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CLASSICAL SOLUTIONS OF INITIAL PROBLEMS FOR QUASILINEAR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Abstract. We consider the initial problem for a quasilinear partial functional differential equation of the first order

$$\partial_t z(t,x) + \sum_{i=1}^{n} f_i(t,x,z_{(t,x)})\partial_{x_i} z(t,x) = G(t,x,z_{(t,x)}),$$
$$z(t,x) = \varphi(t,x) \;\; ((t,x) \in [-h_0,0] \times R^n)$$

where $z_{(t,x)}\colon [-h_0,0]\times[-h,h] \to R$ is a function defined by $z_{(t,x)}(\tau,\xi) = z(t+\tau, x+\xi)$ for $(\tau,\xi) \in [-h_0,0]\times[-h,h]$. Using the method of bicharacteristics and the fixed-point theorem we prove, under suitable assumptions, a theorem on the local existence and uniqueness of classical solutions of the problem and its continuous dependence on the initial condition.

Keywords: partial functional differential equations, classical solutions, local existence, bicharacteristics.

Mathematics Subject Classification: 35R10, 35L45.

1. INTRODUCTION

In addition to classical solutions, the following classes of generalized solutions to hyperbolic functional differential problems are considered in literature. Initial boundary value problems for almost linear systems for unknown functions of two independent variables were considered in [8]. A continuous function is a solution of a mixed problem if it satisfies an integral functional system by integrating along bicharacteristics. Existence theorems and differential inequalities related to almost linear functional problems can be found in [4]. Distributional solutions of almost linear problems were investigated in [9]. The method used in this paper is constructive; the existence result is based on a difference scheme.

The class of Carathéodory solutions consists of all functions which are continuous and have their partial derivatives almost everywhere in a domain. The set of all points where the differential functional equation is not fulfilled is of Lebesgue measure zero. The existence and uniqueness results for quasilinear systems with initial or initial-boundary conditions in the class of Carathéodory solutions can be found in [5, 10].

Existence results are based on the following method. Functional differential problems are equivalent, under natural assumptions on given functions, to integral equations which are obtained from original problems by integrating along bicharacteristics.

Carathéodory solutions to initial problems for nonlinear equations were considered in [6]. An essential extension of some ideas concerning classical solutions of hyperbolic functional differential problems is given in [2, 3], where the Cinquini Cibrario solutions are considered. This class of solutions is placed between classical solutions and solutions in the Carathéodory sense.

The monograph [7] contains an exposition of existence an uniqueness of generalized and classical solutions to hyperbolic functional differential problems.

We formulate the functional differential problem. Let $a > 0$, $h_0 \in R_+$, $R_+ = [0, +\infty)$, and $h = (h_1, \ldots, h_n) \in R_+^n$ be given. We define the sets

$$E = [0, a] \times R^n, \quad D = [-h_0, 0] \times [-h, h]$$

and $E_0 = [-h_0, 0] \times R^n$. Suppose that $z\colon E_0 \cup E \to R$ and $(t, x) \in E$ are fixed. We define the function $z_{(t,x)}\colon D \to R$ as follows

$$z_{(t,x)}(\tau, \xi) = z(t + \tau, x + \xi), \ (\tau, \xi) \in D.$$

The function $z_{(t,x)}$ is the restriction of z to the set $[t - h_0, t] \times [x - h, x + h]$ and this restriction is shifted to the set D. Elements of the space $C(D, R)$ will be denoted by w, $\bar{w}$ and so on. Put $\Omega = E \times C(D, R)$ and let

$$f = (f_1, \ldots, f_n)\colon \Omega \to R^n, \quad G\colon \Omega \to R, \quad \varphi\colon E_0 \to R$$

be given functions. We will deal with the following initial problem

$$\partial_t z(t, x) + \sum_{i=1}^{n} f_i(t, x, z_{(t,x)}) \partial_{x_i} z(t, x) = G(t, x, z_{(t,x)}), \tag{1}$$

$$z(t, x) = \varphi(t, x) \text{ on } E_0, \tag{2}$$

where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$.

A function $\tilde{z} \in C^1([-h_0, \xi] \times R^n, R)$, where $0 < \xi \le a$, is a classical solution of (1), (2) if it satisfies equation (1) and condition (2) holds on E_0.

The aim of this paper is to prove a theorem on the existence and continuous dependence of classical solutions for (1), (2).

2. BICHARACTERISTICS

The following function spaces will be needed in our considerations. Write $E_t^* = [-h_0, t] \times R^n$ and $E_t = [0, t] \times R^n$ where $0 \le t \le a$. We will denote by $\|\cdot\|_t$ the supremum norm in the spaces $C(E_t^*, R)$ and $C(E_t^*, R^n)$. We will denote by $M_{n\times n}$ the class of all $n \times n$ matrices with real elements. For $x \in R^n$, $X \in M_{n\times n}$, where $x = (x_1, \ldots, x_n)$ and

$$X = [x_{ij}]_{i,j=1,\ldots,n}$$

we put

$$\|x\| = \sum_{i=1}^{n} |x_i| \text{ and } \|X\| = \max_{1\le j\le n} \sum_{i=1}^{n} |x_{ij}|$$

The product of two matrices is denoted by '$*$'. If $X \in M_{n\times n}$ then X^T is the transpose matrix. We use the symbol '$\circ$' to denote the scalar product in R^n.

Let us denote by $\|\cdot\|_0$ the supremum norm in the space $C(D, R)$. Let $C^1(D, R)$ be the set of all functions $w\colon D \to R$ such that the derivatives $\partial_t w$, $(\partial_{x_1} w, \ldots, \partial_{x_n} w) = \partial_x w$ exist and $\partial_t w \in C(D, R)$, $\partial_x w \in C(D, R^n)$. For $w \in C^1(D, R)$ we put

$$\|w\|_{\mathrm{I}} = \|w\|_0 + \max\{|\partial_t w(t, x)| + \|\partial_x w(t, x)\| : (t, x) \in D\}.$$

We denote by $C^{1.L}(D, R)$ the class of all $w \in C^1(D, R)$ such that $\|w\|_{\mathrm{I.L}} < +\infty$ where

$$\|w\|_{\mathrm{I.L}} = \|w\|_{\mathrm{I}}$$
$$+ \sup\Big\{ \frac{|\partial_t w(t, x) - \partial_t w(\bar t, \bar x)| + \|\partial_x w(t, x) - \partial_x w(\bar t, \bar x)\|}{|t - \bar t| + \|x - \bar x\|} :$$
$$(t, x), (\bar t, \bar x) \in D, (t, x) \ne (\bar t, \bar x) \Big\}.$$

We will consider the spaces $\Omega^{(1)} = E \times C^1(D, R)$ and $\Omega^{(1.L)} = E \times C^{1.L}(D, R)$. Let Θ be the class of all functions $\gamma \in C(R_+, R_+)$ which are nondecreasing on R_+.

Now we define some further function spaces. Given $s = (s_0, s_1, s_2) \in R_+^3$, we denote by $C^{1.L}[s]$ the set of all functions $\varphi \in C(E_0, R)$ such that:

(i) there exists $(\partial_{x_1}\varphi, \ldots, \partial_{x_n}\varphi) = \partial_x\varphi$, $\partial_t\varphi$ and $\partial_x\varphi \in C(E_0, R^n)$, $\partial_t\varphi \in C(E_0, R)$,

(ii) the estimates $|\varphi(t, x)| \le s_0$, and

$$|\partial_t\varphi(t, x)| + \|\partial_x\varphi(t, x)\| \le s_1,$$
$$|\partial_t\varphi(t, x) - \partial_t\varphi(\bar t, \bar x)| + \|\partial_x\varphi(t, x) - \partial_x\varphi(\bar t, \bar x)\| \le s_2[|t - \bar t| + \|x - \bar x\|]$$

are satisfied on E_0.

Let $\varphi \in C^{1.L}[s]$ be given and let $0 < c \le a$, $d = (d_0, d_1, d_2) \in R_+^3$, $d_i \ge s_i$ for $i = 0, 1, 2$.

We consider the space $C^{1.L}_{\varphi.c}[d]$ of all functions $z\colon E^*_c \to R$ such that:

(i) $z \in C(E^*_c, R)$ and $z(t,x) = \varphi(t,x)$ on E_0,

(ii) there exist $\partial_t z$ and $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$ on E^*_c and the estimates

$$|z(t,x)| \le d_0, \quad |\partial_t z(t,x)| + \|\partial_x z(t,x)\| \le d_1,$$
$$|\partial_t z(t,x) - \partial_t z(\bar t, \bar x)| + \|\partial_x z(t,x) - \partial_x z(\bar t, \bar x)\| \le d_2[|t - \bar t| + \|x - \bar x\|],$$

are satisfied on E^*_c.

We denote by $CL(D,R)$ the set of all linear continuous functions defined on $C(D,R)$ and having its values in R^n and by $\|\cdot\|_*$ the norm in $CL(D,R)$.

We will prove that under suitable assumptions on f, G and φ, and for sufficiently small c with $0 < c \le a$, there exists a solution $\bar z$ of problem (1), (2) such that $\bar z \in C^{1.L}_{\varphi.c}[d]$.

We begin with assumptions on f.

Assumptions H$[f]$. Suppose that the function $f = (f_1, \ldots, f_n)^T\colon \Omega \to R^n$ in the variables (t,x,w), is such that:

1) $f \in C(\Omega, R^n)$ and the derivatives

$$\partial_t f(t,x,w) = (\partial_t f_1(t,x,w), \ldots, \partial_t f_n(t,x,w))^T,$$
$$\partial_x f(t,x,w) = [\partial_{x_j} f_i(t,x,w)]_{i,j=1,\ldots,n},$$

and the Fréchet derivative

$$\partial_w f(t,x,w) = (\partial_w f_1(t,x,w), \ldots, \partial_w f_n(t,x,w))^T,$$

exist for $(t,x,w) \in \Omega^{(1)}$,

2) there are $\alpha, \beta \in \Theta$ such that

$$\|f(t,x,w)\| \le \alpha(\|w\|_0) \text{ on } \Omega,$$
$$\|\partial_t f(t,x,w)\|,\ \|\partial_x f(t,x,w)\|,\ \|\partial_w f(t,x,w)\|_* \le \beta(\|w\|_{\mathrm{I}}) \text{ on } \Omega^{(1)},$$

and there is a $\gamma \in \Theta$ such that for $(t,x,w) \in \Omega^{(1.L)}$, $(\bar x, \bar w) \in R^n \times C(D,R)$ we have

$$\|\partial_x f(t,x,w) - \partial_x f(t,\bar x,\bar w)\| \le \gamma(\|w\|_{\mathrm{I.L}})[\|x - \bar x\| + \|w - \bar w\|_{\mathrm{I}}],$$
$$\|\partial_w f(t,x,w) - \partial_w f(t,\bar x,\bar w)\|_* \le \gamma(\|w\|_{\mathrm{I.L}})[\|x - \bar x\| + \|w - \bar w\|_{\mathrm{I}}].$$

Suppose that $\varphi \in C^{1.L}[s]$ and $z \in C^{1.L}_{\varphi.c}[d]$. We consider the Cauchy problem

$$\eta'(\tau) = f\left(\tau, \eta(\tau), z_{(\tau,\eta(\tau))}\right), \quad \eta(t) = x, \tag{3}$$

and denote by $g[z](\cdot,t,x) = (g_1[z](\cdot,t,x), \ldots, g_n[z](\cdot,t,x))$ its classical solution.

The function $g[z](\cdot,t,x)$ is the bicharacteristic of equation (1) corresponding to z. Write

$$P[z](\tau,t,x) = \left(\tau, g[z](\tau,t,x), z_{(\tau, g[z](\tau,t,x))}\right).$$

We prove a lemma on bicharacteristics.

Lemma 2.1. *Suppose that Assumption* H$[f]$ *is satisfied and let*

$$\varphi, \bar{\varphi} \in C^{1.L}[s], \quad z \in C^{1.L}_{\varphi.c}[d], \quad \bar{z} \in C^{1.L}_{\bar{\varphi}.c}[d],$$

be given. Then the solutions $g[z](\cdot, t, x)$ *and* $g[\bar{z}](\cdot, t, x)$ *exist on the interval* $[0, a]$ *and are unique. Moreover, the following estimates hold*

$$\|\partial_t g[z](\tau, t, x)\| \leq C, \quad \|\partial_x g[z](\tau, t, x)\| \leq C, \tag{4}$$

for $\tau \in [0, c]$, $(t, x) \in E_c$, *and*

$$\begin{aligned} &\|\partial_t g[z](s, t, x) - \partial_t g[z](s, \bar{t}, \bar{x})\| \leq Q[|t - \bar{t}| + \|x - \bar{x}\|], \\ &\|\partial_x g[z](s, t, x) - \partial_x g[z](s, \bar{t}, \bar{x})\| \leq Q[|t - \bar{t}| + \|x - \bar{x}\|] \end{aligned} \tag{5}$$

for $\tau \in [0, c]$, $(t, x), (\bar{t}, \bar{x}) \in E_c$ *and*

$$\|g[z](\tau, t, x) - g[\bar{z}](\tau, t, x)\| \leq \bar{A} \int_0^t \|z - \bar{z}\|_{(s)} \; ds, \tag{6}$$

for $\tau \in [0, c]$, $\tau \leq t$, $(t, x) \in E_c$, *where*

$$C = \max\{1, \alpha(d_0)\} \exp\{cB\}, \quad Q = \left\{(1 + C)B + c\widetilde{C}\right\} \exp[cB], \quad \bar{A} = \beta(\tilde{d}) \exp[cB]$$

and

$$B = \beta(\tilde{d})(1 + d_1), \quad \widetilde{C} = C^2 \left[\gamma(|d|)(1 + d_1 + d_2)(1 + d_1) + d_2 \beta(\tilde{d})\right]$$

and

$$\tilde{d} = d_0 + d_1, \quad |d| = d_0 + d_1 + d_2.$$

Proof. Let $z \in C^{1.L}_{\varphi.c}[d]$. The existence and uniqueness of solutions of (3) follow from the theorem on classical solutions of initial problems. From another classical theorem on differentiation of solutions with respect to the initial data it follows that the derivatives $\partial_t g[z] = (\partial_t g_1[z], \ldots, \partial_t g_n[z])^T$ and

$$\partial_x g[z] = \left[\partial_{x_j} g_i[z]\right]_{i,j=1,\ldots,n}$$

exist and fulfil the integral equations

$$\begin{aligned} &\partial_t g[z](\tau, t, x) = -f\left(t, x, z_{(t,x)}\right) + \\ &+ \int_t^\tau \left[\partial_x f\big(P[z](s, t, x)\big) + \partial_w f\big(P[z](s, t, x)\big) * (\partial_x z)_{(s, g[z](s,t,x))}\right] * \partial_t g[z](s, t, x) ds \end{aligned} \tag{7}$$

and

$$\partial_x g[z](\tau,t,x) = I +$$
$$+ \int_t^\tau \left[\partial_x f\big(P[z](s,t,x)\big) + \partial_w f\big(P[z](s,t,x)\big) * (\partial_x z)_{(s,g[z](s,t,x))}\right] * \partial_x g[z](s,t,x)ds \tag{8}$$

where I denotes the identity matrix. oreover, for $P \in \Omega$ and $(\tau, y) \in E$, we denote

$$\partial_w f(P) * (\partial_x z)_{(\tau,y)} = \big(\partial_w f_1(P), \ldots, \partial_w f_n(P)\big)^T * \big((\partial_{x_1} z)_{(\tau,y)}, \ldots, (\partial_{x_n} z)_{(\tau,y)}\big) =$$
$$= \left[\partial_w f_i(P)\, (\partial_{x_j} z)_{(\tau,y)}\right]_{i,j=1,\ldots,n} \quad \in M_{n\times n}.$$

Note that

$$\left\|\partial_w f(P) * (\partial_x z)_{(\tau,y)}\right\| \le \left\|\partial_w f(P)\right\| \cdot \left\| (\partial_x z)_{(\tau,y)}\right\|$$

for $P \in \Omega$, $(\tau, y) \in E$.

It follows from (7), (8), Assumption H$[f]$ and the definition of space $C^{1.L}_{\varphi.c}[d]$ that the functions $\partial_t g[z](\cdot,t,x)$, $\partial_x g[z](\cdot,t,x)$ satisfy integral inequalities

$$\|\partial_t g[z](\tau,t,x)\| \le \alpha(d_0) + B\left|\int_t^\tau \|\partial_t g[z](s,t,x)\|\; ds\right|,$$
$$\|\partial_x g[z](\tau,t,x)\| \le 1 + B\left|\int_t^\tau \|\partial_x g[z](s,t,x)\|\; ds\right|,$$

and from the Gronwall theorem we get (4). Then using (4) we get

$$\|\partial_t g[z](\tau,t,x) - \partial_t g[z](\tau,\bar t,\bar x)\| \le$$
$$\le \left(B + c\widetilde{C}\right)[|t-\bar t| + \|x-\bar x\|] + CB\,|t-\bar t| + B\left|\int_t^\tau \|\partial_t g[z](s,t,x) - \partial_t g[z](s,\bar t,\bar x)\|\; ds\right|$$

and

$$\|\partial_x g[z](\tau,t,x) - \partial_x g[z](\tau,\bar t,\bar x)\| \le$$
$$\le c\widetilde{C}\,[|t-\bar t| + \|x-\bar x\|] + CB\,|t-\bar t| + B\left|\int_t^\tau \|\partial_x g[z](s,t,x) - \partial_x g[z](s,\bar t,\bar x)\|\; ds\right|.$$

and from the Gronwall theorem we get

$$\|\partial_t g[z](\tau,t,x) - \partial_t g[z](\tau,\bar t,\bar x)\| \le Q_2[|t-\bar t| + \|x-\bar x\|] + Q_0\,|t-\bar t| \le$$
$$\le (Q_0+Q_2)[|t-\bar t| + \|x-\bar x\|],$$

and

$$\|\partial_x g[z](\tau,t,x)-\partial_x g[z](\tau,\bar{t},\bar{x})\| \le Q_1[|t-\bar{t}|+\|x-\bar{x}\|]+Q_0\,|t-\bar{t}| \le \\ \le (Q_0+Q_1)[|t-\bar{t}|+\|x-\bar{x}\|],$$

where

$$Q_0 = CB\exp[cB], \qquad Q_1 = c\widetilde{C}\exp[cB], \\ Q_2 = Q_1 + B\exp[cB].$$

Taking maximum on the right hand sides of the above inequalities we get (5).

Now we prove (6). The function $g[z](\tau,t,x)$ satisfies the following relation:

$$g[z](\tau,t,x) = x + \int_t^\tau f\left(s, g[z](s,t,x), z_{(s,g[z](s,t,x))}\right) ds.$$

Suppose that $\tau \le t$. Then

$$\|g[z](\tau,t,x)-g[\bar{z}](\tau,t,x)\| \le \\ \le B\left|\int_t^\tau \|g[z](s,t,x)-g[\bar{z}](s,t,x)\|\, ds\right| + \beta(\tilde{d})\int_0^t \|z-\bar{z}\|_{(s)}\, ds.$$

Again from the Gronwall inequality we obtain

$$\|g[z](\tau,t,x)-g[\bar{z}](\tau,t,x)\| \le \beta(\tilde{d})\exp[cB]\int_0^t \|z-\bar{z}\|_{(s)}\, ds.$$

This completes the proof. □

Assumptions H$[f,G]$. Suppose that Assumption H$[f]$ is satisfied and function $G\colon \Omega \to R$ in the variables (t,x,w), is such that:

1) $G \in C(\Omega,R)$ and the derivative $\partial_x G(t,x,w) = (\partial_{x_1} G(t,x,w),\dots,\partial_{x_n} G(t,x,w))$ and the Fréchet derivative $\partial_w G(t,x,w)$ exist for $(t,x,w) \in \Omega^{(1)}$,
2) for $\alpha,\beta,\gamma \in \Theta$ as in Assumption H$[f]$ there is

$$\|G(t,x,w)\| \le \alpha(\|w\|_0) \text{ on } \Omega, \\ \|\partial_x G(t,x,w)\|,\ \|\partial_w G(t,x,w)\|_* \le \beta(\|w\|_{\mathrm{I}}) \text{ on } \Omega^{(1)},$$

and for $(t,x,w) \in \Omega^{(1.L)}$, $(\bar{x},\bar{w}) \in R^n \times C(D,R)$ there is

$$\|\partial_x G(t,x,w)-\partial_x G(t,\bar{x},\bar{w})\| \le \gamma(\|w\|_{\mathrm{I.L}})[\|x-\bar{x}\|+\|w-\bar{w}\|_{\mathrm{I}}], \\ \|\partial_w G(t,x,w)-\partial_w G(t,\bar{x},\bar{w})\|_* \le \gamma(\|w\|_{\mathrm{I.L}})[\|x-\bar{x}\|+\|w-\bar{w}\|_{\mathrm{I}}].$$

We define the operator W on $C^{1.L}_{\varphi.c}[d]$ by the formula

$$\begin{aligned} W[z](t,x) = \varphi(0, g[z](0,t,x)) + \int_0^t G\big(P[z](s,t,x)\big)\, ds \quad &\text{for } (t,x) \in E_c, \\ W[z](t,x) = \varphi(t,x) \quad &\text{for } (t,x) \in E_0. \end{aligned} \tag{9}$$

Remark 2.2. *The right-hand side of* (9) *is obtained in the following way. We consider* (1) *along bicharacteristics:*

$$\partial_t z(\tau, g[z](\tau,t,x)) + \partial_x z(\tau, g[z](\tau,t,x)) * f\left(\tau, g[z](\tau,t,x), z_{(\tau, g[z](\tau,t,x))}\right) = \\ = G\left(\tau, g[z](\tau,t,x), z_{(\tau, g[z](\tau,t,x))}\right)$$

from which, using (3), *we get*

$$\frac{d}{dt} z(\tau, g[z](\tau,t,x)) = G\left(\tau, g[z](\tau,t,x), z_{(\tau, g[z](\tau,t,x))}\right).$$

By integrating the above equation with respect to τ, we get the right-hand side of (9).

Assumptions H$[\varphi, c, d]$. Suppose that:

1) the constants c, $d = (d_0, d_1, d_2)$ satisfy the conditions

$$\begin{gathered} d_0 \geq s_0 + c\,\alpha(d_0), \\ d_1 \geq 2 s_1 C + \alpha(d_0) + 2cCB, \\ d_2 \geq B + 2\left[s_1 Q + s_2 C^2 + B(C + cQ) + c\widetilde{C}\right] \end{gathered}$$

2) the following consistency condition holds true for $x \in R^n$:

$$\partial_t \varphi(0,x) + \sum_{i=1}^n f_i(0, x, \varphi_{(0,x)}) \partial_{x_i} \varphi(0,x) = G(0, x, \varphi_{(0,x)}). \tag{10}$$

Let us denote

$$(\partial_x z)_{(\tau,y)} * \partial_x g[z](\tau,t,x) = \Big((\partial_{x_1} z)_{(\tau,y)}, \dots, (\partial_{x_n} z)_{(\tau,y)}\Big) * \partial_x g[z](\tau,t,x) = \\ = \left(\sum_{i=1}^n \partial_{x_1} g_i[z](\tau,t,x) \cdot (\partial_{x_i} z)_{(\tau,y)}, \dots, \sum_{i=1}^n \partial_{x_n} g_i[z](\tau,t,x) \cdot (\partial_{x_i} z)_{(\tau,y)}\right) \\ \in\ C(D, R^n)$$

for $(\tau, y) \in E$ and $(\tau, t, x) \in [0,a] \times E$. Moreover, for $P \in \Omega$ and $(\tau, y) \in E$, we denote

$$\partial_w G(P)\, (\partial_x z)_{(\tau,y)} = \big(\partial_w G(P)\, (\partial_{x_1} z)_{(\tau,y)}, \dots, \partial_w G(P)\, (\partial_{x_n} z)_{(\tau,y)}\big) \quad \in R^n.$$

Lemma 2.3. *Suppose that Assumptions* H$[f,G]$, H$[\varphi,c,d]$ *are satisfied. Then the operator* W *maps* $C^{1.L}_{\varphi.c}[d]$ *into itself.*

Proof. Let $z \in C^{1.L}_{\varphi.c}[d]$. Write $V_0 = \partial_t W[z]$ and $V[z] = \partial_x W[z]$. From (9) it follows that

$$\begin{aligned} V_0(t,x) = {} & \partial_x\varphi(0,g[z](0,t,x)) * \partial_t g[z](0,t,x) + G\left(t,x,z_{(t,x)}\right) + \\ & + \int_0^t \partial_x G(P[z](s,t,x)) * \partial_t g[z](s,t,x)\,ds + \\ & + \int_0^t \partial_w G(P[z](s,t,x))\,(\partial_x z)_{(s,g[z](s,t,x))} * \partial_t g[z](s,t,x)\,ds \end{aligned} \tag{11}$$

and

$$\begin{aligned} V[z](t,x) = {} & \partial_x\varphi(0,g[z](0,t,x)) * \partial_x g[z](0,t,x) + \\ & + \int_0^t \partial_x G(P[z](s,t,x)) * \partial_x g[z](s,t,x)\,ds + \\ & + \int_0^t \partial_w G(P[z](s,t,x))\,(\partial_x z)_{(s,g[z](s,t,x))} * \partial_x g[z](s,t,x)\,ds. \end{aligned} \tag{12}$$

It follows from the above integral equations that

$$|W[z](t,x)| \le s_0 + c\alpha(d_0), \tag{13}$$

$$|\partial_t W[z](t,x)| \le s_1 C + \alpha(d_0) + cCB \tag{14}$$

and

$$\|\partial_x W[z](t,x)\| \le s_1 C + cCB. \tag{15}$$

From (13) and adding inequalities (14), (15) we get

$$|W[z](t,x)| \le d_0, \qquad |V_0(t,x)| + \|V[z](t,x)\| \le d_1 \tag{16}$$

for $(t,x) \in E_c$. It follows from (11) that for (t,x), $(\bar t,\bar x) \in E_c$ there is

$$\begin{aligned} & |V_0(t,x) - V_0(\bar t,\bar x)| \le \\ & \quad \le |\partial_x\varphi(0,g[z](0,t,x)) * \partial_t g[z](0,t,x) - \partial_x\varphi(0,g[z](0,\bar t,\bar x)) * \partial_t g[z](0,\bar t,\bar x)| + \\ & \qquad + \left|G\left(t,x,z_{(t,x)}\right) - G\left(\bar t,\bar x,z_{(\bar t,\bar x)}\right)\right| + \\ & + \int_0^t |\partial_x G(P[z](s,t,x)) * \partial_t g[z](s,t,x) - \partial_x G(P[z](s,\bar t,\bar x)) * \partial_t g[z](s,\bar t,\bar x)|\,ds + \end{aligned}$$

$$+\int_0^t \Big|\partial_w G(P[z](s,t,x))\Big[(\partial_x z)_{(s,g[z](s,t,x))} * \partial_t g[z](s,t,x)\Big] + \\ -\partial_w G(P[z](s,\bar t,\bar x))\Big[(\partial_x z)_{(s,g[z](s,\bar t,\bar x))} * \partial_t g[z](s,\bar t,\bar x)\Big]\Big|\, ds + \\ +\left|\int_t^{\bar t}\Big[\|\partial_x G(P[z](s,\bar t,\bar x)) * \partial_t g[z](s,\bar t,\bar x)\| + \\ + \big\|\partial_w G(P[z](s,\bar t,\bar x)) * (\partial_x z)_{(s,g[z](s,\bar t,\bar x))} * \partial_t g[z](s,\bar t,\bar x)\big\|\Big]\, ds\right|.$$

From the above inequality, Assumption H$[f,G]$ and Lemma 2.1 it follows that

$$|V_0(t,x) - V_0(\bar t,\bar x)| \le \\ \le \Big[s_1 Q + s_2 C^2 + B(1+cQ) + c\widetilde{C}\Big]\cdot[|t-\bar t| + \|x-\bar x\|] + BC\,|t-\bar t|,$$

for (t,x), $(\bar t,\bar x)\in E_c$ and, consequently,

$$|V_0(t,x) - V_0(\bar t,\bar x)| \le \\ \le \Big[s_1 Q + s_2 C^2 + B(1+C+cQ) + c\widetilde{C}\Big]\cdot[|t-\bar t| + \|x-\bar x\|] \quad (17)$$

for (t,x), $(\bar t,\bar x)\in E_c$. Now we write the Lipschitz condition for the function $V[z]$. From (12) it follows that

$$|V[z](t,x) - V[z](\bar t,\bar x)| \le \\ \le \Big|\partial_x\varphi(0,g[z](0,t,x)) * \partial_x g[z](0,t,x) - \partial_x\varphi(0,g[z](0,\bar t,\bar x)) * \partial_x g[z](0,\bar t,\bar x)\Big| + \\ + \int_0^t \Big|\partial_x G(P[z](s,t,x)) * \partial_x g[z](s,t,x) - \partial_x G(P[z](s,\bar t,\bar x)) * \partial_x g[z](s,\bar t,\bar x)\Big|\, ds + \\ + \int_0^t \Big|\partial_w G(P[z](s,t,x))\Big[(\partial_x z)_{(s,g[z](s,t,x))} * \partial_x g[z](s,t,x)\Big] + \\ - \partial_w G(P[z](s,\bar t,\bar x))\Big[(\partial_x z)_{(s,g[z](s,\bar t,\bar x))} * \partial_x g[z](s,\bar t,\bar x)\Big]\Big|\, ds + \\ + \left|\int_t^{\bar t}\Big[\|\partial_x G(P[z](s,\bar t,\bar x)) * \partial_x g[z](s,\bar t,\bar x)\| + \\ + \big\|\partial_w G(P[z](s,\bar t,\bar x)) * (\partial_x z)_{(s,g[z](s,\bar t,\bar x))} * \partial_x g[z](s,\bar t,\bar x)\big\|\Big]\, ds\right|$$

for (t,x), $(\bar t,\bar x)\in E_c$.

In a way similar to that used to prove (17) we get

$$|V[z](t,x) - V[z](\bar{t},\bar{x})| \leq$$
$$\leq \Big[s_1 Q + s_2 C^2 + B(C + cQ) + c\widetilde{C}\Big] \cdot [|t-\bar{t}| + \|x-\bar{x}\|] \quad (18)$$

for (t,x), $(\bar{t},\bar{x}) \in E_c$. Adding inequalities (17) and (18) we get

$$|V_0(t,x) - V_0(\bar{t},\bar{x})| + \|V[z](t,x) - V[z](\bar{t},\bar{x})\| \leq$$
$$\leq \Big\{B + 2\Big[s_1 Q + s_2 C^2 + B(C + cQ) + c\widetilde{C}\Big]\Big\} \cdot [|t-\bar{t}| + \|x-\bar{x}\|] \quad (19)$$

for (t,x), $(\bar{t},\bar{x}) \in E_c$. From (19) and Assumption H$[\varphi, c, d]$ we get

$$|V_0(t,x) - V_0(\bar{t},\bar{x})| + \|V[z](t,x) - V[z](\bar{t},\bar{x})\| \leq d_2[|t-\bar{t}| + \|x-\bar{x}\|] \quad (20)$$

for (t,x), $(\bar{t},\bar{x}) \in E_c$. We can see from (16) and (20) that the function $W[z]$ satisfies condition (ii) from the definition of $C^{1.L}_{\varphi.c}[d]$. Moreover, from (9) it follows that

$$W[z](t,x) = \varphi(t,x) \text{ on } E_c.$$

Now we prove that $W[z] \in C^1(E_c^*, R)$. From (11) and (7) it follows that

$$V_0(0,x) = \partial_x\varphi(0,x) * \partial_t g[z](0,0,x) + G(0,x,z_{(0,x)}) =$$
$$= -\partial_x\varphi(0,x) * f(0,x,z_{(0,x)}) + G(0,x,z_{(0,x)}) =$$
$$= -\partial_x\varphi(0,x) * f(0,x,\varphi_{(0,x)}) + G(0,x,\varphi_{(0,x)}).$$

Then from consistency condition (10) we get

$$V_0(0,x) = \partial_t\varphi(0,x).$$

From (12) and (8) it follows that

$$V[z](0,x) = \partial_x\varphi(0,x) * \partial_x g[z](0,0,x) = \partial_x\varphi(0,x) * I = \partial_x\varphi(0,x).$$

Then the function $W[z]$ satisfies condition (i) of the definition of the space $C^{1.L}_{\varphi.c}[d]$. This completes the proof of Lemma 2.3. □

Theorem 2.4. *Suppose that* $\varphi \in C^{1.L}s$ *and Assumptions* H$[f,G]$, H$[\varphi,c,d]$ *are satisfied. Then there exists exactly one solution* $\bar{z} \in C^{1.L}_{\varphi.c}[d]$ *of problem* (1), (2).

If $\bar{\varphi} \in C^{1.L}s$ *and* $\bar{v} \in C^{1.L}_{\bar{\varphi}.c}[d]$ *is a solution of equation* (1) *with the initial boundary condition* $z(t,x) = \bar{\varphi}(t,x)$ *on* E_0 *then there is* $\Lambda_c \in R_+$ *such that*

$$\|v - \bar{v}\|_t \leq \Lambda_c\Big[\,\|\varphi - \bar{\varphi}\|_{(0)}\Big], \quad 0 \leq t \leq c. \quad (21)$$

Proof. We prove that there exists exactly one $\bar{z} \in C^{1.L}_{\varphi.c}[d]$ satisfying the equation $z = W[z]$. Lemma 2.3 shows that $W\colon C^{1.L}_{\varphi.c}[d] \to C^{1.L}_{\varphi.c}[d]$. It follows that there is an $A > 0$ such that for $z, \tilde{z} \in C^{1.L}_{\varphi.c}[d]$ there holds

$$\|W[z](t,x) - W[\bar{z}](t,x)\| \leq A \int_0^t \|z - \tilde{z}\|_{(s)} \, ds. \tag{22}$$

Now we define the norm in the space $C^{1.L}_{\varphi.c}[d]$ as follows

$$\|z\|_\lambda = \max\{\|z(t,x)\| \, e^{-\lambda t} \colon \ (t,x) \in E_c\},$$

where $\lambda > A$. It is easy to see that $(C^{1.L}_{\varphi.c}[d], \|\cdot\|_\lambda)$ is a Banach space. Now we prove that there exists $q \in [0,1)$ such that

$$\|W[z] - W[\tilde{z}]\|_\lambda \leq q \, \|z - \tilde{z}\|_\lambda \ \text{ for } z, \tilde{z} \in C^{1.L}_{\varphi.c}[d]. \tag{23}$$

According to (22), there is

$$\begin{aligned}
&\|W[z](t,x) - W[\tilde{z}](t,x)\| \leq \\
&\qquad \leq A \int_0^t \|z - \tilde{z}\|_{(s)} \, ds = A \int_0^t \|z - \tilde{z}\|_{(s)} \, e^{-\lambda s} e^{\lambda s} ds \leq \\
&\qquad \leq A \, \|z - \tilde{z}\|_\lambda \int_0^t e^{\lambda s} ds = \frac{A}{\lambda} \, \|z - \tilde{z}\|_\lambda \, (e^{\lambda t} - 1) \leq \frac{A}{\lambda} \, \|z - \tilde{z}\|_\lambda \, e^{\lambda t}
\end{aligned}$$

for $(t,x) \in E_c$. Then

$$\|W[z](t,x) - W[\tilde{z}](t,x)\| \, e^{-\lambda t} \leq \frac{A}{\lambda} \, \|z - \tilde{z}\|_\lambda \,, \quad (t,x) \in E_c.$$

It follows that estimate (23) holds with $q = A\lambda^{-1}$. By the Banach fixed point theorem, there exists the unique fixpoint of W. Denoting this fixpoint by $\bar{z}$ we prove that it is a solution of equation (1). For $(t,x) \in E_c$, there is

$$\bar{z}(t,x) = \varphi(0, g[\bar{z}](0,t,x)) + \int_0^t G(s, g[\bar{z}](s,t,x), \bar{z}_{(s, g[\bar{z}](s,t,x))}) ds.$$

For a given $x \in R^n$, let us put $y = g[\bar{z}](0,t,x)$. It follows from Lemma 2.1 that $g[\bar{z}](s,t,x) = g[\bar{z}](s,0,y)$ for $s,t \in [0,c]$ and $x = g[\bar{z}](t,0,y)$. Then we get

$$\bar{z}(t, g[\bar{z}](t,0,y)) = \varphi(0,y) + \int_0^t G\left(s, g[\bar{z}](s,0,y), \bar{z}_{(s, g[\bar{z}](s,0,y))}\right) ds. \tag{24}$$

The relations $y = g[\bar{z}](0, t, x)$ and $x = g[\bar{z}](t, 0, y)$ are equivalent for $x, y \in R^n$. By differentiating (24) with respect to t and putting again $x = g[\bar{z}](t, 0, y)$ we conclude that $\bar{z}$ satisfies (1). Since $\bar{z}$ satisfies initial condition (2), it is a solution of our problem.

Now we prove relation (21). The function $\bar{v}$ satisfies integral functional equation

$$z(t, x) = W[z](t, x)$$

and initial condition (2) with $\bar{\varphi}$ instead of φ. It follows easily that there are Λ_0, $\Lambda_1 \in R_+$ such that the integral inequality

$$\|v - \bar{v}\|_t \leq \Lambda_0 \Big[\|\varphi - \bar{\varphi}\|_{(0)} \Big] + \Lambda_1 \int_0^t \Big[\|v - \bar{v}\|_{(\tau)} \Big] d\tau, \quad 0 \leq t \leq c,$$

is satisfied. Using the Gronwall inequality, we obtain (21) with $\Lambda_c = \Lambda_0 \exp\Big(\Lambda_1 c\Big)$. This proves the Theorem. □

3. DIFFERENTIAL EQUATIONS WITH DEVIATED VARIABLES

Suppose that $\alpha_0\colon [0, a] \to R$, $\alpha'\colon E \to R^n$ are given functions and that conditions

$$-h_0 \leq \alpha_0(t) - t \leq 0, \quad -h \leq \alpha'(t, x) - x \leq h, \quad (t, x) \in E. \tag{25}$$

are satisfied. We consider operators f, G defined by

$$\begin{aligned} f(t, x, w) &= \widetilde{f}(t, x, w(\alpha_0(t) - t, \alpha'(t, x) - x)), \\ G(t, x, w) &= \widetilde{G}(t, x, w(\alpha_0(t) - t, \alpha'(t, x) - x)), \end{aligned} \tag{26}$$

where $\widetilde{f}\colon E \times R \to R^n$, $\widetilde{G}\colon E \times R \to R$, $(t, x, w) \in \Omega$.

In this case, (1) is equivalent to the differential equation with deviated variables

$$\partial_t z(t, x) + \sum_{i=1}^{n} \widetilde{f}_i(t, x, z(\alpha(t, x)))\partial_{x_i} z(t, x) = \widetilde{G}(t, x, z(\alpha(t, x))), \tag{27}$$

where $\alpha(t, x) = (\alpha_0(t), \alpha'(t, x))$. Now we formulate our existence result for problem (27), (2).

Assumptions H$[\widetilde{f}]$. Suppose that the function $\widetilde{f}\colon E \times R \to R^n$ in the variables (t, x, p) satisfies the conditions:

1) $\widetilde{f} \in C(E \times R, R^n)$ and there is $B_0 \in \Theta$ such that

$$\Big\| \widetilde{f}(t, x, p) \Big\| \leq B_0(|p|) \text{ on } E \times R$$

2) the partial derivatives

$$\partial_t \widetilde{f}(Q), \quad (\partial_{x_1} \widetilde{f}(Q), \ldots, \partial_{x_n} \widetilde{f}(Q)) = \partial_x \widetilde{f}(Q), \quad \partial_p \widetilde{f}(Q), \quad Q = (t, x, p),$$

exist for $(t, x, p) \in E \times R$ and there is a $\bar{B} \in R_+$ such that

$$\left\|\partial_t \widetilde{f}(t, x, p)\right\|, \quad \left\|\partial_x \widetilde{f}(t, x, p)\right\|, \quad \left\|\partial_p \widetilde{f}(t, x, p)\right\| \leq \bar{B}$$

for $(t, x, p) \in E \times R$,

3) there is $\bar{C} \in R_+$ such that the functions $\partial_t \widetilde{f}$, $\partial_x \widetilde{f}$, $\partial_p \widetilde{f}$ satisfy the Lipschitz condition with respect to $(x, p) \in R^n \times R$.

Assumptions H$[\widetilde{f}, \widetilde{G}]$. Suppose that Assumption H$[\widetilde{f}]$ is satisfied and the function $\widetilde{G}\colon E \times R \to R$ in the variables (t, x, p) satisfies the conditions:

1) $\widetilde{G} \in C(E \times R, R)$ and the partial derivatives

$$(\partial_{x_1} \widetilde{G}(Q), \ldots, \partial_{x_n} \widetilde{G}(Q)) = \partial_x \widetilde{G}(Q), \;\; \partial_p \widetilde{G}(Q), \;\; Q = (t, x, p),$$

exist for $(t, x, p) \in E \times R$,

2) for $B_0 \in \Theta$, $\bar{B} \in R_+$, as in Assumption H$[\widetilde{f}]$, there is

$$\left|\widetilde{G}(t, x, p)\right| \leq B_0(|p|), \quad \left\|\partial_x \widetilde{G}(t, x, p)\right\|, \quad \left\|\partial_p \widetilde{G}(t, x, p)\right\| \leq \bar{B}$$

for $(t, x, p) \in E \times R$ and the functions $\partial_x \widetilde{G}$, $\partial_p \widetilde{G}$ satisfy the Lipschitz condition with respect to $(x, p) \in R^n \times R$ with Lipschitz constant $\bar{C} \in R_+$ as in Assumption H$[\widetilde{f}]$.

Assumptions H$[\alpha]$. Suppose that the functions $\alpha_0\colon [0, a] \to R$, $\alpha'\colon E \to R^n$ are such that:

1) $0 \leq \alpha_0(t) \leq t$ for $t \in [0, a]$,

2) the derivatives $\frac{d}{dt}\alpha_0(t)$, $\partial_t \alpha'(t, x)$, $\partial_x \alpha'(t, x)$ exist for $(t, x) \in E$ and there is $\bar{r}_0 \in R_+$ such that

$$\left|\frac{d}{dt}\alpha_0(t)\right| \leq r_0, \quad \|\partial_t \alpha'(t, x)\| \leq r_0, \quad \left\|\left[\partial_x \alpha'(t, x)\right]^T\right\| \leq r_0,$$
$$\left\|\left[\partial_x \alpha'(t, x) - \partial_x \alpha'(t, \bar{x})\right]^T\right\| \leq r_1 \|x - \bar{x}\|$$

for $(t, x), (t, \bar{x}) \in E$.

Theorem 3.1. *Suppose that Assumptions* $\mathrm{H}[\widetilde{f},\widetilde{G}]$, $\mathrm{H}[\alpha]$ *are satisfied and* $\varphi \in C^{1.L}s$ *and that*

$$\partial_t \varphi(0,x) = \sum_{i=1}^{n} \widetilde{f}_i(0,x,\varphi(\alpha(0,x)))\partial_{x_i}\varphi(0,x) = \widetilde{G}\big(0,x,\varphi(\alpha(0,x))\big) \tag{28}$$

for $x \in R^n$.

Then there are a $c \in (0,a]$ *and* $v\colon E_c^* \to R$ *such that* v *is a solution of* (27), (2).

Proof. Write

$$\tilde{\alpha}_0(t) = \alpha_0(t) - t, \quad \tilde{\alpha}'(t,x) = \alpha'(t,x) - x, \quad (t,x) \in E$$

and $\tilde{\alpha}(t,x) = (\tilde{\alpha}_0(t), \tilde{\alpha}'(t,x))$. Then the operators f, G are defined by

$$f(t,x,w) = \widetilde{f}(t,x,w(\tilde{\alpha}(t,x))), \quad G(t,x,w) = \widetilde{G}(t,x,w(\tilde{\alpha}(t,x))), \quad (t,x,w) \in \Omega.$$

We see at once that

$$\begin{aligned}
\partial_t f(t,x,w) &= \partial_t \widetilde{f}(t,x,w(\tilde{\alpha}(t,x))) + \\
&\quad + \partial_p \widetilde{G}(t,x,w(\tilde{\alpha}(t,x)))\Big[\partial_t w(\tilde{\alpha}(t,x))\frac{d}{dt}\tilde{\alpha}_0(t) + \partial_x w(\tilde{\alpha}(t,x)) * \partial_t \tilde{\alpha}(t,x)\Big] \\
\partial_x f(t,x,w) &= \partial_x \widetilde{f}(t,x,w(\tilde{\alpha}(t,x))) + \partial_p \widetilde{f}(t,x,w(\tilde{\alpha}(t,x)))\, \partial_x w(\tilde{\alpha}(t,x)) * \partial_x \tilde{\alpha}'(t,x), \\
&\qquad \partial_w f(t,x,w)\bar{w} = \widetilde{f}(t,x,w(\tilde{\alpha}(t,x)))\bar{w}(\tilde{\alpha}(t,x))
\end{aligned}$$

and

$$\begin{aligned}
\partial_x G(t,x,w) &= \partial_x \widetilde{G}(t,x,w(\tilde{\alpha}(t,x))) + \partial_p \widetilde{G}(t,x,w(\tilde{\alpha}(t,x)))\, \partial_x w(\tilde{\alpha}(t,x)) * \partial_x \tilde{\alpha}'(t,x), \\
&\qquad \partial_w G(t,x,w)\bar{w} = \widetilde{G}(t,x,w(\tilde{\alpha}(t,x)))\bar{w}(\tilde{\alpha}(t,x)),
\end{aligned}$$

where $(t,x,w) \in \Omega^{(1)}$ and $\bar{w} \in C(D,R)$.

It is clear that

$$\begin{aligned}
\|\partial_t f(t,x,w)\| &\le \bar{B}[1 + (1+r_0)\,\|w\|_{\mathrm{I}}], \\
\|\partial_x f(t,x,w)\| &\le \bar{B}[1 + (1+r_0)\,\|w\|_{\mathrm{I}}]
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_w f(t,x,w)\|_* &\le \bar{B}, \\
\|\partial_w G(t,x,w)\|_* &\le \bar{B}, \\
\|\partial_x G(t,x,w)\| &\le \bar{B}[1 + (1+r_0)\,\|w\|_{\mathrm{I}}].
\end{aligned}$$

We conclude from Assumptions H$[\widetilde{f},\widetilde{G}]$ and H$[\alpha]$ that

$$\begin{aligned}
&\|\partial_x f(t,x,w)-\partial_x f(t,\bar{x},\bar{w})\| \le \\
&\quad\le \Big[\bar{C}[1+(1+r_0)\,\|w\|_{\mathrm{I.L}}]^2+\bar{B}[r_1+(1+r_0)^2]\,\|w\|_{\mathrm{I.L}}\ \Big]\,\|x-\bar{x}\|+ \\
&\qquad\quad+[\bar{C}+(1+r_0)(\bar{B}+\bar{C}\,\|w\|_{\mathrm{I.L}})]\ \|w-\bar{w}\|_{\mathrm{I}}\,, \\
&\|\partial_w f(t,x,w)-\partial_w f(t,\bar{x},\bar{w})\|_* \le \bar{C}\Big[(1+(1+r_0)\,\|w\|_{\mathrm{I.L}})\,\|x-\bar{x}\|+\|w-\bar{w}\|_{\mathrm{I}}\Big], \\
&\|\partial_x G(t,x,w)-\partial_x G(t,\bar{x},\bar{w})\| \le \\
&\quad\le \Big[\bar{C}[1+(1+r_0)\,\|w\|_{\mathrm{I.L}}]^2+\bar{B}[r_1+(1+r_0)^2]\,\|w\|_{\mathrm{I.L}}\ \Big]\,\|x-\bar{x}\|+ \\
&\qquad\quad+[\bar{C}+(1+r_0)(\bar{B}+\bar{C}\,\|w\|_{\mathrm{I.L}})]\ \|w-\bar{w}\|_{\mathrm{I}}\,, \\
&\|\partial_w G(t,x,w)-\partial_w G(t,\bar{x},\bar{w})\|_* \le \bar{C}\Big[(1+(1+r_0)\,\|w\|_{\mathrm{I.L}})\,\|x-\bar{x}\|+\|w-\bar{w}\|_{\mathrm{I}}\Big].
\end{aligned}$$

It follows that all the assumptions of Theorem 2.4 are satisfied and the assertion follows.

Remark 3.2. *Suppose that the functions* $\widetilde{f}\colon E\times R\to R^n$, $\widetilde{G}\colon E\times R\to R$ *are given and the operators* f,G *are defined by*

$$f(t,x,w)=\widetilde{f}(t,x,\int_D w(\tau,s)ds d\tau),$$
$$G(t,x,w)=\widetilde{G}(t,x,\int_D w(\tau,s)ds d\tau).$$

Then (1) *reduces to the differential integral equation*

$$\partial_t z(t,x)+\sum_{i=1}^n \widetilde{f}_i(t,x,\int_D z(t+\tau,x+s)dsd\tau)\,\partial_{x_i} z(t,x)= \\ =\widetilde{G}(t,x,\int_D z(t+\tau,x+s)\,ds\,d\tau). \tag{29}$$

The existence result for problem (29), (2) *can be easily deduced from Theorem 2.4.*

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