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BIPARTITE EMBEDDING OF (p, q)-TREES

Abstract. A bipartite graph $G = (L, R; E)$ where $V(G) = L \cup R$, $|L| = p$, $|R| = q$ is called a $(p, q)$-tree if $|E(G)| = p + q - 1$ and $G$ has no cycles. A bipartite graph $G = (L, R; E)$ is a subgraph of a bipartite graph $H = (L', R'; E')$ if $L \subseteq L'$, $R \subseteq R'$ and $E \subseteq E'$.

In this paper we present sufficient degree conditions for a bipartite graph to contain a $(p, q)$-tree.

Keywords: bipartite graph, tree, embedding graph.

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1. TERMINOLOGY

We shall use standard graph theory notation. We consider only finite, undirected graphs. All graphs will be assumed to have neither loops nor multiple edges.

Let $G = (L, R; E)$ be a bipartite graph with a partition $L$, $R$ and an edge set $E$. That means, $L$ and $R$ are two disjoint sets of independent vertices of the graph $G$ such that $L \cup R = V(G)$. We call $L = L(G)$ and $R = R(G)$ the left and right set of bipartition. Note that the graphs $G = (L, R; E)$ and $G' = (R, L; E)$ are different.

For a vertex $x \in V(G)$, $N(x, G)$ denotes the set of its neighbors in $G$. The degree $d(x, G)$ of the vertex $x$ in $G$ is the cardinality of the set $N(x, G)$.

$\Delta_L(G)$ ($\delta_L(G)$) and $\Delta_R(G)$ ($\delta_R(G)$) are maximum (minimum) vertex degree in the set $L(G)$ and $R(G)$, respectively. A vertex $x$ of $G$ is said to be pendant if $d(x, G) = 1$. For subsets $A$ and $B$ of $V(G)$, let $N(A, B; G)$ denote the set of edges $xy \in E(G)$ such that $x \in A$ and $y \in B$. $K_{p,q}$ is the complete $(p,q)$-bipartite graph. $\bar{G}$ is the complement of $G$ into $K_{p,q}$.

A bipartite graph $G = (L, R; E)$ is a subgraph of a bipartite graph $H = (L', R'; E')$ if $L \subseteq L'$, $R \subseteq R'$ and $E \subseteq E'$. If $G$ is a subgraph of $H$, then we write $G \leq H$. Observe that the meaning of the word subgraph is different from the usual.
one (see [3] and [1] page 1282). For instance, the graph $K_{1,2} = (\{a\}, \{b, c\}; \{ab, ac\})$ is not a subgraph of $K_{2,1} = (\{d, e\}, \{f\}; \{df, ef\})$. We say that a bipartite graph $G = (L, R; E)$ is bipartite embeddable or simply embeddable into bipartite graph $H = (L', R'; E')$ if there is an injection $f$ such that $f : L \cup R \rightarrow L' \cup R'$, $f(L) = L'$ and for every edge $xy \in E$, $f(x)f(y)$ is an edge of $H$. The function $f$ is called the bipartite embedding (or embedding) of $G$ into $H$. In other words, a bipartite graph $G = (L, R; E)$ is said to be embeddable into bipartite graph $H = (L', R'; E')$ when there exists a pair $(f_1, f_2)$ of injective mappings $f_1 : L \rightarrow L'$ and $f_2 : R \rightarrow R'$ such that if $x \in L$ and $y \in R$ are adjacent in $G$, then $f_1(x)$ and $f_2(y)$ are adjacent in $H$ (see [3]). It follows easily that $G$ is embeddable into $H$ if and only if $G$ is a subgraph of $H$. Note that $K_{1,2}$ is not embeddable into $K_{2,1}$.

A $(p,q)$-bipartite graph $G$ is called a $(p,q)$-tree if $G$ is connected and $|E(G)| = p + q - 1$. Thus each $(p,q)$-tree is a tree and for each tree $T$ there exist integers $p$ and $q$ such that $T$ is a $(p,q)$-tree. If $G$ is a $(p,q)$-bipartite and $|E(G)| = p + q - k$ and $G$ has no cycles then $G$ is called a $(p,q,k)$-forest. So, a $(p,q,1)$-forest is a $(p,q)$-tree. Let $T$ be a $(p,q)$-tree and $y \in V(T)$. Let us denote by $U_y$ the set of all $z \in N(y, T)$ such that $d(z, T) = 1$. We shall call $U_y$ the bough with the center $y$. The vertex $x \in V(T)$ is called penultimate vertex if $U_x \neq \emptyset$ and $d(x, T) = |U_x| + 1$ and there is the longest path $P$ in $T$ such that $x \in V(P)$.

2. RESULTS

First we shall give some results concerning the subgraphs of general graphs.

In 1963, Erdős and Sós (see [5]) stated the following conjecture, which was proved by Brandt in [2].

**Theorem 1.** Let $G$ be a graph with $n$ vertices and more than

$$f(k, n) = \max \left\{ \binom{2k - 1}{2}, \binom{k - 1}{2} + (k - 1)(n - k + 1) \right\}$$

edges. Then $G$ contains every forest with $k$ edges and without isolated vertices as a subgraph.

The following well-known result was atributed by Chvátal to graph-theoretical folklore [4]:

**Theorem 2.** Suppose $G$ is a graph with the minimum degree not less than $k$. Then $G$ contains every tree with $k$ edges.

S. Brandt in [2] proved:

**Theorem 3.** Suppose $F$ is a forest with $k$ edges and order $n$ and $G$ is a graph with at least $n$ vertices. If $\delta(G) \geq k$, then $F$ is a subgraph of $G$. 
We shall consider bipartite embedding problem, analogous to the classical embedding problem, the first general condition for a bipartite graph to be a subgraph of another bipartite graph was given by Rado in [6] (See also [3]).

In this paper we present sufficient degree conditions for a bipartite graph to contain every \((p,q)\)-tree.

The following lemma, proved in Section 3, is an easy bipartite equivalent of Theorem 2.

**Lemma A.** Let \(G = (L', R'; E')\) be a \((p', q')\)-bipartite graph such that \(\delta_L(G) \geq q\) and \(\delta_R(G) \geq p\). Then every \((p,q)\)-tree \(T = (L, R; E)\) is a subgraph of \(G\).

Observe that if \(\Delta_L(T) = q\) (or \(\Delta_R(T) = p\)), then Lemma A is best possible in the sense that it cannot be improved by decreasing the minimum degree of the graph \(G\).

Hence, now we shall consider a \((p,q)\)–tree \(T\) such that \(K_{1, q}\) is not a subgraph of \(T\).

The main results are the following theorem and its obvious corollaries:

**Theorem B.** Let \(T = (L, R; E)\) be a \((p, q)\)-tree, \(\Delta_L(T) \leq q - 1\) and let \(G = (L', R'; E')\) be a connected \((p', q')\)-bipartite graph such that \(q' \geq q\), \(\delta_L(G) \geq q - 1\) and \(\delta_R(G) \geq p\). Then \(T\) is a subgraph of \(G\).

Note that if \(\Delta_L(T) = q - 1\) (or \(\Delta_R(T) = p\)), then Theorem B is best possible.

Let \(P_k\) be a path with \(k\) edges and let \(k\) be even, \(k \geq 4\). By Theorem 3, \(P_k\) is a subgraph of a graph \(G\) if \(\delta(G) \geq k\), but by Theorem B, \(P_k\) is a subgraph of a bipartite graph \(G'\), if \(\delta(G') \geq k/2\).

**Corollary C.** Let \(T = (L, R; E)\) be a \((p, q)\)-tree, \(\Delta_L(T) \leq q - 1\) and let \(G = (L', R'; E')\) be a \((p', q')\)-bipartite graph such that \(\delta_L(G) \geq q - 1\), \(\delta_R(G) \geq p\) and every connected component \(G_1\) has at least \(p\) and \(q\) vertices in \(L(G_1)\) and \(R(G_1)\), respectively. Then \(T\) is a subgraph of each component of \(G\).

**Corollary D.** Let \(F = (L, R; E)\) be a \((p, q, k)\)-forest, \(k \geq 2\) and let \(G = (L', R'; E')\) be a \((p', q')\)-bipartite graph such that \(q' \geq q\), \(\delta_L(G) \geq q - 1\), \(\delta_R(G) \geq p\). Then \(F\) is a subgraph of \(G\).

3. PROOFS

To prove Lemma A and Theorem B, we shall need two lemmas.

**Lemma 3.1** Let \(T = (L, R; E)\) be a \((p, q)\)-tree, let \(U_y \neq \emptyset\) be a bough in \(T\) and let \(G\) be a \((p', q')\)-bipartite graph, \(\delta_L(G) \geq q\) and \(\delta_R(G) \geq p\). If \(T \setminus U_y \leq G\) then \(T \leq G\).
Proof. Let $T = (L, R; E)$ be a $(p, q)$-tree, $y \in V(T)$, $U_y \neq \emptyset$ and let $G = (L', R'; E')$ be a $(p', q')$-bipartite graph verifying the assumptions of the lemma. Without loss of generality we may assume that $y \in L$. Let us denote by $T_1$ the tree $T \setminus U_y$. Let $|U_y| = k$. If $k = q$ then $T = K_{1,q}$ and $T \leq G$. We now assume that $k \leq q - 1$. Note that $T_1 = (L_1, R_1; E_1)$ is a $(p, q - k)$-tree and $1 \leq d(y, T_1) \leq q - k$. By assumptions of the lemma, there exists an embedding $f$ of $T_1$ into $G$. Let $f(y) = z$. We will denote by $N^*(z)$ the set $\{w \in N(z, G) \mid w \in f[R_1]\}$. Hence $|N^*(z)| \leq q - k$. Since $\delta_L(G) \geq q$, there are $k$ vertices $w_i$ such that $w_i \in (N(z, G) \setminus N^*(z))$. If $W^* = \{w_i, i = 1, \ldots, k\}$ then the function $f^*$ such that $f^*(v) = f(v)$ for $v \in V(T_1)$ and $f^*[U_y] = W^*$ is an embedding of $T$ into $G$. 

The Proof of Lemma A. The proof is by induction on $p + q$. If $T$ is a $(p, q)$-tree such that $p + q \leq 4$ and $G$ is a $(p', q')$-bipartite graph verifying the assumptions of the lemma, then the lemma is easy to check.

So, let us suppose $p + q \geq 5$ and the lemma is true for all integers $p_1$, $q_1$ with $p_1 + q_1 < p + q$. Let $T$ be a $(p, q)$-tree and let $G$ be a $(p', q')$-bipartite graph such that $\delta_L(G) \geq q$ and $\delta_R(G) \geq p$. There exists a vertex $y$ in $V(T)$ such that $|U_y| = k > 0$. Without loss of generality we may assume that $y \in L$. If $k = q$, then the lemma is obvious. If $k \leq q - 1$, then let us denote by $T_1$ the tree $T \setminus U_y$. Since $T_1$ is $(p, q - k)$-tree it follows, by the induction hypothesis, that $T_1 \leq G$. We obtain an embedding of $T$ into $G$ by Lemma 3.1.

Lemma 3.2 Let $T$ be a $(p, q)$-tree such that $T \neq K_{1,q}$ and $T \neq K_{p,1}$. Then there exist at least two penultimate vertices in $V(T)$.

The proof of Lemma 3.2 is trivial.

The Proof of Lemma B. Let $T = (L, R; E)$ be a $(p, q)$-tree such that $\Delta_L(T) \leq q - 1$ and let $G = (L', R'; E')$ be a $(p', q')$-bipartite graph verifying assumptions of Theorem B. The proof will be divided into two steps.

Case 1. Let us first assume that there exists a penultimate vertex, say $y$, in $L$.

Let $|U_y| = k > 0$, $\{x\} = N(y, T) \setminus U_y$ and let us denote by $T_1$ the tree $T \setminus U_y = (L_1, R_1; E_1)$. $T_1$ is a $(p, q - k)$-tree. By Lemma A, there exists an embedding $f$ of $T_1$ into $G$. Let $f(y) = w$, $f(x) = z$, $f[L_1] = L'_1$ and $f[R_1] = R'_1$. If $d(w, G) \geq q$ or $R'_1 \not\subset N(w, G)$, then there are $k$ vertices, $v_1, \ldots, v_k \in (N(w, G) \setminus R'_1)$. The function $f^*$ such that

$$f^*(v) = f(v), \text{ for } v \in V(T_1)$$

$$f^*(x_i) = v_i, \text{ for } x_i \in U_y, \quad i = 1, \ldots, k$$

is an embedding of $T$ into $G$. So, we may assume that $d(w, G) = q - 1$ and $R'_1 \subset N(w, G)$. Write $R'_2 = N(w, G) \setminus R'_1$. 

Subcase 1.1 There exists a vertex $w_1 \in N(z, G)$ such that $d(w_1, G) \geq q$ or $|N(w_1, G) \cap R'_1| < q - k$.

Then, the vertex $w_1$ has $k$ neighbors, say $z'_1, \ldots, z'_k$, which are not elements of $R'_1$. Thus we conclude that the function $f_1^*$ given by

$$
\begin{align*}
  f_1^*(v) &= f(v), \text{ for } v \in V(T_1) \setminus \{y\}, \\
  f_1^*(y) &= w_1, \\
  f_1^*(x_i) &= z'_i, \text{ for } x_i \in U_y, \quad i = 1, \ldots, k,
\end{align*}
$$

and, if $w_1 \in f[L_1]$ and $w_1 = f(v^*)$ then $f_1^*(v^*) = w$, is the embedding of $(p, q)$-trees $123$.

Subcase 1.2 Now we assume that for each vertex $w' \in N(z, G)$ there is $|N(w', G) \cap R'_1| = q - k$ and $d(w', G) = q - 1$.

Observe that in this case $G$ has a subgraph $H$ such that $H$ is a $(p'_1, q - k)$-complete bipartite graph, $L(H) = N(z, G)$, $R(H) = R'_1$ and $p'_1 = d(z, G) \geq p$.

Subcase 1.2.1 There is a vertex $w'_1 \in N(z, G)$ such that $N(w'_1, G) \neq N(w, G)$.

Thus there exist vertices $z_1 \in R' \setminus N(w, G)$, $z_2 \in R'_2$ such that $z_1w'_1 \in E'$ and $z_2w'_1 \notin E'$. By Lemma 3.2, there is a penultimate vertex $y' \neq y$ in $V(T)$. First we assume that $y' \in L$. We will denote by $F_2$ the forest $T \setminus U_{y'} \setminus \{y, y', x'_1\}$, where $x'_1 \in U_{y'}$. By Lemma A, $F_2 \leq H_1 = H \setminus \{z_3, w, w'_1\}$, where $z_3 \in R(H)$. If $f_2$ is an embedding of $F_2$ into $H_1$ then the embedding $f_2^*$ of $T$ into $G$ is defined as follows:

$$
\begin{align*}
  f_2^*(v) &= f_2(v), \text{ for } v \in V(F_2), \\
  f_2^*[U_{y'}] &= R'_2 \cup \{z_3\}, \\
  f_2^*(y') &= w'_1, \\
  f_2^*(x'_1) &= z_1, \\
  f_2^*(y) &= w.
\end{align*}
$$

Let now $y' \in R$ and $|U_{y'}| = k'$ and let $x' \in (N(y', T) \setminus U_{y'})$. Let us denote by $T_3$ the tree $T \setminus U_{y'} \setminus U_y \setminus \{y, x', y'\}$, and by $H_2$ a bipartite graph such that $L(H_2) = L(H) \setminus \{w, w'_1\} \setminus L'_3$, where $L'_3 \subset N(z_1, G) \setminus \{w, w'_1\}$, $|L'_3| = k'$, $R(H_2) = R(H) \setminus \{z_3\}$, and $z_3 \in R(H)$. By Lemma A, there is an embedding $f_3$ of $T_3$ into $H_2$. Let $f_3^*$ be given as follows:

$$
\begin{align*}
  f_3^*(v) &= f_3(v), \quad v \in V(T_3), \\
  f_3^*(y) &= w, \\
  f_3^*(x') &= w'_1, \\
  f_3^*[U_{y'}] &= R'_2 \cup \{z_3\}, \\
  f_3^*(y') &= z_1.
\end{align*}
$$
\[ f_3^*[U_{y'}] = L_3'. \]

Therefore, \( T \leq G \).

**Subcase 1.2.2** Each vertex \( w' \in N(z, G) \) has the degree \( q - 1 \) and \( N(w, G) = N(w', G) \). It follows that \( G \) has a subgraph \( H_3 = K_{p', q-1} \), where \( L(H_3) = N(z, G) \), \( R(H_3) = N(w, G) \). Observe that \( R \setminus R(H_3) \neq \emptyset \) and \( N(L(H_3), R \setminus R(H_3); G) = \emptyset \). Let \( z_4 \) be a vertex in \( R \setminus R(H_3) \). By assumption of the theorem, there are vertices \( z_5 \in R(H_3) \) and \( w_2 \in L(G) \setminus L(H_3) \) such that \( z_4w_2 \in E(G) \) and \( w_2z_5 \in E(G) \). It is easily seen that \( T_4 = (T \setminus U_{y'} \setminus \{x, y\}) \leq (H_3 \setminus \{z_5\} \setminus R_4') \), where \( R_4' \subset (N(w_2, G) \setminus \{z_5\}) \) and \( |R_4'| = k \) and \( z_4 \in R_4' \). Obviously, \( T \leq G \), again.

**Case 2** Let us assume there is no penultimate vertex in \( L \).

Thus, by Lemma 3.2, there exist at least two penultimate vertices in \( R \). Let \( y_1 \) be a penultimate vertex in \( R \) and let \( \{x''\} = N(y_1, T) \setminus U_{y_1} \).

Consider a tree \( T_5 \) obtained from the tree \( T \) by deleting pendant vertices \( x_1, x_2, \ldots, x_k \), so that the vertex \( x'' \) may be penultimate vertex in \( L \).

By Case 1, \( T_5 \leq G \) and by assumption \( \delta_R(G) \geq p \) we deduce that \( T \leq G \) and the theorem is proved.

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