Abstract. A 2-dominating set of a graph $G$ is a set $D$ of vertices of $G$ such that every vertex not in $S$ is dominated at least twice. The minimum cardinality of a 2-dominating set of $G$ is the 2-domination number $\gamma_2(G)$. We show that if $G$ is a nontrivial connected cactus graph with $k(G)$ even cycles ($k(G) \geq 0$), then $\gamma_2(G) \geq \gamma_1(G) - k(G)$, and if $G$ is a graph of order $n$ with at most one cycle, then $\gamma_2(G) \geq (n + \ell - s)/2$ improving Fink and Jacobson’s lower bound for trees with $\ell > s$, where $\gamma_1(G)$, $\ell$ and $s$ are the total domination number, the number of leaves and support vertices of $G$, respectively. We also show that if $T$ is a tree of order $n \geq 3$, then $\gamma_2(T) \leq \beta(T) + s - 1$, where $\beta(T)$ is the independence number of $T$.

Keywords: 2-domination number, total domination number, independence number, cactus graphs, trees.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v$, $\deg_G(v)$, is the number of vertices adjacent to $v$. A vertex of degree one is called a leaf and its neighbor is called a support vertex. If $u$ is a support vertex, then $L_u$ will denote the set of leaves attached at $u$. A graph $G$ is called a cactus graph if each edge of $G$ is contained in at most one cycle. A cactus graph having one cycle is called a unicycle graph and a connected cactus graph with no cycles is called a tree. A tree $T$ is a double star if it contains exactly two vertices that are not leaves. A double star with $p$ and $q$ leaves attached at each support vertex, respectively, is denoted by $S_{p,q}$. For a graph $G$ we denote by $n(G)$, $\ell(G)$ and $s(G)$ the number of vertices, leaves and support vertices of $G$, respectively (we use $n$, $\ell$ and $s$ if there is no ambiguity).
We are interested in a variation of domination in graphs, called 2-domination. A subset $D$ of $V(G)$ is a 2-dominating set if every vertex not in $S$ is adjacent to at least 2 vertices of $D$. The 2-domination number $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set of $G$. Note that every graph $G$ has a 2-dominating set since $V(G)$ is such a set. The concept of 2-domination was introduced by Fink and Jacobson [5, 6], and studied for example in [1, 2].

The independence number $\alpha(G)$ is the maximum cardinality among the independent sets of vertices of $G$. The concept of domination and its variations see the books of Haynes, Hedetniemi and Slater [8, 9].

In this paper we show that if $G$ is a nontrivial connected cactus graph with $k(G)$ even cycles ($k(G) \geq 0$), then $\gamma_2(G) \geq \gamma_t(G) - k(G)$, and if $G$ is a graph of order $n$ with at most one cycle, then $\gamma_2(G) \geq (n + \ell - s)/2$. Finally, we show that every tree $T$ of order at least three satisfies $\gamma_2(T) \leq \beta(T) + s - 1$.

2. LOWER BOUNDS

Before presenting our main results, we make a couple of straightforward observations.

Observation 1. Every 2-dominating set of a graph $G$ contains every leaf.

Observation 2. Let $T$ be a tree obtained from a nontrivial tree $T'$ by adding a star $K_{1,p}$ with the center vertex $v$ attached by an edge $vw$ at a vertex $w$ of $T'$. Then:

1) $\gamma_2(T') \leq \gamma_2(T) - |L_v|$, with equality if $p \geq 2$ or $w$ is a leaf in $T'$.

2) $\beta(T') = \beta(T) - |L_v|$.

Proof. 1) Let $D$ be a $\gamma_2(T)$-set. By Observation 1, $D$ contains $L_v$ and without loss of generality $v \notin D$ (else replace $v$ by $w$ in $D$). Thus $D - L_v$ is a 2-dominating set of $T'$ and hence $\gamma_2(T') \leq \gamma_2(T) - |L_v|$. Now if $p \geq 2$, that is $v$ is adjacent to at least two leaves, then every $\gamma_2(T')$-set can be extended to a 2-dominating set of $T$ by adding $L_v$. So $\gamma_2(T) \leq \gamma_2(T') + |L_v|$, which leads the equality. Assume now that $p = 1$ and $w$ is a leaf of $T'$. By Observation 1, $w$ is in every $\gamma_2(T')$-set $D'$, so $D'$ is extended to a 2-dominating set of $T$ by adding the leaf neighbor of $v$. Therefore, $\gamma_2(T) \leq \gamma_2(T') + 1$, implying the equality $\gamma_2(T) = \gamma_2(T') - 1$.

2) Obvious. \hfill \Box

In [7], Haynes et al. showed that the 2-domination number is bounded from below by the total domination number for every nontrivial tree.

Theorem 3 (Haynes et al. [7]). For every nontrivial tree, $\gamma_2(T) \geq \gamma_t(T)$.

Below, we extend this result onto cactus graphs. The total domination and 2-domination numbers of a cycle are easy to compute.
Observation 4. For a cycle $C_n$ on $n \geq 3$ vertices:

i) $\gamma_t(C_n) = n/2$ if $n \equiv 0(\text{mod } 4)$ and $\gamma_t(C_n) = \lceil n/2 \rceil + 1$ otherwise,

ii) $\gamma_2(C_n) = \lceil n/2 \rceil$.

Theorem 5. If $G$ is a nontrivial connected cactus graph with $k(G)$ even cycles ($k(G) \geq 0$), then $\gamma_2(G) \geq \gamma_t(G) - k(G)$, and this bound is sharp.

Proof. If $G$ is a tree, then $k(G) = 0$ and by Theorem 3 the result is valid. If $G$ is a cycle $C_n$, then by Observation 4, the result holds. Thus we assume that $G$ is neither a tree nor a cycle $C_n$. Among all connected cactus graphs with $k(G)$ even cycles that do not satisfy the result, let $G$ be one which contains as few vertices and edges as possible. Let $S$ be a $\gamma_2(G)$-set. Assume first that there are two adjacent vertices $x, y$ on some cycle such that $x, y$ are both in $S$ or both not in $S$. Consider the spanning graph $G'$ obtained by removing the edge $xy$. Then $S$ is a 2-dominating set of $G'$ and hence $\gamma_2(G) = |S| \geq \gamma_2(G')$. Also $\gamma_t(G') \geq \gamma_t(G)$, since every total dominating set of $G'$ is a total dominating set of $G$. Now $G'$ satisfies the result and so $\gamma_2(G') \geq \gamma_t(G') - k(G')$. Since $k(G) \geq k(G')$, it follows that $\gamma_2(G) \geq \gamma_2(G') \geq \gamma_t(G') - k(G') \geq \gamma_t(G) - k(G)$, a contradiction.

Thus we assume that all vertices on the cycles are contained alternately in $S$. This implies that $G$ contains no odd cycle. Let $u, v$ be two adjacent vertices on an even cycle such that $u \in S$ and $v \notin S$. Let $G''$ be the spanning graph of $G$ obtained by removing the edge $uw$. Then $S \cup \{v\}$ is a 2-dominating set of $G''$ and so $\gamma_2(G'') \leq |S| + 1$. There also is $\gamma_t(G'') \geq \gamma_t(G)$ and $k(G) = k(G'') + 1$. Now since $G''$ satisfies $\gamma_2(G'') \geq \gamma_t(G'' - k(G''))$, we obtain $\gamma_2(G) + 1 \geq \gamma_2(G'') \geq \gamma_t(G'' - k(G'')) \geq \gamma_t(G) - k(G) + 1$. Therefore, $\gamma_2(G) \geq \gamma_t(G) - k(G)$, a contradiction.

That this bound is sharp may be seen by considering the cactus graph $G_q$ ($q \geq 1$) formed from $q$ path $P_3$, each one with the center vertex $v_i$, where $1 \leq i \leq q$ and $q$ cycle $C_6$ by adding edges between all center vertices so that the subgraph induced by the center vertices is a path $P_q$. Then we identify a vertex of a cycle $C_6$ with one leaf of each path $P_3$. See Figure 1 for an example of $G_3$. For $G_q$, $\gamma_2(G_q) = 5q$, $\gamma_t(G_q) = 6q$ and $k(G_q) = q$. □

![Fig. 1. The graph $G_3$](image-url)
In [5], Fink and Jacobson have established a lower bound on the 2-domination number for every tree in term of its order.

**Theorem 6** ([5]). If $T$ is a tree of order $n$, then $\gamma_2(T) \geq (n + 1)/2$.

Next we give a lower bound for the 2-domination number in trees that improves Fink and Jacobson’s one if $\ell > s$.

**Lemma 7.** If $T$ is a tree of order $n$ with $\ell$ leaves and $s$ support vertices, then $\gamma_2(T) \geq (n + \ell - s)/2$, and this bound is sharp.

**Proof.** We proceed by induction on the order of $T$. If $\text{diam}(T) \in \{0, 1\}$, then the result is valid. If $\text{diam}(T) = 2$, then $T$ is a star $K_{1,p}$ ($p \geq 2$), where $\gamma_2(T) = p$ and $(n + \ell - s)/2 = p$, so the result is valid. If $\text{diam}(T) = 3$, then $T$ is a $d$ouble star $S_{p,q}$, where $\gamma_2(T) = p + q$ if $\min\{p, q\} \geq 2$ and $\gamma_2(T) = 2 + \max\{p, q\}$ otherwise. Thus again the result is valid. Assume that for every tree $T'$ of order $n'$ with $n > n'$, there is $\gamma_2(T') \geq (n' + \ell' - s')/2$.

Let $T$ be a tree of order $n$. Root $T$ at a vertex $r$ of maximum eccentricity $\text{diam}(T) \geq 4$. Let $v$ be a support vertex of maximum distance from $r$ and $u$ the parent of $v$ in the rooted tree.

Let $T' = T - (L_v \cup \{v\})$. Then $n' = n - (|L_v| + 1)$ and $T'$ is nontrivial. We consider two cases.

**Case 1.** $\deg_T(v) \geq 3$. By Observation 2, since $|L_v| \geq 2$, then $\gamma_2(T) = \gamma_2(T') + |L_v|$.

If $u$ is not a leaf in $T'$, then $\ell' = \ell - |L_v|$ and $s' = s - 1$. Applying the inductive hypothesis to $T'$,

$$\gamma_2(T) - |L_v| = \gamma_2(T') \geq (n' + \ell' - s')/2 = (n + \ell - s)/2 - |L_v|,$$

hence $\gamma_2(T) \geq (n + \ell - s)/2$.

If $u$ is a leaf in $T'$, then $\ell' = \ell - |L_v| + 1$ and $s' \leq s - 1$. Applying the inductive hypothesis to $T'$,

$$\gamma_2(T) - |L_v| = \gamma_2(T') \geq (n' + \ell' - s')/2 \geq (n + \ell - s)/2 - |L_v|,$$

hence $\gamma_2(T) \geq (n + \ell - s)/2$.

**Case 2.** $\deg_T(v) = 2$, that is $|L_v| = 1$. If $u$ is not a leaf in $T'$, then $\ell' = \ell - 1$ and $s' = s - 1$. Again by Observation 2, $\gamma_2(T) - 1 \geq \gamma_2(T')$. Applying the inductive hypothesis to $T'$, we obtain the desired result. Now if $u$ is a leaf in $T'$, then by Observation 2, $\gamma_2(T) - 1 = \gamma_2(T')$. Also $\ell' = \ell$ and $s' \leq s$. Applying the inductive hypothesis to $T'$, the result follows.

That this bound is sharp may be seen in a tree $T$ where every vertex $T$ is either a leaf or a support vertex adjacent to at least two leaves. Clearly, $\ell = \ell + s$ and $\gamma_2(T) = \ell = (n + \ell - s)/2$. □

Notice that in [1], Blidia et al. showed that every nontrivial tree $T$ satisfies $\gamma_2(T) \leq (n + \ell)/2$. So Lemma 7 gives in some sense a best framing for the 2-domination number in trees.
Theorem 8. If $G$ is a graph of order $n$ with at most one cycle, $\ell$ leaves and $s$ support vertices, then $\gamma_2(G) \geq (n + \ell - s)/2$, and this bound is sharp.

Proof. If all the components of $G$ are trees, then by Lemma 7 the result holds. If $G$ is a cycle $C_n$ then $\ell = s = 0$ and by Observation 4, $\gamma_2(C_n) = \lceil n/2 \rceil$, implying that the result is valid. Thus $G$ contains a component $H$ that is a unicycle graph with a cycle $C$ where at least one vertex of $C$ has degree at least three. It suffices to prove the theorem for the subgraph $H$. Let $S$ be a $\gamma_2(H)$-set and assume that $H$ is the smallest connected unicycle graph that does not satisfy the theorem.

Suppose that $H$ contains a support vertex, say $v \notin C$. We further assume that $v$ is at maximum distance from $C$. Then $L_v \subseteq S$ and without loss of generality $v \notin S$ (else replace $v$ by its neighbor, say $w$, in the unique path from $v$ to $C$). Let $H' = H - (L_v \cup \{v\})$. Then $H'$ is a connected unicycle graph with $n(H') = n(H) - (|L_v| + 1)$ and $S - L_v$ is a 2-dominating set of $H'$. Hence $\gamma_2(H) - |L_v| \geq \gamma_2(H')$ and since $H'$ is smaller than $H$, it satisfies the theorem. If $\deg_H(w) \geq 3$ then $\ell(H') = \ell(H) - |L_v|$ and $s(H') = s(H) - 1$.

It follows that

$$\gamma_2(H) - |L_v| \geq \gamma_2(H') \geq (n(H') + \ell(H') - s(H'))/2 =$$

$$= (n(H) - (|L_v| + 1) + \ell(H) - |L_v| - s(H) + 1)/2$$

and, therefore, $\gamma_2(H) \geq (n(H) + \ell(H) - s(H))/2$, contradicting our assumption.

Now if $\deg_H(w) = 2$, then $\ell(H') = \ell(H) - |L_v| + 1$ and $s(H') \leq s(H)$. It follows that

$$\gamma_2(H) - |L_v| \geq \gamma_2(H') \geq (n(H') + \ell(H') - s(H'))/2 \geq$$

$$\geq (n(H) - (|L_v| + 1) + \ell(H) - |L_v| + 1 - s(H))/2$$

and, therefore, $\gamma_2(H) \geq (n(H) + \ell(H) - s(H))/2$, a contradiction.

It remains to examine the case where every support vertex of $H$ is on the cycle $C$. Let $u$ be a support vertex on $C$ such that $u \in S$. Let $H'$ be the graph obtained from $H$ by removing all leaves adjacent to $u$. Then $S - L_u$ is a 2-dominating set of $H'$, $\ell' = \ell - |L_u|$ and $s' = s - 1$. Thus

$$\gamma_2(H) - |L_u| \geq \gamma_2(H') \geq (n(H') + \ell(H') - s(H'))/2 =$$

$$= (n(H) - |L_u| + \ell(H) - |L_u| - s(H) + 1)/2$$

and, therefore, $\gamma_2(H) > (n(H) + \ell(H) - s(H))/2$, a contradiction.

Thus we assume that every support vertex on $C$ is not in $S$. If $C$ is a triangle, that is $C = C_3$ then it is a simple task to check the result depending on whether $C$ contains one, two or three support vertices. Thus we assume that the length of $C$ is at least four. Let $x$ be a support vertex and $y, z$ its two neighbors on $C$. Let $H'$ be the graph obtained from $H$ by removing $x$ and its leaves and by adding a new edge $yz$. Then $S - L_x$ is a 2-dominating set of $H'$, $n(H') = n(H) - (|L_x| + 1)$, $\ell(H') = \ell(H) - |L_x|$ and $s(H') = s(H) - 1$. 


It follows that
\[
\gamma_2(H) - |L_x| \geq \gamma_2(H') \geq (n(H') + \ell(H') - s(H'))/2 = \\
(n(H) - |L_x| - 1 + \ell(H) - |L_x| - s(H) + 1)/2
\]
and, therefore, \(\gamma_2(H) \geq (n(H) + \ell(H) - s(H))/2\), a contradiction.

The unicycle graph \(G\) formed by a cycle \(C\) where each vertex on \(C\) is adjacent to at least two leaves shows that the lower bound of Theorem 8 is attained. \(\square\)

Note that the lower bound in Theorem 8 is not valid for cactus graphs with at least two cycles. To see this, consider the graph \(G_k\) formed by \(k \geq 2\) cycles \(C_4\) by identifying a vertex from each cycle into one vertex. Then \(n(G_k) = 3k + 1, \ell = s = 0\) and \(\gamma_2(G) = k + 1 < (n(G_k) + \ell - s)/2 = (3k + 1)/2\).

3. UPPER BOUND

It is shown in [1] that the 2-domination number is bounded from below by the independence number for every tree \(T\). In this section we establish an upper bound for the 2-domination number in terms of the independence number and the number of support vertices, which gives a good framing for the 2-domination number in trees.

**Theorem 1.** If \(T\) is a tree of order at least three with \(s\) support vertices, then \(\gamma_2(T) \leq \beta(T) + s - 1\) and this bound is sharp.

**Proof.** We proceed by induction on the number of vertices of \(T\). If \(\text{diam}(T) = 2\) then \(T\) is a star \(K_{1,p}\) \((p \geq 2)\) where \(\gamma_2(T) = \beta(T) = p\) and \(s = 1\), so the result holds. If \(\text{diam}(T) = 3\) then \(T\) is a double star \(S_{p,q}\) with \(q \geq p\) where \(\gamma_2(T) = p + q\) if \(p \geq 2\) and \(\gamma_2(T) = q + 2\) otherwise, \(\beta(T) = p + q\) and \(s = 2\). Thus the result is valid. Assume that for every tree \(T'\) of order \(n'\) with \(n > n' > 3\), there is \(\gamma_2(T') \leq \beta(T') + s' - 1\).

Let \(T\) be a tree of order \(n\). Root \(T\) at a vertex \(r\) of maximum eccentricity \(\text{diam}(T) \geq 4\). Let \(v\) be a support vertex of maximum distance from \(r\) and \(u\) the parent of \(v\) in the rooted tree.

Let \(T' = T - (\{v\} \cup L_v)\). Since \(\text{diam}(T) \geq 4\), the order of \(T'\) is at least three. We consider two cases.

**Case 1.** \(\deg_T(v) \geq 3\). By Observation 2, \(\gamma_2(T) - |L_v| = \gamma_2(T'), \beta(T) - |L_v| = \beta(T')\) and \(s' \leq s\). Applying our induction to \(T'\), we obtain:

\[
\gamma_2(T) - |L_v| = \gamma_2(T') \leq \beta(T') + s' - 1 \leq \beta(T) - |L_v| + s - 1.
\]

Hence \(\gamma_2(T) \leq \beta(T) + s - 1\).

**Case 2.** \(\deg_T(v) = 2\). Then \(v\) is adjacent to exactly one leaf, say \(v'\), so \(|L_v| = 1\). We again consider two cases.

**Case 2.1.** \(\deg_T(u) = 2\). Then \(s' \leq s\), and by Observation 2, \(\gamma_2(T) - 1 = \gamma_2(T')\) and \(\beta(T) - 1 = \beta(T')\). Applying the inductive hypothesis to \(T'\), we obtain the desired inequality.
Case 2.2. \( \deg_T(u) \geq 3 \). Then \( s' = s - 1 \) and by Observation 2, \( \beta(T) - 1 = \beta(T') \).

Also \( \gamma_2(T) \leq \gamma_2(T') + 2 \), since every \( \gamma_2(T') \)-set can be extended to a 2-dominating set of \( T \) by adding \( \{v, v'\} \). By induction on \( T' \)

\[
\gamma_2(T) \leq \gamma_2(T') + 2 \leq \beta(T') + s' + 1 = (\beta(T) - 1) + (s - 1) + 1,
\]
hence \( \gamma_2(T) \leq \beta(T) + s - 1 \).

The upper bound is sharp for the path \( P_n \) of even order \( n \geq 4 \).

In [4], Favaron proved that every tree \( T \) of order \( n \) with \( \ell \) leaves satisfies \( \beta(T) \geq (n + \ell)/3 \). Using Lemma 7 and Theorem 1, we obtain the following corollary for the independence number, which in some sense improves Favaron’s one [4] for trees.

**Corollary 2.** If \( T \) is a tree of order at least 3 with \( \ell \) leaves and \( s \) support vertices, then \( \beta(T) \geq (n + \ell - 3s + 2)/2 \).

REFERENCES


Mustapha Chellali
m_chellali@yahoo.com

University of Blida
Department of Mathematics
B. P. 270, Blida, Algeria

Received: October 31, 2005.