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SOME ANALYTICAL PROPERTIES OF DISSOLVING OPERATORS RELATED WITH THE CAUCHY PROBLEM FOR A CLASS OF NONAUTONOMOUS PARTIAL DIFFERENTIAL EQUATIONS. PART 1

Abstract. The analytical properties of dissolving operators related with the Cauchy problem for a class of nonautonomous partial differential equations in Hilbert spaces are studied using theory of bi-linear forms in respectvely rigged Hilbert spaces triples. Theorems specifying the existence of a dissolving operator for a class of adiabatically perturbed nonautonomous partial differential equations are stated. Some applications of the results obtained are discussed.

Keywords: dissolving operators, bilinear forms, Cauchy problem, semigroups, evolution equations.

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1. THE PROBLEM SETTING

Consider a real separable Hilbert space $\mathcal{H}$ and endow it with a Hilbert-Schmidt type rigged Gelfand triple $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ with the corresponding norms $\|\cdot\|_0$ and $\|\cdot\|_\pm$.

Assume that in $\mathcal{H}_+$ there is given a family of bilinear forms $\mathcal{H}_+ \times \mathcal{H} \to C^1([0,T];\mathbb{R})$ with the following properties:

i) $|h(t)(\psi,\varphi)| \leq c\|\varphi\|_+\|\psi\|_+$ for all $t \in [0,T]$ and some $c > 0$;

ii) the bilinear form $h(t)$, $t \in 0,T]$, is symmetric and such that for $\psi,\varphi \in \mathcal{H}_+$ and some $\alpha,\lambda > 0$ $\ h(t)(\varphi,\varphi) \geq \alpha\|\varphi\|_+^2 - \lambda\|\varphi\|^2$ uniformly in $t \in [0,T]$.

Conditions i) and ii) are sufficient for defining a family of bounded symmetric partial differential operators $H(t) : \mathcal{H}_+ \to \mathcal{H}_+$ such that for any $\psi,\varphi \in H_+$,

$h(t)(\psi,\varphi) = (H(t)\psi,\varphi)$ for all $t \in [0,T]$, where $H(t) : \sum_{|\beta|=0}^{n} \frac{\partial^{|\alpha|}}{\partial x^\alpha} a_\alpha(x;t) \frac{\partial |\alpha|}{\partial x^\alpha}$, $x \in \mathbb{R}^m$, is some symmetric partial differential expression of order $2n \in \mathbb{Z}_+$.

The family of bounded partial differential symmetric operators $H(t) : \mathcal{H}_+ \to \mathcal{H}_+$, $t \in [0,T]$, defined above is characterized by the important property: they are simultaneously unbounded symmetric operators in the Hilbert space $\mathcal{H}$ with dense domains dom $H(t) \subset \mathcal{H}$ depending on the evolution parameter $t \in [0,T]$. Moreover, this family of partial differential operators, being symmetric, possesses the set of corresponding self-adjoint extensions, which allow us to effectively study solutions to the evolution equations

$$\partial\psi/\partial t = -H(t)\psi \tag{1.0}$$

in the Hilbert space $\mathcal{H} \ni \psi$, by means of the related spectral properties of the corresponding self-adjoint operators $H(t)$, $t \in [0,T]$. The following theorem, which is a slight generalization of the Lions–Magenes theorem [1], is of importance for our further considerations.

Theorem 1.1. *There exists a self-adjoint family of operators $U(t,s)$, $t \geq s \in [0,T]$, acting in $\mathcal{H}$, with the following properties:*

1) $U(t,\tau) \cdot U(\tau,s) = U(t,s)$, $U(t,t) = 1$ *for any* $t \geq \tau \geq s \in [0,T]$;
2) $U(t,s) : H_+ \to H_+$ *for any* $t \geq s \in [0,T]$, *and* $\|U(t,s)\|_{++} \leq A$ *for some* $A > 0$, *uniformly in* $t,s \in [0,T]$;
3) *for any* $\psi \in \mathcal{H}_+$ *the function* $[0,T] \ni t \to U(t,s)\psi \in \mathcal{H}_+$ *is continuously differentiable in* $\mathcal{H}$;
4) *for any* $\psi \in \mathcal{H}_+$ *and* $t \geq s \in [0,T]$ *the following identity*
$$\frac{\partial}{\partial t}(U(t,s)\psi) = -H(t)U(t,s)\psi \tag{1.1}$$
holds in $\mathcal{H}_-$.

Take now any linear operator $K : \mathcal{H}_+ \to \mathcal{H}_+$ and denote by $K^+ : \mathcal{H}_- \to \mathcal{H}_-$ its adjoint in $\mathcal{H}_-$ and by $K^* : \mathcal{H} \to \mathcal{H}$ its usual adjoint to $K : \mathcal{H} \to \mathcal{H}$. Then, after [2], we denoted the operator $\hat{K} := (K^*)^+$ for which the following properties hold: $\hat{K}|_{\mathcal{H}_+} = K$ and $\|\hat{K}\|_{-,-} = \|K\|_{++}$. Making use of these notations, one can formulate the following Duhamel type theorem.

Theorem 1.2. *Let two families of operators $H_1(t)$ and $H_2(t)$, $t \in [0,T]$, be such that the related forms $h_1(t)(\psi,\phi) := (H_i(t)\psi,\phi)$, $i = \overline{1,2}$, satisfy all of the conditions i) and ii) above with the same constants. Then for any $\varphi \in \mathcal{H}_+$ the following relationship holds in H_- :*

$$U_1(t,s)\psi - U_2(t,s)\psi = -\int_s^t \hat{U}_1(t,\tau)[H_1(\tau) - H_2(\tau)]U_2(t,s)\psi d\tau, \tag{1.2}$$

where $U_k(t,s)$, $k=\overline{1,2}$, for $t \geq s \in [0,T]$ are the corresponding dissolving operators, satisfying the related equations similar to (1.1).

In the case when a given family of operators $H(t;\varepsilon)$ in $\mathcal{H}_+$ for $t \in [0,T]$ additionally depends on a parameter $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \subset \mathbb{R}$, $\varepsilon_0 > 0$, one can formulate the following important perturbation theorem generalizing similar one proved by T. Kato [3].

Theorem 1.3. *Assume that for $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \subset \mathbb{R}$ the family of operators $H(t,\varepsilon) : \mathcal{H}_+ \to \mathcal{H}_+$ is such that related forms $h(t;\varepsilon)(\psi,\varphi) := (H(t;\varepsilon)\psi, \varphi)$, $\psi, \varphi \in \mathcal{H}_+$, satisfy both conditions i) and ii) above. Then the following statements hold:*

1) *if for any $\varphi \in \mathcal{H}_+$, almost everywhere in $[0,T]$, $\lim_{\varepsilon \to 0} \|H(t;\varepsilon)\psi - H(t)\psi\|_- = 0$, then*
$$\lim_{\varepsilon \to 0} \|U(t,s;\varepsilon)\psi - U(t,s)\psi\| = 0 \tag{1.3}$$
in $\mathcal{H}$, where $U(t,s;\varepsilon)$, $t \geq s \in [0,T]$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, is the family of related dissolving operators of the Cauchy problem for equation (1.1) with the operator $H(t;\varepsilon) : \mathcal{H} \to \mathcal{H}$, $t \in [0,T]$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

2) *if there exists such a function $g \in L_1(0,T;\mathbb{R}_+)$ that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$*
$$\|\varepsilon^{-1}(H(t;\varepsilon) - H(t))\|_{+-} \leq g(t) \tag{1.4}$$
for all $t \in [0,T]$ and, moreover, there exists such a weakly measurable family of operators $H^{(1)}(t) : \mathcal{H}_+ \to \mathcal{H}_-$ that for any $\psi \in \mathcal{H}_+$, almost everywhere in $[0,T]$
$$\lim_{\varepsilon \to 0} \|\varepsilon^{-1}(H(t;\varepsilon) - H(t))\psi - H^{(1)}(t)\psi\|_- = 0, \tag{1.5}$$
then for $\psi \in \mathcal{H}_+$ there exists the derivative
$$U^{(1)}(t,s)\psi = -\frac{d}{d\varepsilon} U(t,s;\varepsilon)\psi|_{\varepsilon=0} \tag{1.6}$$
and the related formula
$$U^{(1)}(t,s)\psi = -\int_s^t \hat{U}(t,\tau) H^{(1)}(\tau) U(\tau,s)\psi d\tau \tag{1.7}$$
is satisfied for all $t \geq s \in [0,T]$.

A proof of this theorem is a rather simple consequence of the previous theorem 1.2.

2. THE ADIABATICALLY PERTURBED NONAUTONOMOUS PARTIAL DIFFERENTIAL EQUATIONS: NONSTATIONARY SPECTRAL ANALYSIS

Consider now, as before, the following family of order $2n \in \mathbb{Z}_+$ symmetric partial differential expressions

$$H(\varepsilon, t) := \sum_{|\beta|=0}^{n} \frac{\partial^{|\beta|}}{\partial x^\beta} a_\beta(x; \tau_0 + \varepsilon t) \frac{\partial^{|\beta|}}{\partial x^\beta} \tag{2.1}$$

in the Hilbert space $\mathcal{H} = L_2(\Omega; \mathbb{R})$, $\Omega \subset \mathbb{R}^m$, where $t \in \mathbb{R}_+$, coefficient functions $a_\alpha \in C^{|\alpha|}(\Omega \times \mathbb{S}^1; \mathbb{R})$, $|\beta| = \overline{0, n}$, $\tau_0 \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \subset \mathbb{R}$, $\varepsilon_0 > 0$, is a small adiabatic parameter, tending i to zero in applications. We can associate the following linear partial differential equation in $\mathcal{H}$ with expression (2.1):

$$d\psi/dt = -H(\varepsilon t) \tag{2.2}$$

and pose the following mixed Cauchy problem:

$$\psi|_{t=0} = \psi^{(0)} \in \mathcal{H}, \qquad \psi|_{\partial\Omega} = 0 \tag{2.3}$$

A suitable solution $\psi_\varepsilon := \psi(t; \varepsilon) \in \mathcal{H}$ to problems (2.2) and (2.3) will depend on a small parameter $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. This dependence is important for calculating different related bilinear functionals on solutions to (2.2), (2.3) at $t \in \mathbb{R}_+$ as $\varepsilon \to 0$, like the following ones:

$$\bar{h}(t) = \lim_{\varepsilon \to 0} \frac{(\psi_\varepsilon, H(\varepsilon t)\psi_\varepsilon)}{(\psi_\varepsilon, \psi_\varepsilon)} = \lim_{\varepsilon \to 0} \frac{d}{dt} \ln(\psi_\varepsilon, \psi_\varepsilon)^{\frac{1}{2}}, \tag{2.4}$$

$$\bar{b}(t) = \lim_{\varepsilon \to 0} \frac{(\psi_\varepsilon, B(\varepsilon t)\psi_\varepsilon)}{(\psi_\varepsilon, \psi_\varepsilon)},$$

where $B(\varepsilon t) : \mathcal{H} \to \mathcal{H}$ is some family of symmetric partial differential operators commuting with the family of operators (2.1) at each $t \in \mathbb{R}_+$.

The problem of existence of limits (2.4) as $\varepsilon \to 0$ is, to be known, very important [4, 5, 6] in numerous applications, investigating stability properties of related phenomena in mechanics, applied physics, mathematical biology and so on.

For studying Cauchy problem (2.2) and (2.3), we will use the results formulated in Section 1. Before doing this, note here that Cauchy problem (2.2), (2.3) at $\varepsilon = 0$ is that for the constant symmetric partial differential expression

$$H_0 := \sum_{|\alpha|=0}^{n} \frac{\partial^{|\alpha|}}{\partial x^\alpha} a_\alpha(x; \tau_0) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \tag{2.5}$$

whose evolution properties are assumed to be known. Namely, owing to the Kato results [3], there exists in $\mathcal{H}$ a strongly continuous self-adjoint semigroup $U_0(t)\colon \mathcal{H} \to \mathcal{H}$, $t \in \mathbb{R}_+$, satisfying the following evolution differential equation H_-:

$$\frac{\partial}{\partial t}(U_0(t)\psi) = -H_0 U_0(t)\psi \tag{2.6}$$

for any $\psi \in \mathcal{H}_+$ and $t \in \mathbb{R}_+$.

Assume now, for simplicity, that ε-dependence of the operator $H(\varepsilon t) : \mathcal{H} \to \mathcal{H}$ is differential for all $t \in \mathbb{R}_+$ and the following limit holds for all $t \in \mathbb{R}_+$:

$$\lim_{\varepsilon\to 0} \|\varepsilon(\int\limits_0^t H'(\varepsilon s)ds)\psi\|_{-,-} = 0. \tag{2.7}$$

Then, based on theorem 1.3, one can formulate the following statement.

Statement 2.1 There exists a family of invertible self-adjoint operators $U(t,s;\varepsilon) : \mathcal{H} \to \mathcal{H}$ for all $t \geq s \in \mathbb{R}_+$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, such that for any $\psi \in \mathcal{H}_+$

$$\lim_{\varepsilon\to 0} \|U(t,s;\varepsilon)\psi - U_0(t,s)\varphi\| = 0 \tag{2.8}$$

in $\mathcal{H}$. Moreover, if there exists a function $g \in L_1(0,T;\mathbb{R}_+)$ satisfying the inequality

$$\|(\int\limits_0^t H(\varepsilon s)ds)\|_{+,-} \leq g(t) \tag{2.9}$$

and there exists the limit

$$\lim_{\varepsilon\to 0} \|(\int\limits_0^t H'(\varepsilon s)ds)\psi - H_0^{(1)}\psi\|_{-} = 0 \tag{2.10}$$

for some autonomous operator $H_0^{(1)} : \mathcal{H}_+ \to \mathcal{H}_-$, where $t \in \mathbb{R}_+$ and $\psi \in \mathcal{H}_+$, the following expression

$$\frac{d}{d\varepsilon}U(t,s;\varepsilon)|_{\varepsilon=0} = U_0^{(1)}(t,s) \tag{2.11}$$

holds and the related evolution formula

$$U_0^{(1)}(t-s)\psi = -\int\limits_s^t \hat{U}_0(t-\tau)H_0^{(1)}(\tau)U_0(\tau-s)\psi d\tau \tag{2.12}$$

is satisfied for all $t \geq s \in \mathbb{R}_+$.

This result can be effectively applied to our problem of calculating limit expressions like (2.4) as $\varepsilon \to 0$. To this end it is necessary to describe preliminarily the set of symmetric partial differential expressions (2.1) satisfying the conditions of statement 2.1, listed above. Moreover, the explicit ε-dependence of the self-adjoint family of corresponding dissolving operators for problems (2.2), (2.3) can be studied very thoroughly if a self-adjoint extension of symmetric operators $H(\varepsilon t) : \mathcal{H} \to \mathcal{H}$ at each $t \in \mathbb{R}_+$ with dense dom$H(t) \subset \mathcal{H}$ is specified and the related expansions in their time-dependent eigenfunctions are found and described by means of the spectral measure [2, 5] technique. These aspects of the present work are planned to be analyzed in detail somewhere else together with investigation of some examples important for applications.

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