Opuscula Mathematica • Vol. 26 • No. 1 • 2006

Beata Orchel

BIPARTITE EMBEDDING OF (p, q)-TREES

Abstract. A bipartite graph $G = (L, R; E)$ where $V(G) = L \cup R$, $|L| = p$, $|R| = q$ is called a (p, q)-tree if $|E(G)| = p + q - 1$ and G has no cycles. A bipartite graph $G = (L, R; E)$ is a subgraph of a bipartite graph $H = (L', R'; E')$ if $L \subseteq L'$, $R \subseteq R'$ and $E \subseteq E'$.

In this paper we present sufficient degree conditions for a bipartite graph to contain a (p, q)-tree.

Keywords: bipartite graph, tree, embedding graph.

Mathematics Subject Classification: 05C35.

1. TERMINOLOGY

We shall use standard graph theory notation. We consider only finite, undirected graphs. All graphs will be assumed to have neither loops nor multiple edges.

Let $G = (L, R; E)$ be a bipartite graph with a partition L, R and an edge set E. That means, L and R are two disjoint sets of independent vertices of the graph G such that $L \cup R = V(G)$. We call $L = L(G)$ and $R = R(G)$ the left and right set of bipartition. Note that the graphs $G = (L, R; E)$ and $G' = (R, L; E)$ are different.

For a vertex $x \in V(G)$, $N(x, G)$ denotes the set of its neighbors in G. The degree $d(x, G)$ of the vertex x in G is the cardinality of the set $N(x, G)$.

$\Delta_L(G)$ ($\delta_L(G)$) and $\Delta_R(G)$ ($\delta_R(G)$) are maximum (minimum) vertex degree in the set $L(G)$ and $R(G)$, respectively. A vertex x of G is said to be pendant if $d(x, G) = 1$. For subsets A and B of $V(G)$, let $N(A, B; G)$ denote the set of edges $xy \in E(G)$ such that $x \in A$ and $y \in B$. $K_{p,q}$ is the complete (p, q)-bipartite graph. $\bar{G}$ is the complement of G into $K_{p,q}$.

A bipartite graph $G = (L, R; E)$ is a *subgraph* of a bipartite graph $H = (L', R'; E')$ if $L \subseteq L'$, $R \subseteq R'$ and $E \subseteq E'$. If G is a subgraph of H, then we write $G \leq H$. Observe that the meaning of the word *subgraph* is different from the usual one (see [3] and [1] page 1282). For instance, the graph $K_{1,2} = (\{a\},\{b,c\};\{ab,ac\})$ is not a subgraph of $K_{2,1} = (\{d,e\},\{f\};\{df,ef\})$. We say that a bipartite graph $G = (L,R;E)$ is *bipartite embeddable* or simply *embeddable* into bipartite graph $H = (L',R';E')$ if there is an injection f such that $f : L \cup R \to L' \cup R'$, $f(L) = L'$ and for every edge $xy \in E$, $f(x)f(y)$ is an edge of H. The function f is called the *bipartite embedding* (or *embedding*) of G into H. In other words, a bipartite graph $G = (L,R;E)$ is said to be *embedded* into bipartite graph $H = (L',R' : E')$ when there exists a pair (f_1,f_2) of injective mappings $f_1 : L \to L'$ and $f_2 : R \to R'$ such that if $x \in L$ and $y \in R$ are adjacent in G, then $f_1(x)$ and $f_2(y)$ are adjacent in H (see [3]). It follows easily that G is embeddable into H if and only if G is a subgraph of H. Note that $K_{1,2}$ is not embeddable into $K_{2,1}$.

A (p,q)-bipartite graph G is called a (p,q)-*tree* if G is connected and $|E(G)| = p+q-1$. Thus each (p,q)-tree is a tree and for each tree T there exist integers p and q such that T is a (p,q)-tree. If G is a (p,q)-bipartite and $|E(G)| = p+q-k$ and G has no cycles then G is called a (p,q,k)-*forest*. So, a $(p,q,1)$-forest is a (p,q)-tree. Let T be a (p,q)-tree and $y \in V(T)$. Let us denote by U_y the set of all $z \in N(y,T)$ such that $d(z,T) = 1$. We shall call U_y the *bough with the center y*. The vertex $x \in V(T)$ is called *penultimate vertex* if $U_x \neq \emptyset$ and $d(x,T) = |U_x| + 1$ and there is the longest path P in T such that $x \in V(P)$.

2. RESULTS

First we shall give some results concerning the subgraphs of general graphs.

In 1963, Erdös and Sós (see [5]) stated the following conjecture, which was proved by Brandt in [2].

Theorem 1. *Let G be a graph with n vertices and more than*

$$f(k,n) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}$$

edges. Then G contains every forest with k edges and without isolated vertices as a subgraph.

The following well-known result was atributed by Chvátal to graph-theoretical folklore [4]:

Theorem 2. *Suppose G is a graph with the minimum degree not less than k. Then G contains every tree with k edges.*

S. Brandt in [2] proved:

Theorem 3. *Suppose F is a forest with k edges and order n and G is a graph with at least n vertices. If $\delta(G) \geq k$, then F is a subgraph of G.*

We shall consider bipartite embedding problem, analogous to the classical embedding problem, the first genaral condition for a bipartite graph to be a subgraph of another bipartite graph was given by Rado in [6] (See also [3]).

In this paper we present sufficient degree conditions for a bipartite graph to contain every (p,q)-tree.

The following lemma, proved in Section 3, is an easy bipartite equivalent of Theorem 2.

Lemma A. *Let $G = (L', R'; E')$ be a (p', q')-bipartite graph such that $\delta_L(G) \geq q$ and $\delta_R(G) \geq p$. Then every (p,q)–tree $T = (L, R; E)$ is a subgraph of G.*

Observe that if $\Delta_L(T) = q$ (or $\Delta_R(T) = p$), then Lemma A is best possible in the sense that it cannot be improved by decreasing the minimum degree of the graph G.

Hence, now we shall consider a (p,q) − tree T such that $K_{1,q}$ is not a subgraph of T.

The main results are the following theorem and its obvious corollaries:

Theorem B. *Let $T = (L, R; E)$ be a (p,q)-tree, $\Delta_L(T) \leq q-1$ and let $G = (L', R'; E')$ be a connected (p', q')-bipartite graph such that $q' \geq q$, $\delta_L(G) \geq q-1$ and $\delta_R(G) \geq p$. Then T is a subgraph of G.*

Note that if $\Delta_L(T) = q-1$ (or $\Delta_R(T) = p$), then Theorem B is best possible.

Let P_k be a path with k edges and let k be even, $k \geq 4$. By Theorem 3, P_k is a subgraph of a graph G if $\delta(G) \geq k$, but by Theorem B, P_k is a subgraph of a bipartite graph G', if $\delta(G') \geq k/2$.

Corollary C. *Let $T = (L, R; E)$ be a (p,q)-tree, $\Delta_L(T) \leq q-1$ and let $G = (L', R'; E')$ be a (p', q')-bipartite graph such that $\delta_L(G) \geq q-1$, $\delta_R(G) \geq p$ and every connected component G_1 has at least p and q vertices in $L(G_1)$ and $R(G_1)$, respectively. Then T is a subgraph of each component of G.*

Corollary D. *Let $F = (L, R; E)$ be a (p,q,k)-forest, $k \geq 2$ and let $G = (L', R'; E')$ be a (p', q')-bipartite graph such that $q' \geq q$, $\delta_L(G) \geq q-1$, $\delta_R(G) \geq p$. Then F is a subgraph of G.*

3. PROOFS

To prove Lemma A and Theorem B, we shall need two lemmas.

Lemma 3.1 *Let $T = (L, R; E)$ be a (p,q)-tree, let $U_y \neq \emptyset$ be a bough in T and let G be a (p', q')-bipartite graph, $\delta_L(G) \geq q$ and $\delta_R(G) \geq p$. If $T \setminus U_y \leq G$ then $T \leq G$.*

Proof. Let $T = (L, R; E)$ be a (p, q)-tree, $y \in V(T)$, $U_y \neq \emptyset$ and let $G = (L', R'; E')$ be a (p', q')-bipartite graph verifying the assumptions of the lemma. Without loss of generality we may assume that $y \in L$. Let us denote by T_1 the tree $T \setminus U_y$. Let $|U_y| = k$. If $k = q$ then $T = K_{1,q}$ and $T \leq G$. We now assume that $k \leq q - 1$. Note that $T_1 = (L_1, R_1; E_1)$ is a $(p, q - k)$-tree and $1 \leq d(y, T_1) \leq q - k$. By assumptions of the lemma, there exists an embedding f of T_1 into G. Let $f(y) = z$. We will denote by $N^*(z)$ the set $\{w \in N(z, G)$ such that $w \in f[R_1]\}$. Hence $|N^*(z)| \leq q - k$. Since $\delta_L(G) \geq q$, there are k vertices w'_i such that $w'_i \in (N(z, G) \setminus N^*(z))$. If $W^* = \{w'_i, i = 1, \ldots, k\}$ then the function f^* such that $f^*(v) = f(v)$ for $v \in V(T_1)$ and $f^*[U_y] = W^*$ is an embedding of T into G. □

The Proof of Lemma A. The proof is by induction on $p + q$. If T is a (p, q)-tree such that $p + q \leq 4$ and G is a (p', q')-bipartite graph verifying the assumptions of the lemma, then the lemma is easy to check.

So, let us suppose $p + q \geq 5$ and the lemma is true for all integers p_1, q_1 with $p_1 + q_1 < p + q$. Let T be a (p, q)-tree and let G be a (p', q')-bipartite graph such that $\delta_L(G) \geq q$ and $\delta_R(G) \geq p$. There exists a vertex y in $V(T)$ such that $|U_y| = k > 0$. Without loss of generality we may assume that $y \in L$. If $k = q$, then the lemma is obvious. If $k \leq q - 1$, then let us denote by T_1 the tree $T \setminus U_y$. Since T_1 is $(p, q - k)$-tree it follows, by the induction hypothesis, that $T_1 \leq G$. We obtain an embedding of T into G by Lemma 3.1. □

Lemma 3.2 *Let T be a (p, q)-tree such that $T \neq K_{1,q}$ and $T \neq K_{p,1}$. Then there exist at least two penultimate vertices in $V(T)$.*

The proof of Lemma 3.2 is trivial.

The Proof of Lemma B. Let $T = (L, R; E)$ be a (p, q)-tree such that $\Delta_L(T) \leq q - 1$ and let $G = (L', R'; E')$ be a (p', q')-bipartite graph verifying assumptions of Theorem B. The proof will be divided into two steps.

Case 1. *Let us first assume that there exists a penultimate vertex, say y, in L.*

Let $|U_y| = k > 0$, $\{x\} = N(y, T) \setminus U_y$ and let us denote by T_1 the tree $T \setminus U_y = (L_1, R_1; E_1)$. T_1 is a $(p, q - k)$-tree. By Lemma A, there exists an embedding f of T_1 into G. Let $f(y) = w$, $f(x) = z$, $f[L_1] = L'_1$ and $f[R_1] = R'_1$. If $d(w, G) \geq q$ or $R'_1 \not\subset N(w, G)$, then there are k vertices, $v_1, \ldots, v_k \in (N(w, G) \setminus R'_1)$. The function f^* such that

$$f^*(v) = f(v), \text{ for } v \in V(T_1)$$
$$f^*(x_i) = v_i, \text{ for } x_i \in U_y, \quad i = 1, \ldots, k$$

is an embedding of T into G. So, we may assume that $d(w, G) = q - 1$ and $R'_1 \subset N(w, G)$. Write $R'_2 = N(w, G) \setminus R'_1$.

Subcase 1.1 *There exists a vertex $w_1 \in N(z,G)$ such that $d(w_1,G) \geq q$ or $|N(w_1,G) \cap R'_1| < q-k$.*

Then, the vertex w_1 has k neighbors, say $z'_1, \ldots, z'_k$, which are not elements of R'_1. Thus we conclude that the function f^*_1 given by

$$\begin{aligned} f^*_1(v) &= f(v), \text{ for } v \in V(T_1) \setminus \{y\}, \\ f^*_1(y) &= w_1, \\ f^*_1(x_i) &= z'_i, \text{ for } x_i \in U_y, \quad i = 1, \ldots, k, \end{aligned}$$

and, if $w_1 \in f[L_1]$ and $w_1 = f(v^*)$ then $f^*_1(v^*) = w$, is the embedding of T into G.

Subcase 1.2 *Now we assume that for each vertex $w' \in N(z,G)$ there is $|N(w',G) \cap R'_1| = q-k$ and $d(w',G) = q-1$.*

Observe that in this case G has a subgraph H such that H is a $(p'_1, q-k)$-complete bipartite graph, $L(H) = N(z,G)$, $R(H) = R'_1$ and $p'_1 = d(z,G) \geq p$.

Subcase 1.2.1 *There is a vertex $w'_1 \in N(z,G)$ such that $N(w'_1,G) \neq N(w,G)$.*

Thus there exist vertices $z_1 \in R' \setminus N(w,G)$, $z_2 \in R'_2$ such that $z_1w'_1 \in E'$ and $z_2w'_1 \notin E'$. By Lemma 3.2, there is a penultimate vertex $y' \neq y$ in $V(T)$. First we assume that $y' \in L$. We will denote by F_2 the forest $T \setminus U_y \setminus \{y, y', x'_1\}$, where $x'_1 \in U_{y'}$. By Lemma A, $F_2 \leq H_1 = H \setminus \{z_3, w, w'_1\}$, where $z_3 \in R(H)$. If f_2 is an embedding of F_2 into H_1 then the embedding f^*_2 of T into G is defined as follows:

$$\begin{aligned} f^*_2(v) &= f_2(v), \text{ for } v \in V(F_2), \\ f^*_2[U_y] &= R'_2 \cup \{z_3\}, \\ f^*_2(y') &= w'_1, \\ f^*_2(x'_1) &= z_1, \\ f^*_2(y) &= w. \end{aligned}$$

Let now $y' \in R$ and $|U_{y'}| = k'$ and let $x' \in (N(y',T) \setminus U_{y'})$. Let us denote by T_3 the tree $T \setminus U_{y'} \setminus U_y \setminus \{y, x', y'\}$, and by H_2 a bipartite graph such that $L(H_2) = L(H) \setminus \{w, w'_1\} \setminus L'_3$, where $L'_3 \subset N(z_1,G) \setminus \{w, w'_1\}$, $|L'_3| = k'$, $R(H_2) = R(H) \setminus \{z_3\}$, and $z_3 \in R(H)$. By Lemma A, there is an embedding f_3 of T_3 into H_2. Let f^*_3 be given as follows:

$$\begin{aligned} f^*_3(v) &= f_3(v), \quad v \in V(T_3), \\ f^*_3(y) &= w, \\ f^*_3(x') &= w'_1, \\ f^*_3[U_y] &= R'_2 \cup \{z_3\}, \\ f^*_3(y') &= z_1, \end{aligned}$$

$$f_3^*[U_{y'}] = L_3'.$$

Therefore, $T \leq G$.

Subcase 1.2.2 Each vertex $w' \in N(z, G)$ has the degree $q-1$ and $N(w, G) = N(w', G)$. It follows that G has a subgraph $H_3 = K_{p_1', q-1}$, where $L(H_3) = N(z, G)$, $R(H_3) = N(w, G)$. Observe that $R \setminus R(H_3) \neq \emptyset$ and $N(L(H_3), R \setminus R(H_3); G) = \emptyset$. Let z_4 be a vertex in $R \setminus R(H_3)$. By assumption of the theorem, there are vertices $z_5 \in R(H_3)$ and $w_2 \in L(G) \setminus L(H_3)$ such that $z_4 w_2 \in E(G)$ and $w_2 z_5 \in E(G)$. It is easily seen that $T_4 = (T \setminus U_y \setminus \{x, y\}) \leq (H_3 \setminus \{z_5\} \setminus R_4')$, where $R_4' \subset (N(w_2, G) \setminus \{z_5\})$ and $|R_4'| = k$ and $z_4 \in R_4'$. Obviously, $T \leq G$, again.

Case 2 *Let us assume there is no penultimate vertex in* L.

Thus, by Lemma 3.2, there exist at least two penultimate vertices in R. Let y_1 be a penultimate vertex in R and let $\{x''\} = N(y_1, T) \setminus U_{y_1}$.

Consider a tree T_5 obtained from the tree T by deleting pendant vertices x_1, $x_2, \ldots, x_k$, so that the vertex x'' may be penultimate vertex in L.

By Case 1, $T_5 \leq G$ and by assumption $\delta_R(G) \geq p$ we deduce that $T \leq G$ and the theorem is proved. □

Acknowledgements

The autor gratefully acknowledges the many helpful suggestions of Professor A. Paweł Wojda during the preparation of the paper.

REFERENCES

[1] Bollobás B., *Extremal Graph Theory*, in: Handbook of Combinatorics R. I. Graham, M. Gröschel, L. Lovász, (ed.), The MIT Press, Cambridge, Massachusetts, 1995.

[2] Brandt S., *Subtrees and subforests of graphs*, J. Combin. Theory Ser. B. **61** (1994), 63-70.

[3] Catlin Paul A., *Subgraphs of graphs, 1*, Discrete Mathematics **10** (1974), 225–233.

[4] Chvátal V., *Tree — complete graph Ramsey numbers*, J. Graph Theory **1** (1977), 93.

[5] Erdös P., *Extremal problems in graph theory*, in: Theory of Graphs and its Applications M. Fiedler (ed.), Academic Press, 1964, 29–36.

[6] Rado R., *A theorem on general measure functions*, Proc. London Math. Soc **44** (1938) (2).

Beata Orchel
orchel@agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Kraków, Poland

Received: August 28, 2005.