Mustapha Chellali

BOUNDS ON THE 2-DOMINATION NUMBER IN CACTUS GRAPHS

Abstract. A 2-dominating set of a graph G is a set D of vertices of G such that every vertex not in S is dominated at least twice. The minimum cardinality of a 2-dominating set of G is the 2-domination number $\gamma_2(G)$. We show that if G is a nontrivial connected cactus graph with $k(G)$ even cycles ($k(G) \geq 0$), then $\gamma_2(G) \geq \gamma_t(G) - k(G)$, and if G is a graph of order n with at most one cycle, then $\gamma_2(G) \geqslant (n + \ell - s)/2$ improving Fink and Jacobson's lower bound for trees with $\ell > s$, where $\gamma_t(G)$, ℓ and s are the total domination number, the number of leaves and support vertices of G, respectively. We also show that if T is a tree of order $n \geqslant 3$, then $\gamma_2(T) \leqslant \beta(T) + s - 1$, where $\beta(T)$ is the independence number of T.

Keywords: 2-domination number, total domination number, independence number, cactus graphs, trees.

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1. INTRODUCTION

Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and edge set $E(G)$. The *degree* of a vertex v, $\deg_G(v)$, is the number of vertices adjacent to v. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. If u is a support vertex, then L_u will denote the set of leaves attached at u. A graph G is called a *cactus graph* if each edge of G is contained in at most one cycle. A cactus graph having one cycle is called a *unicycle graph* and a connected cactus graph with no cycles is called a *tree*. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with p and q leaves attached at each support vertex, respectively, is denoted by $S_{p,q}$. For a graph G we denote by $n(G)$, $\ell(G)$ and $s(G)$ the number of vertices, leaves and support vertices of G, respectively (we use n, ℓ and s if there is no ambiguity).

We are interested in a variation of domination in graphs, called 2-domination. A subset D of $V(G)$ is a 2-*dominating set* if every vertex not in S is adjacent to at least 2 vertices of D. The 2-*domination number* $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set of G. Note that every graph G has a 2-dominating set since $V(G)$ is such a set. The concept of 2-domination was introduced by Fink and Jacobson [5, 6], and studied for example in [1, 2]. The *independence number* $\beta(G)$ of a graph G is the maximum cardinality among the independent sets of vertices of G and the *total domination number* $\gamma_t(G)$ of a graph G is the minimum cardinality of a dominating set whose induced subgraph contains no isolated vertices. For more details on domination and its variations see the books of Haynes, Hedetniemi and Slater [8, 9].

In this paper we show that if G is a nontrivial connected cactus graph with $k(G)$ even cycles ($k(G) \geq 0$), then $\gamma_2(G) \geq \gamma_t(G) - k(G)$, and if G is a graph of order n with at most one cycle, then $\gamma_2(G) \geqslant (n + \ell - s)/2$. Finally, we show that every tree T of order at least three satisfies $\gamma_2(T) \leqslant \beta(T) + s - 1$.

2. LOWER BOUNDS

Before presenting our main results, we make a couple of straightforward observations.

Observation 1. *Every* 2*-dominating set of a graph* G *contains every leaf.*

Observation 2. *Let* T *be a tree obtained from a nontrivial tree* T' *by adding a star* $K_{1,p}$ *with the center vertex* v *attached by an edge* vw *at a vertex* w *of* T'. *Then:*

1) $\gamma_2(T') \leqslant \gamma_2(T) - |L_v|$, *with equality if* $p \geqslant 2$ *or* w *is a leaf in* T'.
2) $\beta(T') = \beta(T) - |L_v|$.

Proof. 1) Let D be a $\gamma_2(T)$-set. By Observation 1, D contains L_v and without loss of generality $v \notin D$ (else replace v by w in D). Thus $D - L_v$ is a 2-dominating set of T' and hence $\gamma_2(T') \leqslant \gamma_2(T) - |L_v|$. Now if $p \geqslant 2$, that is v is adjacent to at least two leaves, then every $\gamma_2(T')$-set can be extended to a 2-dominating set of T by adding L_v. So $\gamma_2(T) \leqslant \gamma_2(T') + |L_v|$, which leads the equality. Assume now that $p = 1$ and w is a leaf of T'. By Observation 1, w is in every $\gamma_2(T')$-set D', so D' is extended to a 2-dominating set of T by adding the leaf neighbor of v. Therefore, $\gamma_2(T) \leqslant \gamma_2(T') + 1$, implying the equality $\gamma_2(T') = \gamma_2(T) - 1$.

2) Obvious. □

In [7], Haynes *et al.* showed that the 2-domination number is bounded from below by the total domination number for every nontrivial tree.

Theorem 3 (Haynes *et al.* [7]). *For every nontrivial tree,* $\gamma_2(T) \geq \gamma_t(T)$.

Below, we extend this result onto cactus graphs. The total domination and 2-domination numbers of a cycle are easy to compute.

Observation 4. *For a cycle C_n on $n \geq 3$ vertices:*

i) $\gamma_t(C_n) = n/2$ *if* $n \equiv 0(mod\,4)$ *and* $\gamma_t(C_n) = \lfloor n/2 \rfloor + 1$ *otherwise,*

ii) $\gamma_2(C_n = \lceil n/2 \rceil$.

Theorem 5. *If G is a nontrivial connected cactus graph with $k(G)$ even cycles ($k(G) \geq 0$), then $\gamma_2(G) \geq \gamma_t(G) - k(G)$, and this bound is sharp.*

Proof. If G is a tree, then $k(G) = 0$ and by Theorem 3 the result is valid. If G is a cycle C_n, then by Observation 4, the result holds. Thus we assume that G is neither a tree nor a cycle C_n. Among all connected cactus graphs with $k(G)$ even cycles that do not satisfy the result, let G be one which contains as few vertices and edges as possible. Let S be a $\gamma_2(G)$-set. Assume first that there are two adjacent vertices x, y on some cycle such that x, y are both in S or both not in S. Consider the spanning graph G' obtained by removing the edge xy. Then S is a 2-dominating set of G' and hence $\gamma_2(G) = |S| \geq \gamma_2(G')$. Also $\gamma_t(G') \geq \gamma_t(G)$, since every total dominating set of G' is a total dominating set of G. Now G' satisfies the result and so $\gamma_2(G') \geq \gamma_t(G') - k(G')$. Since $k(G) \geq k(G')$, it follows that $\gamma_2(G) \geq \gamma_2(G') \geq \gamma_t(G') - k(G') \geq \gamma_t(G) - k(G)$, a contradiction.

Thus we assume that all vertices on the cycles are contained alternately in S. This implies that G contains no odd cycle. Let u, v be two adjacent vertices on an even cycle such that $u \in S$ and $v \notin S$. Let G'' be the spanning graph of G obtained by removing the edge uv. Then $S \cup \{v\}$ is a 2-dominating set of G'' and so $\gamma_2(G'') \leq |S| + 1$. There also is $\gamma_t(G'') \geq \gamma_t(G)$ and $k(G) = k(G'') + 1$. Now since G'' satisfies $\gamma_2(G'') \geq \gamma_t(G'') - k(G'')$, we obtain $\gamma_2(G) + 1 \geq \gamma_2(G'') \geq \gamma_t(G'') - k(G'') \geq \gamma_t(G) - k(G) + 1$. Therefore, $\gamma_2(G) \geq \gamma_t(G) - k(G)$, a contradiction.

That this bound is sharp may be seen by considering the cactus graph G_q ($q \geq 1$) formed from q path P_5, each one with the center vertex v_i where $1 \leq i \leq q$ and q cycle C_6 by adding edges between all center vertices so that the subgraph induced by the center vertices is a path P_q. Then we identify a vertex of a cycle C_6 with one leaf of each path P_5. See Figure 1 for an example of G_3. For G_q, $\gamma_2(G_q) = 5q$, $\gamma_t(G_q) = 6q$ and $k(G_q) = q$. □

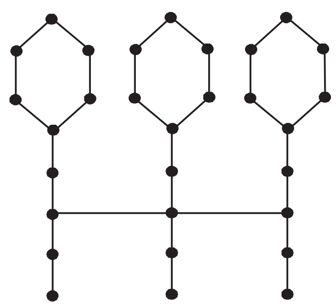

Fig. 1. The graph G_3

In [5], Fink and Jacobson have established a lower bound on the 2-domination number for every tree in term of its order.

Theorem 6 ([5]). *If T is a tree of order n, then $\gamma_2(T) \geqslant (n+1)/2$.*

Next we give a lower bound for the 2-domination number in trees that improves Fink and Jacobson's one if $\ell > s$.

Lemma 7. *If T is a tree of order n with ℓ leaves and s support vertices, then $\gamma_2(T) \geqslant (n+\ell-s)/2$, and this bound is sharp.*

Proof. We proceed by induction on the order of T. If $\mathrm{diam}(T) \in \{0,1\}$, then the result is valid. If $\mathrm{diam}(T) = 2$, then T is a star $K_{1,p}$ ($p \geqslant 2$), where $\gamma_2(T) = p$ and $(n+\ell-s)/2 = p$, so the result is valid. If $\mathrm{diam}(T) = 3$, then T is a vdouble star $S_{p,q}$, where $\gamma_2(T) = p+q$ if $\min\{p,q\} \geqslant 2$ and $\gamma_2(T) = 2+\max\{p,q\}$ otherwise. Thus again the result is valid. Assume that for every tree T' of order n' with $n > n'$, there is $\gamma_2(T') \geqslant (n'+\ell'-s')/2$.

Let T be a tree of order n. Root T at a vertex r of maximum eccentricity $\mathrm{diam}(T) \geqslant 4$. Let v be a support vertex of maximum distance from r and u the parent of v in the rooted tree.

Let $T' = T - (L_v \cup \{v\})$. Then $n' = n - (|L_v|+1)$ and T' is nontrivial. We consider two cases.

Case 1. $\deg_T(v) \geqslant 3$. By Observation 2, since $|L_v| \geqslant 2$, then $\gamma_2(T) = \gamma_2(T') + |L_v|$. If u is not a leaf in T', then $\ell' = \ell - |L_v|$ and $s' = s-1$. Applying the inductive hypothesis to T',

$$\gamma_2(T) - |L_v| = \gamma_2(T') \geqslant (n'+\ell'-s')/2 = (n+\ell-s)/2 - |L_v|\,,$$

hence $\gamma_2(T) \geqslant (n+\ell-s)/2$.

If u is a leaf in T', then $\ell' = \ell - |L_v| + 1$ and $s' \leqslant s$. Applying the inductive hypothesis to T',

$$\gamma_2(T) - |L_v| = \gamma_2(T') \geqslant (n'+\ell'-s')/2 \geqslant (n+\ell-s)/2 - |L_v|\,,$$

hence $\gamma_2(T) \geqslant (n+\ell-s)/2$.

Case 2. $\deg_T(v) = 2$, that is $|L_v| = 1$. If u is not a leaf in T', then $\ell' = \ell - 1$ and $s' = s-1$. Again by Observation 2, $\gamma_2(T) - 1 \geqslant \gamma_2(T')$. Applying the inductive hypothesis to T', we obtain the desired result. Now if u is a leaf in T', then by Observation 2, $\gamma_2(T) - 1 = \gamma_2(T')$. Also $\ell' = \ell$ and $s' \leqslant s$. Applying the inductive hypothesis to T', the result follows.

That this bound is sharp may be seen in a tree T where every vertex T is either a leaf or a support vertex adjacent to at least two leaves. Clearly, $n = \ell + s$ and $\gamma_2(T) = \ell = (n+\ell-s)/2$. □

Notice that in [1], Blidia *et al.* showed that every nontrivial tree T satisfies $\gamma_2(T) \leqslant (n+\ell)/2$. So Lemma 7 gives in some sense a best framing for the 2-domination number in trees.

Theorem 8. *If G is a graph of order n with at most one cycle, ℓ leaves and s support vertices, then $\gamma_2(G) \geqslant (n + \ell - s)/2$, and this bound is sharp.*

Proof. If all the components of G are trees, then by Lemma 7 the result holds. If G is a cycle C_n then $\ell = s = 0$ and by Observation 4, $\gamma_2(C_n) = \lceil n/2 \rceil$, implying that the result is valid. Thus G contains a component H that is a unicycle graph with a cycle C where at least one vertex of C has degree at least three. It suffices to prove the theorem for the subgraph H. Let S be a $\gamma_2(H)$-set and assume that H is the smallest connected unicycle graph that does not satisfy the theorem.

Suppose that H contains a support vertex, say $v \notin C$. We further assume that v is at maximum distance from C. Then $L_v \subset S$ and without loss of generality $v \notin S$ (else replace v by its neighbor, say w, in the unique path from v to C). Let $H' = H - (L_v \cup \{v\})$. Then H' is a connected unicycle graph with $n(H') = n(H) - (|L_v| + 1)$ and $S - L_v$ is a 2-dominating set of H'. Hence $\gamma_2(H) - |L_v| \geqslant \gamma_2(H')$ and since H' is smaller than H, it satisfies the theorem. If $\deg_H(w) \geqslant 3$ then $\ell(H') = \ell(H) - |L_v|$ and $s(H') = s(H) - 1$.

It follows that

$$\begin{aligned}\gamma_2(H) - |L_v| \geqslant \gamma_2(H') &\geqslant (n(H') + \ell(H') - s(H'))/2 = \\ &= \big(n(H) - \big(|L_v| + 1\big) + \ell(H) - |L_v| - s(H) + 1\big)/2\end{aligned}$$

and, therefore, $\gamma_2(H) \geqslant (n(H) + \ell(H) - s(H))/2$, contradicting our assumption.

Now if $\deg_H(w) = 2$, then $\ell(H') = \ell(H) - |L_v| + 1$ and $s(H') \leqslant s(H)$. It follows that

$$\begin{aligned}\gamma_2(H) - |L_v| \geqslant \gamma_2(H') &\geqslant (n(H') + \ell(H') - s(H'))/2 \geqslant \\ &\geqslant (n(H) - (|L_v| + 1) + \ell(H) - |L_v| + 1 - s(H))/2\end{aligned}$$

and, therefore, $\gamma_2(H) \geqslant (n(H) + \ell(H) - s(H))/2$, a contradiction.

It remains to examine the case where every support vertex of H is on the cycle C. Let u be a support vertex on C such that $u \in S$. Let H' be the graph obtained from H by removing all leaves adjacent to u. Then $S - L_u$ is a 2-dominating set of H', $\ell' = \ell - |L_u|$ and $s' = s - 1$. Thus

$$\begin{aligned}\gamma_2(H) - |L_u| \geqslant \gamma_2(H') &\geqslant (n(H') + \ell(H') - s(H'))/2 = \\ &= (n(H) - |L_u| + \ell(H) - |L_u| - s(H) + 1)/2\end{aligned}$$

and, therefore, $\gamma_2(H) > (n(H) + \ell(H) - s(H))/2$, a contradiction.

Thus we assume that every support vertex on C is not in S. If C is a triangle, that is $C = C_3$ then it is a simple task to check the result depending on whether C contains one, two or three support vertices. Thus we assume that the length of C is at least four. Let x be a support vertex and y, z its two neighbors on C. Let H' be the graph obtained from H by removing x and its leaves and by adding a new edge yz. Then $S - L_x$ is a 2-dominating set of H', $n(H') = n(H) - (|L_x| + 1)$, $\ell(H') = \ell(H) - |L_x|$ and $s(H') = s(H) - 1$.

It follows that

$$\gamma_2(H) - |L_x| \geqslant \gamma_2(H') \geqslant (n(H') + \ell(H') - s(H'))/2 = \\ = (n(H) - |L_x| - 1 + \ell(H) - |L_x| - s(H) + 1)/2$$

and, therefore, $\gamma_2(H) \geqslant (n(H) + \ell(H) - s(H))/2$, a contradiction.

The unicycle graph G formed by a cycle C where each vertex on C is adjacent to at least two leaves shows that the lower bound of Theorem 8 is attained. □

Note that the lower bound in Theorem 8 is not valid for cactus graphs with at least two cycles. To see this, consider the graph G_k formed by $k \geqslant 2$ cycles C_4 by identifying a vertex from each cycle into one vertex. Then $n(G_k) = 3k+1$, $\ell = s = 0$ and $\gamma_2(G) = k + 1 < (n(G_k) + \ell - s)/2 = (3k + 1)/2$.

3. UPPER BOUND

It is shown in [1] that the 2-domination number is bounded from below by the independence number for every tree T. In this section we establish an upper bound for the 2-domination number in terms of the independence number and the number of support vertices, which gives a good framing for the 2-domination number in trees.

Theorem 1. *If T is a tree of order at least three with s support vertices, then $\gamma_2(T) \leqslant \beta(T) + s - 1$ and this bound is sharp.*

Proof. We proceed by induction on the number of vertices of T. If $\operatorname{diam}(T) = 2$ then T is a star $K_{1,p}$ $(p \geqslant 2)$ where $\gamma_2(T) = \beta(T) = p$ and $s = 1$, so the result holds. If $\operatorname{diam}(T) = 3$ then T is a double star $S_{p,q}$ with $q \geqslant p$ where $\gamma_2(T) = p+q$ if $p \geqslant 2$ and $\gamma_2(T) = q + 2$ otherwise, $\beta(T) = p + q$ and $s = 2$. Thus the result is valid. Assume that for every tree T' of order n' with $n > n' \geqslant 3$, there is $\gamma_2(T') \leqslant \beta(T') + s' - 1$.

Let T be a tree of order n. Root T at a vertex r of maximum eccentricity $\operatorname{diam}(T) \geqslant 4$. Let v be a support vertex of maximum distance from r and u the parent of v in the rooted tree.

Let $T' = T - (\{v\} \cup L_v)$. Since $\operatorname{diam}(T) \geqslant 4$, the order of T' is at least three. We consider two cases.

Case 1. $\deg_T(v) \geqslant 3$. By Observation 2, $\gamma_2(T) - |L_v| = \gamma_2(T')$, $\beta(T) - |L_v| = \beta(T')$ and $s' \leqslant s$. Applying our induction to T', we obtain:

$$\gamma_2(T) - |L_v| = \gamma_2(T') \leqslant \beta(T') + s' - 1 \leqslant \beta(T) - |L_v| + s - 1.$$

Hence $\gamma_2(T) \leqslant \beta(T) + s - 1$.

Case 2. $\deg_T(v) = 2$. Then v is adjacent to exactly one leaf, say v', so $|L_v| = 1$. We again consider two cases.

Case 2.1. $\deg_T(u) = 2$. Then $s' \leqslant s$, and by Observation 2, $\gamma_2(T) - 1 = \gamma_2(T')$ and $\beta(T) - 1 = \beta(T')$. Applying the inductive hypothesis to T', we obtain the desired inequality.

Case 2.2. $\deg_T(u) \geqslant 3$. Then $s' = s - 1$ and by Observation 2, $\beta(T) - 1 = \beta(T')$. Also $\gamma_2(T) \leqslant \gamma_2(T') + 2$, since every $\gamma_2(T')$-set can be extended to a 2-dominating set of T by adding $\{v, v'\}$. By induction on T'

$$\gamma_2(T) \leqslant \gamma_2(T') + 2 \leqslant \beta(T') + s' + 1 = \big(\beta(T) - 1\big) + (s - 1) + 1,$$

hence $\gamma_2(T) \leqslant \beta(T) + s - 1$.

The upper bound is sharp for the path P_n of even order $n \geq 4$. □

In [4], Favaron proved that every tree T of order n with ℓ leaves satisfies $\beta(T) \geqslant (n + \ell)/3$. Using Lemma 7 and Theorem 1, we obtain the following corollary for the independence number, which in some sense improves Favaron's one [4] for trees.

Corollary 2. *If T is a tree of order at least 3 with ℓ leaves and s support vertices, then $\beta(T) \geqslant (n + \ell - 3s + 2)/2$.*

REFERENCES

[1] Blidia M., Chellali M., Favaron O., *Independence and 2-domination in trees.* Australasian J. Comb. **33** (2005), 317–327.

[2] Blidia M., Chellali M., Volkmann L., *Bounds of the 2-domination number of graphs* Utilitas Math., to appear.

[3] Favaron O., *On a conjecture of Fink and Jacobson concerning k-domination and k-dependence.* J. Combinat. Theory Ser. B **39** (1985), 101–102.

[4] Favaron O. *A bound on the independent domination number of a tree*, Int. J. Graph Theory **1**(**1**) (1992), 19–27.

[5] Fink J. F., Jacobson M. S., *n-domination in graphs*, in: Alavi Y. and Schwenk A. J. (eds), Graph Theory with Applications to Algorithms and Computer Science, New York, Wiley, 1985, 283–300.

[6] Fink J. F., Jacobson M. S., *On n-domination, n-dependence and forbidden subgraphs.* Graph Theory with Applications to Algorithms and Computer Science. New York, Wiley, 1985, 301–312.

[7] Haynes T. W., Hedetniemi S. T., Henning M. A., Slater P. J., *H-forming sets in graphs*, Discrete Mathematics **262** (2003), 159–169.

[8] Haynes T. W., Hedetniemi S. T., Slater P. J., *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[9] Haynes T. W., Hedetniemi S. T., Slater P. J. (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

Mustapha Chellali
m_chellali@yahoo.com

University of Blida
Department of Mathematics
B. P. 270, Blida, Algeria

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