b], \quad h(t) \notin [0,\, b], \end{cases}$$

and analogously defined $\phi^g(t)$ belong to the space $\mathbf{L}_2$. We do not suppose that desired solution x must give a continuous extension of the function ϕ to the segment $[0,\, b]$. If it is required and if ϕ is continuous in the exterior of the interval $(0,\, b)$, then one can put $\alpha = \phi(0)$, $\beta = \phi(b)$.

This problem is naturally considered in the space $\mathbf{H}^2$.

Define the operator $S_h\colon\ \mathbf{H}^2 \to \mathbf{L}_2$ by the equality

$$(S_h x)(t) = \begin{cases} x(h(t)), & \text{if } \ h(t) \in [0,\, b], \\ 0, & \text{if } \ h(t) \notin [0,\, b], \end{cases}$$

then $x(h(t)) = (S_h x)(t) + \phi^h(t)$. Substituting this and eliminating the constant term $\frac{1}{2}\int_0^b p(t)\phi^h(t)\phi^g(t)\,dt$, we rewrite the problem in the following form:

$$\frac{1}{2}\int_0^b \ddot{x}^2(t) - p(t)(S_h x)(t)(S_g x)(t) - p(t)\left[\phi^h(t)(S_g x)(t) + \phi^g(t)(S_h x)(t)\right]\,dt \to \min,$$

$$x(0) = \alpha^1, \quad x(b) = \alpha^2.$$

The model problem

$$\ddot{x}(t) = z(t), \quad x(0) = x(b) = 0,$$

has the unique solution $x \in \mathbf{H}^2$:

$$x(t) = (\mathbf{W}z)(t) = \int_0^b W(t,\, s)z(s)\,ds,$$

where

$$W(t,\,s) = -\frac{1}{b}\cdot\begin{cases} s(b-t), & \text{if } \ 0 \le s \le t \le b,\\ t(b-s), & \text{if } \ 0 \le t < s \le b,\\ 0 & \text{otherwise.}\end{cases} \tag{20}$$

When it is useful, we may consider this operator as acting from $\mathbf{L}_2$ to $\mathbf{L}_2$. Then it is a self-adjoint one, because its kernel is symmetric.

Using this, we calculate the kernel of the operator $\mathbf{K}$:

$$K(t,\,s) = \frac{1}{2}\int_0^b p(\theta)\big[W\big(h(\theta),\,t\big)W\big(g(\theta),\,s\big) + W\big(g(\theta),\,t\big)W\big(h(\theta),\,s\big)\big]\,d\theta.$$

Denote

$$\sigma_h(t) = \begin{cases} 1, & \text{if } \ h(t) \in [0,\,b],\\ 0, & \text{if } \ h(t) \notin [0,\,b],\end{cases}$$

then $|W(h(s),\,t)| \le \frac{b}{4}\sigma_h(s)$ for all $(t,\,s)$.

Obviously,

$$r_+(\mathbf{K}) \le \|\mathbf{K}\| \le b \sup_{t,\,s\in[0,\,b]} |K(t,\,s)| \le \frac{b^3}{16}\int_0^b p(\theta)\sigma_h(\theta)\sigma_g(\theta)\,d\theta.$$

Therefore, due to Theorem 3, *in order for problem* (19) *to have a unique minimum point, it is sufficient that*

$$\int_0^b p(s)\sigma_h(s)\sigma_g(s)\,ds < \frac{16}{b^3}. \tag{21}$$

In the situation considered, the projector $\mathbf{P}_1 = I$. Equation (15) is a Fredholm type integral equation of second kind. The method of successive iterations converges if (21) holds.

Now consider the problem with $p = \text{const}$. The critical value $p_{\text{cr}} = \frac{1}{r_+(\mathbf{K}_1)}$, where $\mathbf{K}_1$ is the integral operator with the kernel

$$K_1(t,\,s) = \frac{1}{2}\int_0^b \big[W\big(h(\theta),\,t\big)W\big(g(\theta),\,s\big) + W\big(g(\theta),\,t\big)W\big(h(\theta),\,s\big)\big]\,d\theta.$$

Example 1. Let $b = 1$, $h(t) = g(t) = t - \tau$.

Then the symmetric kernel K_1 takes the form

$$K_1(t,\,s) = \begin{cases} \frac{1}{6}\big(ts^3 + t^3 s + 2ts\,(1-\tau^3) - s^3 - 3t^2 s\big), & \text{if } \ 0 < s < t < 1-\tau,\\ \frac{1}{6}(1-t)\big(-s^3 + s(1 - 3\tau^2 + 2\tau^3)\big), & \text{if } \ 0 < s < 1-\tau < t < 1,\\ \frac{1}{3}(1-t)(1-s)(1-\tau)^3, & \text{if } \ 1-\tau < s < t < 1.\end{cases}$$

The calculation described in Theorem 4 was performed in the class of piecewise polynomials with rational coefficients. So the only error is made by the truncation of iterations.

First, let $\tau = 0$. We put $y_0(t) \equiv 1$ and calculate $y_{i+1} = \mathbf{K}_1 y_i$ for $i \geq 1$. Then $\frac{\|y_i\|}{\|y_{i-1}\|} \to r(\mathbf{K}_1)$ (see Theorem 4). The Table 1 demonstrates a good rate of convergence.

Table 1

i	$\dfrac{\|y_i\|}{\|y_{i-1}\|}$	Estimate p_{cr}
1	0.009242710078	108.193375267
2	0.010265894796	97.409920897
3	0.010265982241	97.409091160
4	0.010265982255	97.409091034
5	0.010265982255	97.409091034

The corresponding eigenvalue was found to be positive; so $p_{\text{cr}} = \frac{1}{r_+(\mathbf{K}_1} = \frac{1}{r(\mathbf{K}_1)} \approx$ 97.409091034. This value coincides, in all visible digits, with the value $p_{\text{cr}} = \pi^4$, which may be obtained by classic methods of variational calculus.

There are some other values of p_{cr} (Table 2).

Table 2

τ	0.	0.05	0.1	0.2
p_{cr}	97.409	97.489	98.040	102.314
τ	0.3	0.4	0.5	0.6
p_{cr}	113.839	138.069	187.378	298.151

7. THE STRESSED PILLAR WITH THREE SUPPORT POINTS

Consider the problem of the critical force for the pillar.

The vertical elastic shank is stressed by the vertical contracting force P (Fig. 1). If $P < P_{\text{cr}}$, then the pillar is stable. If P is greater, the pillar bends. According to the Lagrange variational principle, the shape of the stable pillar is a graph of unique solution of the problem

$$\mathcal{I}_P(x) = \frac{1}{2} \int_0^b \psi^2(t)\ddot{x}^2(t) - P\dot{x}^2(t)\, dt \to \inf,$$

$$x(0) = x(b) = 0,$$

$$x(c) = 0,$$

where $\psi^2(t) = E(t)I(t)$, $E(t)$ is the Young modulus of the material and $I(t)$ is the geometric moment of inertia of the shank section at the level $t \in [0, b]$.

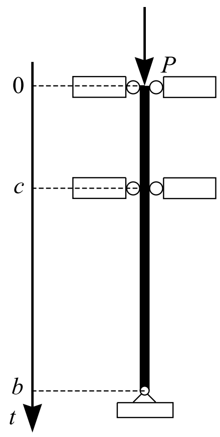

Fig. 1

We assume that ψ is measurable, $\operatorname*{ess\,sup}_{t\in[0,\,b]}\psi(t) < \infty$, and $\operatorname*{ess\,inf}_{t\in[0,\,b]}\psi(t) > 0$.

To construct a W-substitution, consider the model boundary value problem

$$\psi\ddot{x} = z, \quad (z \in \mathbf{L}_2)$$
$$x(0) = x(b) = 0.$$

It has the unique solution

$$x(t) = \left(\mathbf{W}\frac{1}{\psi}z\right)(t) \stackrel{\text{def}}{=} \int\limits_0^b W(t,\,s)\frac{1}{\psi(s)}z(s)\,ds,$$

where function W is defined by (20).

Using this substitution, we get the functional $\mathcal{J}_P$ in $\mathbf{L}_2$:

$$\mathcal{J}_P(z) = \langle z,\,z\rangle - P\left\langle \frac{d}{dt}\mathbf{W}\frac{1}{\psi}z,\, \frac{d}{dt}\mathbf{W}\frac{1}{\psi}z\right\rangle.$$

By direct calculation of kernels we get $\left(\frac{d}{dt}\mathbf{W}\right)^*\left(\frac{d}{dt}\mathbf{W}\right) = -\mathbf{W}$; so

$$\mathcal{J}_P(z) = \langle z,\,z\rangle + P\left\langle \frac{1}{\psi}\mathbf{W}\frac{1}{\psi}z,\, z\right\rangle.$$

Let $w(s) = W(c,\,s)$, then problem (11) in $\mathbf{L}_2$ for our example is the following one:

$$\mathcal{J}_P(z) \to \min,$$
$$\left\langle \frac{1}{\psi}w,\, z\right\rangle = 0.$$

So

$$P_{\text{cr}} = \left[r_+ \left(-\mathbf{P}_1 \frac{1}{\psi} \mathbf{W} \frac{1}{\psi} \mathbf{P}_1 \right) \right]^{-1},$$

where

$$\mathbf{P}_1 z = z - \left\| \frac{1}{\psi} w \right\|^{-2} \left\langle \frac{1}{\psi} w,\, z \right\rangle \frac{1}{\psi} w.$$

Example 2. Let $b = 3$, $c = 1$, and

$$\psi(t) = \begin{cases} 1 & \text{for} \quad t \in [0,\, 1), \\ 2 & \text{for} \quad t \in [1,\, 2), \\ 1 & \text{for} \quad t \in [2,\, 3]. \end{cases}$$

The numerical estimation of $r_+ \left(-\mathbf{P}_1 \frac{1}{\psi} \mathbf{W} \frac{1}{\psi} \mathbf{P}_1 \right) \approx 0.223944$, obtained using piecewise polynomials with rational coefficients, leads to the value $P_{\text{cr}} \approx 4.46539$. In the same way we have got approximatively an eigenfunction that represents the shape of the pillar at the time when it loses the stability (Fig. 2).

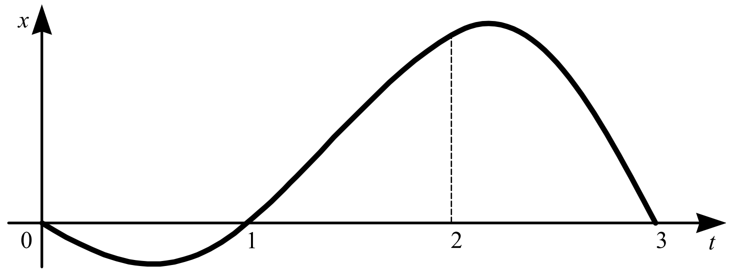

Fig. 2

For the case of no additional support, the exact value for this pillar is obtained in paper [2]: $P_{\text{cr}} = \arccos^2(\frac{\sqrt{13}-2}{9}) \approx 1.936110$.

Some other examples of applied problems solved by the considered method may be found in [8, 7, 1].

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