Opuscula Mathematica • Vol. 25 • No. 2 • 2005

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SOLVING EQUATIONS BY TOPOLOGICAL METHODS

Abstract. In this paper we survey most important results from topological fixed point theory which can be directly applied to differential equations. Some new formulations are presented. We believe that our article will be useful for analysts applying topological fixed point theory in nonlinear analysis and in differential equations.

Keywords: Lefschetz number, fixed points, CAC-maps, condensing maps, ANR-spaces, fixed point index.

Mathematics Subject Classification: Primary 55M20, 47H11, 47H10, 54H25.

0. INTRODUCTION

The problem of solving equations belongs to the most important questions in mathematics. Practically, any part of mathematics deals with equations. Therefore, we can consider algebraic, differential, integral, analytical and many other types of equations. In our lecture we shall concentrate on topological approach to equations. What do we mean by equations?

Let X be an arbitrary nonempty set and let $f\colon X \to X$ be a given function. By the equation (with the left hand side f) we shall understand the following question: is it true that there exists an element $x \in X$ such that

$$f(x) = x. \tag{0.1}$$

The equation (0.1) is given in a global form, because our function f maps X into itself.

So let us assume that U is a nonempty subset of X and let $f\colon U \to X$ be a given function. For such a function f, we shall consider the following equation

$$f(x) = x. \tag{0.2}$$

In both cases a point $\widetilde{x} \in X$ ($\widetilde{x} \in U$) such that $f(\widetilde{x}) = \widetilde{x}$ is called a *solution* of (0.1) (respectively (0.2)).

In this paper we shall say that equation (0.1) is given in a *global form* and (0.2) is given in a *local form*.

We let:

$$S(f) = \{\widetilde{x} \in X \mid f(\widetilde{x}) = \widetilde{x}\} \tag{0.3}$$

and respectively:

$$S(f) = \{\widetilde{x} \in U \mid f(\widetilde{x}) = \widetilde{x}\} \tag{0.4}$$

Then $S(f)$ is called the set of all solutions of (0.1) or (0.2), respectively.

Using topological methods, we are able to obtain the following types of results:

1) existence results, i.e., when the set $S(f)$ is nonempty;
2) topological characterization of the set $S(f)$, i.e., the cardinality of $S(f)$ and the topological structure of $S(f)$ in the case if it is an infinite set;
3) localization of solutions, i.e., we are able to show a special subset $A \subset X$ such that $S(f) \cap A \neq \emptyset$;
4) existence of special solutions, i.e., we are able to prove that for some equations it is possible to obtain periodic solutions or solutions with the multiplicity bigger than 1 and so on.

Note, that using topological methods, we don't consider the problem of numerical computation of solutions. Topological methods deal only with the qualitive part of the theory of equations.

In what follows, we shall explain for what classes of topological spaces and for what classes of mappings the results 1) – 4) are possible both for (0.1) and (0.2).

Finally, we would like to add that the aim of this paper is to survay current results connected with the above mentioned problems. Some direct applications to differential equations are presented in last section. For more complete information see [M-15] and also monographs [M-1]–[M-14].

1. TOPOLOGICAL AND HOMOLOGICAL BACKGROUND

In what follows, we shall use the following notations:

$(\mathbb{R}^n, \|\cdot\|)$ – the euclidean n-dimensional space,
$B^n = B(0,1)$ – the open unit ball in $\mathbb{R}^n$,
$K^n = K(0,1)$ – the closed unit ball in $\mathbb{R}^n$,
$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ – the unit sphere in $\mathbb{R}^n$.

Let $A \subset X$ be a subset of a metric space X, A is called a *retract* (a *neighbourhood retract*) of X if there exists a continuous function $r\colon X \to A$ ($r\colon U \to A$, where $A \subset U$

and U is an open subset of X) such that $r(x) = x$, for every $x \in A$, then the map r is called a retraction map. Note that if A is a retract of X then A is closed.

The notion of retract (neigbourhood retract) is strictly connected with the notion of extension (neighbourhood extension).

Namely, we define:

1.1. A metric space (X, d) has an extension property (neighbourhood extension property) provided for any metric space (Y, d_1), for any closed $B \subset Y$ and for any continuous $f : B \to X$ there exists an extension $\widetilde{f}\colon Y \to X$ ($\widetilde{f}\colon U \to X$, where U is an open neighbourhood of B in Y), i.e., $\widetilde{f}$ is continuous and $\widetilde{f}(y) = f(y)$, for every $y \in B$.

Let us recall the following two results:

Theorem 1.2 (Dugundji Extension Theorem). *Let $(E, \|\cdot\|)$ be a normed space and $C \subset E$ be a convex subset. Then C has an extension property.*

Theorem 1.3. *The unit sphere S^n in $\mathbb{R}^{n+1}$ has neighbourhood extension property.*

For better understanding the above notions, we shall list some properties:

1.4. *If X has an extension property and Y is homeomorphic to X, then Y has extension property (the same is true for the n.e.p.[1]).*

1.5. *If X has an extension property (n.e.p.) and A is a retract of X, then A has extension (n.e.p.) property.*

1.6. *If X has n.e.p. and U is an open subset of X, then U has n.e.p.*

We shall use the following definition which originates from K. Borsuk.

Definition 1.7. *A metric space (X, d) is called an absolute retract (written $X \in$ AR) iff X has an extension property; X is called an absolute neighbourhood retract (written $X \in$ ANR) provided X has n.e.p.*

To understand beter how large the class of ANR-s (AR-s) is we recall

Proposition 1.8 ([M-3]).

1. $X \in$ ANR *if and only if there exists a normed space E and an open subset U of E such that X is homeomorphic to a retract of U,*
2. $X \in$ AR *if and only if there exists a normed space E and a convex subset W of E such that X is homeomorphic to a retract of W.*

[1] neighbourhood extension property

In particular, any open subset in a normed space or any finite polyhedron is an ANR-space; respectively any convex subset is an arbitrary normed space is an AR-space.

The next notion strictly connected with the extension property is the notion of homotopy. Let $f, g\colon X \to Y$ be two continuous mappings between metric spaces. We shall say that f is *homotopic* to g (written $f \sim g$) provided there exists a continuous homotopy $h\colon X \times [0,1] \to Y$ such that:

$$h(x,0) = f(x) \quad \text{for every } x \in X,$$
$$h(x,1) = g(x) \quad \text{for every } x \in X.$$

In other words $f \sim g$ provided the map $\overline{h}\colon X \times \{0\} \cup X \times \{1\} \to Y$ defined as follows:

$$\overline{h}(x,t) = \begin{cases} f(x) & \text{for } t = 0, \\ g(x) & \text{for } t = 1 \end{cases}$$

can be extended over the cylinder $X \times [0,1]$.

Note that if Y is a convex subset of a normed space $(E, \|\cdot\|)$ then any two mappings $f, g\colon X \to Y$ are homotopic $(h(x,t) = (1-t)f(x) + tg(x))$.

A space Y is called *contractible* provided any two (continuous maps) $f, g\colon X \to Y$ are homotopic. In another words X is contractible provided the identity map id_X over X is homotopic to a constans map. Note that any AR-space is contractible and any ANR-space is locally contractible.

One can show that

$$\text{CONVEX SETS} \subset \text{AR} \subset \text{CONTRACTIBLE SETS}.$$

A compact nonempty set B is called an R_δ set provided there exists a decreasing sequence B_n of compact contractible set such that:

$$B_n = \bigcap_n B_n.$$

For compact sets we have

$$\text{CONVEX SETS} \subset \text{AR} \subset \text{CONTRACTIBLE SETS} \subset R_\delta.$$

For more details, we recommend: [M-2, M-3, M-4, M-10].

2. THE GLOBAL CASE

The most general (global) existence theorems are:

(i) the Banach contraction principle (see: [M-1, M-2, M-8, M-9, M-11]),

(ii) the Brouwer fixed point theorem (see: [M-1, M-6, M-8, M-10, 6]),

(iii) the Schauder fixed point theorem (see: [M-1, M-2, M-6, M-8, M-11, M-12, M-13]),

(iv) the Lefschetz fixed point theorem (see: [M-5, M-7, M-10, M-14, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]).

Let (X, d) be a metric space. A map $f\colon X \to X$ is called *contraction* provided there exists $0 \leq \alpha < 1$ such that:

$$d(f(x), f(y)) \leq \alpha d(x, y) \quad \text{for every } x, y \in X. \tag{*}$$

Theorem 2.1 (Banach Contraction Principle). *If (X, d) is a complete space and $f\colon X \to X$ is a contraction map, then there exists exactly one solution of the equation:*

$$f(x) = x.$$

Roughly speaking, we can say that that Theorem 2.1 holds true for a large class of spaces (for an arbitrary complete metric space) and for sufficiently narrow class of mappings (only for mappings satisfying (*)!).

In the Brouwer fixed point theorem we shall have an opposite situation.

Theorem 2.2 (Brouwer fixed point theorem). *If $f\colon K^n \to K^n$ is a continuous map, then the equation $f(x) = x$ has a solution.*

Of course, we don't claim now that there exists exactly one solution.

In Theorem 2.2 the domain of our equation is a very special metric space. Below, we shall try to replace K^n by some other metric spaces.

We start from the following:

Corollary 2.3. *Assume that the metric space (X, d) is homeomorphic to K^n. If $f\colon X \to X$ is a continuous map, then the equation $f(x) = x$ has a solution.*

To obtain Corollary 2.3 from Theorem 2.2 let us denote by $h\colon X \to K^n$ a homeomorphism and let $\overline{h}\colon K^n \to X$ be its inverse. Let f be a given continuous map from X to X.

We have the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\ h\ } & K^n \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ X & \xleftarrow{\ \overline{h}\ } & K^n \end{array}$$

in which $g = h \circ f \circ \overline{h}$. Then g is continuous and by using Theorem 2.1, we get a point $\widetilde{u} \in K^n$ such that $g(\widetilde{u}) = \widetilde{u}$. Then we have

$$\overline{h}(g(\widetilde{u})) = \overline{h}(\widetilde{u}).$$

Let us denote $\overline{h}(\widetilde{u}) = \widetilde{x} \in X$. Then we have

$$\overline{h}\,(g(\widetilde{u})) = \overline{h}\,\big(h\,\big(f\big(\overline{h}(u)\big)\big)\big) = f(\widetilde{x}) = \overline{h}(\widetilde{u}) = \widetilde{x}$$

and consequently $\widetilde{x}$ is a solution of the equation $f(x) = x$.

Our second observation is the following:

Corollary 2.4. *Assume that A is a retract of K^n. If $f\colon A \to A$ is a continuous map, then the equation $f(x) = x$ has a solution.*

Consider the diagram

$$\begin{array}{ccc} K^n & \xrightarrow{\ r\ } & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ K^n & \xleftarrow{\ i\ } & A \end{array}$$

in in which r is a retraction map, i is the inclusion map and $g = i \circ f \circ r$.

Now, the proof of Corollary 2.4 is strictly analogous to the proof of Corollary 2.3.

Theorem 2.5. *The following statements are equivalent:*

1. *S^n is not contractible,*
2. *(Bohl) every continuous map $f\colon K^{n+1} \to \mathbb{R}^{n+1}$ has at least one of the following properties:*
   (i) *the equation $x = f(x)$ has a solution,*
   (ii) *there is an $x \in S^n$ such that $x = \lambda f(x)$, for some $0 < \lambda < 1$*
3. *Brouwer fixed point theorem,*
4. *S^n is not a retract of K^{n+1}.*

Proof. (1.) $\Rightarrow$ (2.) Suppose for every x $f(x) \neq x$ and $y \neq tf(y)$, for all $0 < t < 1$, $y \in S^n$. Then $y \neq tf(y)$ also for $t = 0$ and, by our first hypothesis, for $t = 1$.

Let $r\colon \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ be the map $x \to \frac{x}{\|x\|}$. Then $H\colon S^n \times [0,1] \to S^n$ defined by:

$$H(y,t) = \begin{cases} r(y - 2tf(y)), & 0 \leq t \leq \frac{1}{2}, \\ r((2-2t)y - f((2-2t)y), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

would show that S^n is contractible.

(2.) $\Rightarrow$ (3.) The second possibility in (2) cannot occur, since $f(S^n) \subset K^{n+1}$.

(3.) $\Rightarrow$ (4.) If there were a retraction r, then for the map $x \to -r(x)$ the respective equation would have no solutions.

(4.) $\Rightarrow$ (1.) Assume that $h\colon S^n \times [0,1] \to S^n$ is a homotopy such that $h(y,1) = y$ and $h(y,0) = y_0 \in S^n$. Defining $r\colon K^{n+1} \to S^n$ by

$$r(x) = \begin{cases} y_0, & \|x\| \leq \frac{1}{2} \\ h\left(\dfrac{x}{\|x\|}, 2\|x\| - 1\right), & \|x\| \geq \frac{1}{2} \end{cases}$$

would give a retraction K^{n+1} onto S^n. □

Regarding K^n as the unit ball in $\mathbb{R}^n$, it is natural to ask whether or not Brouwer's theorem is valid for the unit ball of every (possibly infinite-dimensional) normed space. The answer is given in (see [M-8]):

Theorem 2.6. *Let E be a normed space and let K be its closed unit ball. For each continuous map $f\colon K \to K$ the equation $f(x) = x$ has a solution if and only if E is finite-dimensional.*

It is an interesting question to specify a class of mappings for which Theorem 2.6 holds true for an arbitrary normed space. We should do it in the end part of this section. For studying Brouwer's fixed point theorem we recommend: [M-1, M-2, M-8, M-9, M-10].

The result Theorem 2.5 (1) is a particular case of the more general theorem so called Borsuk's Theorem on Antipodes which can be formulated as follows:

Theorem 2.7 (Borsuk's Theorem on Antipodes). *If $f\colon S^{n-1} \to S^{n-1}$, $n \geq 1$ is an odd continuous mapping, i.e., $f(x) = -f(-x)$, then f is not homotopic to a constant map.*

It is well known that:

Theorem 2.8. *The following statements are equivalent:*

1) (Borsuk–Lusternik–Schnirelman) *If $A_1, \ldots, A_{n+1}$ are closed subsets of S^n such that $S^n = A_1 \cup \ldots A_{n+1}$, then there exists i, $1 \leq i \leq n+1$ and a point $x \in S^n$ such that $x \in A_i$ and $(-x) \in A_i$ (the set A_i contains a pair of antipodal points).*
2) (Borsuk) *There is no odd continuous mapping $f\colon S^n \to S^{n-1}$.*
3) *Borsuk's Theorem on Antipodes.*
4) (Borsuk–Ulam) *If $f\colon S^n \to \mathbb{R}^n$ is a continuous map, then there exists $x \in S^n$ such that $f(x) = f(-x)$.*

For more details concerning Theorem 2.5 and Theorem 2.6 we recommend [M-8] (see also: [M-7, M-10, M-11, M-14]).

Finally, let us back to the problem of possible generalization of the Brouwer fixed point theorem.

We shall say a continuous map $f\colon X \to Y$ is compact provided there exists a compact subset $K \subset Y$ such that $f(X) \subset K$. It was observed by Juliusz Paweł Schauder that the class of compact maps is proper for the above mentioned generalization. Namely, we have:

Theorem 2.9 (Schauder fixed point theorem). *Let $X \in \mathrm{AR}$ and $f : X \to X$ be a compact map, then the equation*

$$f(x) = x$$

has a solution.

3. THE GLOBAL CASE. THE LEFSCHETZ FIXED POINT THEOREM

In 1923 S. Lefschetz formulated the famous fixed point theorem so called now the Lefschetz fixed point theorem. Later, in 1928 H. Hopf gave a new proof of the Lefschetz fixed point theorem for self-mappings of polyhedra. Let us remark that Lefschetz formulated his theorem for compact manifolds. In 1967, A. Granas extend the Lefschetz fixed point theorem to the case of absolute neighbourhood retracts. The key to the proof of the theorem is the fact that all compact absolute neighbourhood retracts are homotopically equivalent with polyhedra. Then the case of noncompact absolute neighbourhood retracts was reduced to the compact case by using the generalized trace theory introduced by L. Leray.

In the present section, we would like to present current results concerning this theorem. We shall prove an abstract version of the Lefschetz fixed point theorem (comp. Theorem (3.15)) from which we shall deduce not only well known results but also some new results mainly connected with condensing and k-set contraction mappings. Finally, relative versions of the Lefschetz fixed point theorem are discussed.

It is convenient to introduce the notion of Lefschetz number.

We shall consider the category of pairs of metric spaces and continuous mappings. By a pair of spaces (X, X_0) we understand a pair consisting of a metric space X and one of its subsets X_0. A pair of the form $(X, \emptyset)$ will be identified with the space X. By a map $f\colon (X, X_0) \to (Y, Y_0)$ we understand a continuous map $f\colon X \to Y$ such that $f(X_0) \subset Y_0$. In what follows having a map of pairs

$$f\colon (X, X_0) \to (Y, Y_0)$$

we shall denote by

$$f_X\colon X \to Y \quad \text{and} \quad f_{X_0}\colon X_0 \to Y_0$$

the respective mappings induced by f.

Let H be the Čech homology functor with compact carriers ([7] or [8]) and coefficients in the field of rational numbers Q from the category of all pairs or spaces and all maps between such pairs, to the category of graded vector spaces over Q and linear maps of degree zero. Thus $H(X, X_0) = \{H_q(X, X_0)\}$ is a graded vector space, $H_q(X, X_q)$ being the q-dimensional Čech homology with compact carriers of X. For a map $f\colon (X, X_0) \to (Y, Y_0)$, $H(f)$ is the induced linear map $f_* = \{f_{*q}\}$, where $f_{*q}\colon H_q(X, X_0) \to H_q(Y, Y_0)$.

A non-empty space X is called *acyclic* provided

(i) $H_q(X) = 0$ for all $q \geq 1$,

(ii) $H_q(X) \approx Q$.

Let $u\colon E \to E$ be an endomorphism of an arbitrary vector space. Let us put $N(u) = \{x \in E \mid u^n(x) = 0, \text{ for some } n\}$, where u^n is nth iterate of u and $\widetilde{E} = E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\widetilde{u}\colon \widetilde{E} \to \widetilde{E}$. We call u admissible provided $\dim \widetilde{E} < \infty$. Let $u = \{u_q\}\colon E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$.

We call u a *Leray endomorphism* if:

(i) all u_q are admissible,

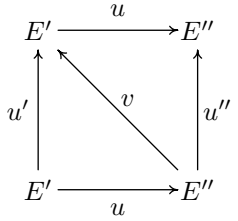
(ii) almost all $\widetilde{E}_q$ are trivial.

For such u, we define the (generalized) Lefschetz number $\Lambda(u)$ by putting

$$\Lambda(u) = \sum_q (-1)^q \operatorname{tr}(\widetilde{u}_q).$$

The following important property of the Leray endomorphisms is a consequence of the well known formula $\operatorname{tr}(u \circ v) = tr(v \circ u)$ for trace.

Proposition 3.1. *Assume that, in the category of graded vector spaces, the following diagram commutes*



then, if u' or u'' is a Leray endomorphism, so is the other; and, in that case, $\Lambda(u') = \Lambda(u'')$.

An endomorphism $u\colon E \to E$ of a graded vector space E is called *weakly-nilpotent* if for every $q \geq 0$ and for every $x \in E_q$, there exists an integer n such that $u_q^n(x) = 0$. Since, for a weakly-nilpotent endomorphism $u\colon E \to E$, we have $N(u) = E$, so

Proposition 3.2. *If $u\colon E \to E$ is a weakly-nilpotent endomorphism, then $\Lambda(u) = 0$.*

Let $f\colon (X, X_0) \to (X, X_0)$ be a map, $f_*\colon H(X, X_0) \to H(X, X_0)$ is a Leray endomorphism. For such f, we define the Lefschetz number $\Lambda(f)$ of f by putting $\Lambda(f) = \Lambda(f_*)$. Clearly, if f and g are homotopic, $f \sim g$, then f is a Lefschetz map if and only if g is a Lefschetz map; and, in this case, $\Lambda(f) = \Lambda(g)$.

Let us observe that if X is an acyclic space or in particular contractible then for every $f\colon X \to X$ the endomorphism $f_*\colon H(X) \to H(X)$ is a Leray endomorphism and $\Lambda(f_*) = 1$.

Consequently, if $X \in \text{AR}$ odr X is a convex subset in a normed space, then for every continuous map $f\colon X \to X$ the Lefschetz number $\Lambda(f) = \Lambda(f_*) = 1$.

We have the following lemma (see: [3, 5]).

Lemma 3.3. *Let $f\colon (X, X_0 \to (X, X_0)$ be a map of pairs. If two of those endomorphisms $f_*\colon H(X, X_0) \to H(X, X_0)$, $(f_X)_*\colon H(X) \to H(X)$, $(f_{X_0})_*\colon H(X_0) \to H(X_0)$ are Leray endomorphisms, then so is the third; and in that case:*

$$\Lambda(f_*) = \Lambda\big((f_X)_*\big) - \Lambda\big((f_{X_0})_*\big)$$

or equivalently:

$$\Lambda(f) = \Lambda(f_X) - \Lambda(f_{X_0}).$$

Definition 3.4. *A continuous map $f\colon X \to X$ is called a Lefschetz map provided the generalized Lefschetz number $\Lambda(f)$ of f is well defined and $\Lambda(f) \neq 0$ implies that the set $S(f) = \{x \in X \mid f(x) = x\}$ is nonempty.*

In 1969, A. Granas ([8], see also [9]) proved:

Theorem 3.5. *Let $X \in \mathrm{ANR}$ and let $f\colon X \to X$ be a continuous and compact map (i.e., $\overline{f(X)}$ is a compact set), then f is a Lefschetz map.*

To formulate the result proved in 1977 by R. Nussbaum ([12]) we need some notations.

We shall need the following Kuratowski (or Hausdorff) (see [M-9, M-10, M-11]) measure of noncompactness. Let X be a complete metric space and A be a bounded subset of X.

We let:

$\gamma(A) = \inf\{r > 0 \mid$ there exists a finite covering of A by subsets of diameter at most $r\}$

or

$\gamma(A) = \inf\{r > 0 \mid$ there exists a finite covering of A by open balls with radius $r\}$.

We have the following properties:

3.6. $0 \leq \gamma(A) \leq \delta(A)$, *where $\delta(A)$ is the diameter of A,*

3.7. $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$,

3.8. $\gamma(N_\varepsilon(A)) \leq \gamma(A) + 2\varepsilon$, *where* $N_\varepsilon(A) = \{x \in E \mid d(x, A) < \varepsilon\}$,

3.9. $\gamma(A) = 0$ *if and only if A is relatively compact,*

3.10. *If $K_1 \supset K_2 \supset \ldots K_n \supset \ldots$, where K_n is closed nonempty for any n and $\lim_{n\to\infty} \gamma(K_n) = 0$, then $K_\infty = \bigcap_{n=1}^{\infty} K_n$ is compact and nonempty.*

Let $f\colon X \to X$ be a map and $A \subset X$ be a subset of X. We shall say that A is f-invariant (invariant under f) provided $f(A) \subset \mathrm{CA}$.

For a map $f\colon X \to X$ a compact f-invariant subset $A \subset X$ is called an *attractor* provided for any open neighbourhood U of A in X and for every compact $K \subset X$ there exists $n = n_K$ such that $f^m(K) \subset U$, for every $m \geq n = n_K$. In what follows we shall denote family of mappings with compact attractor by CA.

Note that, if $f\colon X \to X$ has a compact attractor A, then $S(f) \subset A$.

A continuous mapping $f\colon X \to X$ is called *condensing* (k-set contraction) map provided:

3.11. *if $\gamma(A) \neq 0$, then $\gamma(f(A)) < \gamma(A)$, ($\gamma(f(A)) \leq k \cdot \gamma(A)$ for some $k \in [0, 1)$), where we have assumed that X is a complete metric space and $A \subset X$.*

Of course, any compact map is a k-set contraction map and any k-set contraction map is a condensing map.

Theorem 3.12 ([3, 4, 11, 12]). *Let U be an open subset of a Banach space E. Assume further that $f\colon U \to U$ is a condensing map which has a compact attractor, then f is a Lefschetz map.*

Definition 3.13. *Let $f\colon X \to X$ be a continuous map and X_0 be a f-invariant subset of X. We shall say that X_0 absorbs compact sets provided for any compact set $K \subset X$ there exists a natural number $n = n_K$ such that $f^n(K) \subset X_0$. If for every point $x \in X$ there exists $n = n_x$ such that $f^n(x) \in X_0$, then we shall say that X_0 absorbs points.*

It is easy to prove the following:

Proposition 3.14. *Assume that $f\colon X \to X$ is a continuous map and X_0 is an open subset of X which absorbs points. Then X_0 absorbs compact sets.*

For the proof see: [3]–[5]. Now, we are able to prove the following important result:

Theorem 3.15 (Abstract version of the Lefschetz fixed point theorem). *Let $f\colon (X, X_0) \to (X, X_0)$ be a continuous map of pairs. Assume that $f_{X_0}\colon X_0 \to X_0$ is a Lefschetz map and X_0 absorbs compact sets. Then $f_X\colon X \to X$ is a Lefschetz map.*

Proof. First, we shall observe that $f_*\colon H(X, X_0) \to H(X, X_0)$ is weakly nilpotent and hence $\Lambda(f) = \Lambda(f_*) = 0$.

We let:

$$
\begin{aligned}
&i\colon X_0 \to X, \quad i(x) = x \quad \text{for every } x \in X_0,\\
&\widetilde{H}(X) = H(X)/N((f_X)_*),\\
&\widetilde{H}(X_0) = H(X_0)/N((f_{X_0})_*),\\
&\widetilde{i}_*\colon \widetilde{H}(X_0) \to \widetilde{H}(X), \quad \widetilde{i}_*([a]) = [i_*(a)] \quad \text{for every } [a] \in \widetilde{H}(X_0).
\end{aligned}
$$

Since the considered functor H has compact carriers and X_0 absorbs compact sets we deduce that $\widetilde{i}_*$ is an isomorphism. Consequently from the exactness of the homology sequence for the pair (X, X_0) we infer that $\widetilde{H}(X, X_0) = 0$. Thus $\Lambda(f) = \Lambda(f_*) = 0$ and from Lemma (3.3) we obtain:

$$\Lambda(f) = \Lambda(f_*) = 0.$$

By assumption $f_{X_0}\colon X_0 \to X_0$ is a Lefschetz map. Therefore, in view of Lemma (3.3), we deduce that the Lefschetz number $\Lambda(f_X)$ of f_X is well defined and

$$\Lambda(f) = 0 = \Lambda(f_X) - \Lambda(f_{X_0}).$$

Now, if we assume that $\Lambda(f_X) \neq 0$, then $\Lambda(f_{X_0}) \neq 0$ and hence $S(f_{X_0}) \neq \emptyset$. The proof is completed since $S(f_{X_0}) \subset S(f_X)$. □

In what follows all mappings are assumed to be continuous. Following [4] we recall the notion of compact absorbing contractions (CAC).

Definition 3.16. *A mapping $f\colon X \to X$ is called CAC provided the following conditions are satisfied:*

1. *there exists an open subset U of X such that $\overline{f(U)} \subset U$ and $\overline{f(U)}$ is compact,*
2. *the set U given in* (1) *absorbs points.*

First, we are going to explain how large the class of CAC-mappings is. Evidently, any compact map $f\colon X \to X$ is a CAC-mapping. In fact, the compact set $\overline{f(X)}$ is an attractor of f and we can take X as an open neighbourhood U of $\overline{f(X)}$. More generally, any *eventually compact* map, i.e., the map $f\colon X \to X$ such that there exists n for which $\overline{f^n(X)}$ is compact, has a compact attractor A to be equal $\overline{f^n(X)}$. It is also easy to see that any CAC-map has a compact attractor A, namely $\overline{f(U)}$ (see Definition 3.16.1).

We shall say that a map $f\colon X \to X$ is *asymptotically compact* provided for each $x \in X$ the orbit $\{x, f(x), \ldots, f^n(x), \ldots\}$ is relatively compact and the core

$$C_f = \bigcap_{n=1}^{\infty} \overline{f^n(X)}$$

is nonempty compact.

As is observed in [4] (see Proposition 6.4) any asymptotically compact map $f\colon X \to X$ has a compact attractor A to be equal C_f.

If follows from the above that:

Proposition 3.17.

1. *Any compact map has a compact attractor,*
2. *any eventually compact map has a compact attractor,*
3. *any asymptotically compact map has a compact attractor.*

So the class of mappings with compact attractors is quite large.

To explain the connection between mappings with compact attractors and CAC-mappings we need one more notion.

A map $f\colon X \to X$ is called *locally compact* (LC-map) provided for every $x \in X$ there exists an open neighbourhood U_x of x in X such that $\overline{f(U_x)}$ is compact.

We have:

Proposition 3.18 (see [3]–[5]). *Any locally compact map with compact attractor is a* CAC*-mapping.*

All obtained above information we can illustrate in the following

$$\boxed{\text{CA + LC}} \subset \boxed{\text{CAC}} \subset \boxed{\text{CA}}.$$

We recommend Theorems 4.7, 4.8 in [M-2, M-10, M-15], for further information about considered classes of mappings.

Let us mention the first application of Theorem 3.15:

Theorem 3.19. *Let $X \in$ ANR and $f\colon X \to X$ be a* CAC*-map. Then f is a Lefschetz map.*

Proof. Let $f\colon X \to X$ be a CAC-map, where $X \in$ ANR. We choose an open subset $U \subset X$ according to the point 1 of Definition 3.16. Then $f(U) \subset U$ and $\overline{f(U)} \subset U$ is compact. Therefore, in view of (3.5), the map $\widetilde{f}\colon U \to U$, $\widetilde{f}(x) = f(x)$ is a Lefschetz map. Now our claim follows from the 2 of Definition 3.16, and Theorem 3.15. □

Corollary 3.20. *If $X \in$ AR and $f\colon X \to X$ is a* CAC*-map, then $S(f) \neq \emptyset$.*

Open problem 3.21. Is Theorem 3.19 true for every CA-mapping f?

Now, we are going to discuss the Lefschetz fixed point theorem for condensing mappings.

We prove the following:

Proposition 3.22. *Let (X, d) be a complete bounded space and let $f\colon X \to X$ be a condensing map. Then f is an asymptotically compact map, in particular f has a compact attractor.*

Proof. According to the Proposition 2 in [15] we have

$$\lim_{n\to\infty} \gamma(\overline{f^n(X)}) = 0.$$

It implies, in view of 3.10, that the core

$$C_f = \bigcap_{n=1}^{\infty} \overline{f^n(X)}$$

is compact and nonempty.

Moreover, let $O(x) = \{x, f(x), f^2(x), \ldots\}$ be an orbit of $x \in X$ with respect to f. Then we have: $O(x) = \{x\} \cup f(O(x))$ and consequently, if we assume that $\gamma(O(x)) > 0$, then we get:

$$\gamma(O(x)) = \gamma\left(f\big(O(x)\big)\right) < \gamma(O(x)),$$

a contradiction. So f is asymptotically compact and therefore it has a compact attractor. □

Corollary 3.23. *Let U be an open subset of a Banach space E and let $f\colon U \to U$ be a condensing map. If there exists a closed bounded subset B of E such that $f(U) \subset B \subset U$, then f has a compact attractor.*

In fact, by applying (3.22) to $\widetilde{f}\colon B \to B$, $\widetilde{f}(x) = f(x)$ for every $x \in B$, we get (3.23).

Now, from Theorem 3.12 and Corollary 3.23 we get:

Corollary 3.24. *Let U and $f\colon U \to U$ be the same as in Corollary 3.23. Then f is a Lefschetz map.*

We need the following definition:

Definition 3.25. *A complete, bounded metric space (X, d) is called a special ANR (written $X \in ANR_s$) provided there exists an open U of a Banach space E and two continuous mappings $r\colon U \to X$ and $s\colon X \to U$ such that:*

1. *$r \circ s = id_X$,*
2. *r and s are nonexpansive, i.e., $\gamma(r(B)) \leq \gamma(B)$ and $\gamma(s(A)) \leq \gamma(A)$ for arbitrary two bounded sets A and B.*

We are able to prove the following version of the Lefschetz fixed point theorem:

Theorem 3.26. *Let $X \in \mathrm{ANR}_s$ and let $f\colon X \to X$ be a condensing map. Then f is a Lefschetz map.*

Proof. From Proposition 3.22 we deduce that f has a compact attractor. Let U, $r\colon U \to X$ and $s\colon X \to U$ are according to Definition 3.25.

We define the map $\widetilde{f}\colon U \to U$ by putting:

$$\widetilde{f} = s \circ f \circ r.$$

In, view of 3.25.2., we deduce that $\widetilde{f}$ is a condensing map. Observe that if A is a compact attractor of f, then $s(A)$ is a compact attractor $\widetilde{f}$ (see: 3.25.2.). Consequently $\widetilde{f}\colon U \to U$ is a condensing with compact attractor map. From the other hand we have the following commutative diagram

$$\begin{array}{ccc} U & \xleftarrow{\ s\ } & X \\ {\scriptstyle \widetilde{f}}\downarrow & \nearrow_{f \circ r} & \downarrow{\scriptstyle f} \\ U & \xleftarrow[\ s\ ]{} & X \end{array}$$

Thus $\Lambda(f) = \Lambda(\widetilde{f})$ and our theorem follows from Theorem 3.12. □

Lemma 3.27. *Let $f\colon X \to X$ be a map. Assume further that A is a compact attractor for f and V is an open neighbourhood A in X. Then there exists an open neighbourhood U of A in X such that:*

1. *$f(U) \subset U$,*
2. *$A \subset U \subset V$.*

Proof. Let $U = \bigcap_{n=0}^{\infty} f^{-n}(V)$. Then $f(U) \subset U$ and $A \subset U$. We only need to show that U is an open subset of X. On the contrary, suppose that there exists a sequence $\{x_n\} \subset X \setminus U$ such that $\lim_{n\to\infty} x_n = x$ and $x \in U$. Let $K = \{x_n\} \cup \{x\}$. Then K is a compact set and consequently there exists m such that $f^i(K) \subset V$ for all $i \geq m$. Hence $x_n \in \bigcap_{i=m}^{\infty} f^{-i}(V)$. But $x_n \notin U$ so $x_n \notin \bigcap_{i=m}^{\infty} f^{-i}(V)$ and from the continuity of f follows that $x \notin \bigcap_{i=0}^{m} f^{-i}(V)$ which contradicts the fact that $x \in U$. □

A subset $B \subset X$ is called *nonexpansive retract* provided there exists a continuous map $r\colon X \to B$ such that:

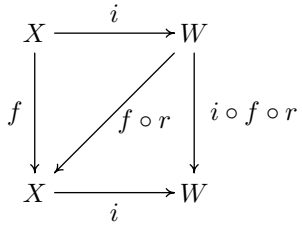
(i) $r(x) = x$ for every $x \in B$,

(ii) $d(r(x), r(y)) \leq d(x, y)$ for every $x, y \in X$.

We prove:

Theorem 3.28. *Assume that X is nonexpansive retract of some open subset W in a Banach space E. Assume further that $f\colon X \to X$ is* CA*-mapping with a compact attractor A. If there exists an open neighbourhood V of A in X such that the restriction $f|_V\colon V \to X$ of f to V is a condensing map, then f is a Lefschetz map.*

Proof. For the proof consider the following diagram



in which $r\colon W \to X$ is the nonexpansive retraction and $i\colon X \to W$ is the inclusion map. Let us put $g = i \circ f \circ r$.

From the commutativity of the above diagram it follows that f is a Lefschetz map iff g is a Lefschetz map. Observe also that A is an attractor for g and moreover, $g|_{r^{-1}}(V) : r^{-1}(V) \to W$ is a condensing map. By applying Lemma (3.27) we get an open subset U of W such that $\widetilde{g} : U \to U$, $\widetilde{g}(u) = g(u)$ is a condensing map with compact attractor A. Consequently it follows from Theorem 3.12 that $\widetilde{g}$ is a Lefschetz map.

Now, in view of (3.15), we deduce that g is a Lefschetz map and the proof is completed. □

Remark 3.29. *Observe that any k-set contraction map is condensing, so Theorems 3.26 and 3.28 remain true for k-set contraction mappings.*

From the point of view of applications in dynamical systems the relative version of the Lefschetz fixed point theorem is important (see: [1, 2, 8, 14]). In the relative

version we get not only the existence of fixed points but also some information of their localization. For the proof of the relative version instead of the Lefschetz number we need the fixed point index for the appropriate class of mappings.

We shall follow the ideas contained in [1]. First we would like to remark the following two facts:

3.30. *the fixed point index is well defined for CAC-mappings on ANR-s (see [1]),*

3.31. *the fixed point index is well defined for condensing CA-mappings on open subset of Banach spaces (see [12, 5]).*

We have the following three versions of the relative Lefschetz fixed theorem:

Theorem 3.32 (comp. [7] or [1]). *Let $X_0 \subset X$ and $X, X_0 \in$ ANR. Assume that $f\colon (X, X_0) \to (X, X_0)$ is a map such that f_X and f_{X_0} are* CAC*-mappings. Then the Lefschetz number $\Lambda(f)$ of f is well defined and $\Lambda(f) \neq 0$ implies that*

$$S(f) \cap (\overline{X \setminus X_0}) \neq \emptyset.$$

Theorem 3.33. *Let W be an open subset of a Banach space E and W_0 be an open subset of W and let $f\colon (W, W_0) \to (W, W_0)$ be a mapping such that:*

3.33.1. *f_W and f_{W_0} are condensing mappings with compact attractors.*

Then the Lefschetz number $\Lambda(f)$ of f is well defined and $\Lambda(f) \neq 0$ implies that

$$S(f) \cap (\overline{W \setminus W_0}) \neq \emptyset.$$

Similarly, for k-set contraction mappings we get:

Theorem 3.34. *Let W and W_0 be the same as in Theorem 3.33 and $f\colon (W, W_0) \to (W, W_0)$ be a mapping such that:*

3.34.1. *f_W and f_{W_0} are k-set contractions with relatively compact orbits.*

Then the Lefschetz number $\Lambda(f)$ of f is well defined and $\Lambda(f) \neq 0$ implies that

$$S(f) \cap (\overline{W \setminus W_0}) \neq \emptyset.$$

Note that the proof of Theorem 3.33 and 3.33 is strictly analogous to the proof of Theorem 3.33 which is presented in full generality in [1].

Finally, let us add some concluding remarks. We would like to point out that the following topics concerning the Lefschetz fixed point theorem are still possible:

(i) non metric case, i.e., for retracts of open sets in admissible spaces in the sense of Klee (comp. [M-2, M-15]),

(ii) periodic fixed point theory (comp. [1, 2, M-15, M-2]),

(iii) the multivalued case (comp. [M-2, M-10, 4, 6]).

4. THE LOCAL CASE

The problem of solving equations become much more complicated when we deal with the local case. Assume that A is a subset of X and

$$f\colon A \to X$$

is a given function.

We would like to study the following equation

$$f(x) = x. \tag{4.1}$$

Note that even in a very simple case the above equation may have no solutions. Namely, let us consider the function $f\colon [0,1] \to \mathbb{R}$ given by $f(x) = \frac{1}{2}x + 2$. Evidently, the equation

$$f(x) = \frac{1}{2}x + 2 = x$$

has no solutions (in $[0,1]$!) in spite of f being a contraction mapping.

Now, we are able to prove the following:

Theorem 4.1 (Local Version of the Banach Contraction Principle). *Let (X,d) be a complete space, $f\colon B(x_0,r) \to X$ be a contraction, i.e.,*

$$d(f(x), f(y)) \leq \alpha d(x,y),$$

where $0 \leq \alpha < 1$ and $x, y \in B(x_0,r)$.

If $d(f(x_0), x_0) < (1-\alpha)r$, then (4.1) has a solution.

Proof. Let $0 \leq s \leq r$ be such that $d(f(x_0), x_0) \leq (1-\alpha)\cdot s$. Then we get that $f(K(x_0,s)) \subset K(x_0,s)$ and since $K(x_0,s)$ as a closed subset of a complete space X is complete we infer, in view of the Banach contraction principle, that (4.1) has a solution. □

The general topological method to solve (4.1) was presented by L. E. J. Brouwer in 1930 and later developed by J. Leray and J. P. Schauder. This method is called the *topological degree method* or more generally the *fixed point index method*. Below we will only sketch this method, for details see: [5, 7, 8, 9, 10].

Let X be an ANR-space and U its open subset. Let:

$$\mathcal{K}(U,X) = \{f\colon U \to X \mid f \text{ is compact and } S(f) \text{ is a compact subset of } U\}.$$

Definition 4.2. *A function*

$$\mathrm{Ind}\colon \mathcal{K}(U,X) \to \mathbb{Z}$$

is called the fixed point index on $\mathcal{K}(U,X)$, provided the following conditions are satisfied:

1. *(Existence) If Ind*$(f) \neq 0$*, then* $S(f) \neq \emptyset$.
2. *(Unity) If* $f\colon U \to X$, $f(x) = x_0$, *is a constant map, then*

$$\operatorname{Ind}(f) = \begin{cases} 1, & \text{if } x_0 \in U, \\ 0, & \text{if } x_0 \notin U. \end{cases}$$

3. *(Additivity) If* $U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$ *and* $f|_{U_i} \in \mathcal{K}(U_i, X)$, $i = 1, 2$, *then*

$$\operatorname{Ind}(f) = \operatorname{Ind}(f|_{U_1}) + \operatorname{Ind}(f|_{U_2}).$$

4. *(Homotopy) If* $f, g \in \mathcal{K}(U, X)$ *are homotopic and there exists a joining compact homotopy* $h\colon U \times [0,1] \to X$ *such that:*

$$\{x \in U \mid \exists t \in [0,1],\ x = h(x,t)\}$$

is compact, then

$$\operatorname{Ind}(f) = \operatorname{Ind}(g).$$

5. *(Normalization) If* $f \in \mathcal{K}(X, X)$, *then*

$$\operatorname{Ind}(f) = \Lambda(f).$$

First of all, let us remark that such a function Ind which satisfies (1)–(5) is unique.

Remark 4.3. *Note that the fixed point index can be also defined for classes of mappings larger than* $\mathcal{K}(U, X)$ *(comp. [10]), in particular, for compact absorbing contractions of ANR-s and for condensing mappings of some particular ANR-s.*

Now, having the fixed point index, in view of (1), we are able to answer when the equation (4.1) has a solution. How to define the fixed point index? There are two different approaches:

(i) homological (see: [M-5, M-7, M-10, M-15]),

(ii) analytical (see: [M-6, M-11, M-12, M-13, M-15]).

We would like to add that for many problems the axiomatic approach is sufficient. Namely, if we assume that there exists a function $\operatorname{Ind} : \mathcal{K}(U, X) \to \mathbb{Z}$ which satisfies (1)–(5), then using this information we are able to solve quite a large class of equations. Moreover, (1) and (5) show us that global results are special cases of the local ones.

Having the fixed point index theory we are getting only an information that there exists a solution of the equation:

$$x = f(x).$$

Now, we want to know how many solutions has the above equation. In which order we shall define so called the Nielsen number $\mathcal{N}(f)$ of f.

Assume that $f\colon X \to X$ is a CAC-mapping and $X \in ANR$. Let $x_1, x_2 \in S(f)$. We shall say that x_1 is *equivalent* to x_2 (written $x_1 \sim x_2$) provided there exists a path $\tau\colon [0,1] \to X$ such that τ is homotopic to $f \circ \tau$ and there exists homotopy $h\colon [0,1] \times [0,1] \to X$ such that:

$$\begin{aligned} h(t,0) &= \tau(t), \\ h(t,1) &= f(\tau(t)), \\ h(0,s) &= x_1, \\ h(1,s) &= x_2, \end{aligned}$$

for every $s, t \in [0,1]$. It is easy to see that "$\sim$" is an equivalence relation in the set $S(f)$. Since $S(f)$ is a compact set one can show that the factor set $S(f)/_\sim$ is finite.

Assume $S(f)/_\sim = \{[x_1], \ldots, [x_k]\}$. We shal l say that the class $[x_i]$ is *essential* provided there exists an open set $U \subset X$ such that:

$$\begin{gathered} y \in [x_i], \quad \text{then } y \in U, \\ S(f) \cap \partial U = \emptyset \end{gathered}$$

and

$$\operatorname{Ind}(f; U, x) \neq \emptyset.$$

Then, we let

$$\mathcal{N}(f) = \#\big\{[x_i] \mid [x_i] \text{ is essential}\big\}.$$

Then $\mathcal{N}(f)$ is called the *Nielsen number* of f.

We have:

Theorem 4.4. *Let $f, g\colon X \to X$ be two* CAC *maps and $X \in$* ANR*. Then the following two conditions are satisfied:*

1. $\#S(f) \geq \mathcal{N}(f)$,
2. *$f \sim g$ (f is homotopic to g in the fixed point index sense), then $\mathcal{N}(f) = \mathcal{N}(g)$.*

We recommend: [M-5, M-2, M-15] for more details concerning the Nielsen fixed point theory.

5. EXAMPLES OF APPLICATIONS TO DIFFERENTIAL EQUATIONS

The aim of this section to show some examples how the topological fixed point theory can be applied to differential equations.

We shall divide this section onto three parts.

5.1. ARONSZAJN TYPE RESULTS

In this part we shall present results about the topological structure of the set of solutions of the Cauchy problem for some nonlinear ordinary differential equations as owed to N. Aronszajn in 1942.

First, we shall formulate a result proved by F. E. Browder and C. P. Gupta (see: [M-2, M-10, 6]) in 1969.

Theorem 5.1. *Let X be a space, $(E, \|\cdot\|)$ a Banach space and $f\colon X \to E$ a proper map, i.e., f is continuous and for every compact $K \subset E$ the set $f^{-1}(K)$ is compact. Assume further that for each $\varepsilon > 0$ a proper map $f_\varepsilon\colon X \to E$ is given and the following two conditions are satisfied:*

1. *$\|f_\varepsilon(x) - f(x)\| < \varepsilon$, for every $x \in X$,*
2. *for any $\varepsilon > 0$ and $u \in E$ such that $\|u\| \leq \varepsilon$, the equation $f_\varepsilon(x) = u$ has exactly one solution.*

Then the set $S = f^{-1}(0)$ is R_δ.

For the proof of this theorem see [M-10]. We recommend [6] and also [2] for some generations of Theorem 5.1.

We need also some technical result originally formulated by S. Szufla in 1979 (see [M-10]) or [6].

Theorem 5.2. *Let $E = C([0,a], \mathbb{R}^m)$ be the Banach space of continuous maps with the usual max-norm and let $X = K(0,r) = \{u \in E \mid \|u\| \leq r\}$ be the closed ball in E. If $F\colon X \to E$ is a compact map and $f\colon X \to E$ is a compact vector field associated with F, i.e., $f(u) = u - F(u)$, such that the following conditions are satisfied:*

1. *there exists an $x_0 \in \mathbb{R}^m$ such that $F(u)(0) = x_0$, for every $u \in K(0,r)$,*
2. *for every $\varepsilon \in [0,a]$ and for every $u, v \in X$ if $u(t) = v(t)$ for each $t \in [0,\varepsilon]$, then $F(u)(t) = F(v)(t)$ for each $t \in [0,\varepsilon]$,*

then there exists a sequence $f_{\varepsilon_n}\colon X \to E$ of continuous proper mappings satisfying the conditions (1) and (2) in Theorem 5.1 with respect to f, where $\varepsilon_n = 1/n$.

Sketch of proof. For the proof it is sufficient to define a sequence $F_n\colon X \to E$ of compact maps such that

$$F(x) = \lim_{n\to\infty} F_n(x), \quad \text{uniformly in } \ x \in X, \tag{i}$$

and

$$f_n\colon X \to E, \quad f_n(x) = x - F_n(x), \quad \text{is a one-to-one map.} \tag{ii}$$

To do this we additionally define the mappings $r_n\colon [0,a] \to [0,a]$ by putting:

$$r_n(t) = \begin{cases} 0, & t \in [0, a/n], \\ t - \dfrac{a}{n}, & t \in (a/n, a]. \end{cases}$$

Now, we are able to define the sequence $\{F_n\}$ as follows:

$$F_n(x)(t) = F(x)(r_n(t)) \quad \text{for } x \in X,\ n = 1, 2, \ldots. \tag{iii}$$

It is easily seen that F_n is a continuous and compact mapping, $n = 1, 2, \ldots$. Since $[r_n(t) - t] \leq a/n$ we deduce from the compactness of F and (iii) that

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \text{uniformly in } x \in X.$$

Now, we shall prove that f_n is a one-to-one map. Assume that for some $x, y \in X$ we have

$$f_n(x) = f_n(y).$$

This implies that

$$x - y = F_n(x) - F_n(y).$$

If $t \in [0, a/n]$, then we have

$$x(t) - y(t) = F(x)(r_n(t)) - F(y)(r_n(t)) = F(x)(0) - F(y)(0).$$

Thus, in view of (1) on Theorems 5.2, we obtain

$$x(t) = y(t) \quad \text{for every } t \in \left[0, \frac{a}{n}\right].$$

Finally, by successively repeating the above procedure n times we infer that

$$x(t) = y(t) \quad \text{for every } t \in [0, a].$$

Therefore, f_n is a one-to-one map and the proof is completed. □

Now, from Theorems 5.1 and 5.2 we obtain:

Corollary 5.3. *Assume that f and F are as in Theorem 5.2. Then $f^{-1}(0) = S(F)$ is an R_δ-set.*

For a given map $g\colon [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ we shall consider the following Cauchy problem

$$\begin{cases} x'(t) = g(t, x(t)), \\ x(0) = x_0. \end{cases} \tag{5.1}$$

In our considerations g is a Carathéodory mapping. By a solution of (5.1) we shall understand an absolutely continuous map $x\colon [0, a] \to \mathbb{R}^n$ such that $x'(t) = g(t, x(t))$ for almost all $t \in [0, a]$ and $x(0) = x_0$. If the right hand side g is continuous, then every solution $x(\cdot)$ is C^1 regular and satisfies $x'(t) = g(t, x(t))$ for every $t \in [0, a]$.

We denote by $S(g, 0, x_0)$ the set of all solutions of the Cauchy problem (5.1).

Theorem 5.4. *Let $g\colon [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ be an integrably bounded Carathéodory mapping. Then $S(g, 0, x_0)$ is R_δ.*

Proof. We define the integral operator

$$F\colon C([0,a],\mathbb{R}^n) \to C([0,a],\mathbb{R}^n)$$

by putting

$$F(u)(t) = x_0 + \int_0^t g\big(r,u(r)\big)\,dr \quad \text{for every } u \text{ and } t. \tag{5.2}$$

Then $S(F) = S(g,0,x_0)$. It is easy to see that F satisfies all the assumptions of Theorem 5.2. Consequently we deduce Theorem 5.4 from Corollary 5.3 and the proof is completed. □

Now, let g be a Carathéodory map with linear growth, i.e., $|g(t,x)| \le \mu(t)(1+|x|)$, where $\mu\colon [0,a] \to \mathbb{R}$ is a Lebesque integrable function. Assume further that $u \in S(g,0,x_0)$. Then we have (cf. (5.2))

$$u(t) = F(u)(t) = x_0 + \int_0^t g\big(r,u(r)\big)\,dr, \tag{5.3}$$

and consequently

$$\|u(t)\| \le \|x_0\| + \int_0^t \mu(r)\,dr + \int_0^t \mu(r)\,\|u(r)\|\,dr. \tag{5.4}$$

Therefore, from the well known Gronwall inequality we obtain

$$\|u(t)\| \le \big(\|x_0\| + \gamma\big)\exp(\gamma) \quad \text{for every } t,$$

where $\gamma = \int_0^a \mu(r)\,dr$.

We define

$$g_0\colon [0,a]\times\mathbb{R}^n \to \mathbb{R}^n$$

by putting

$$g_0(t,x) = \begin{cases} g(t,x), & \text{if } \|x\| \le M \text{ and } t \in [0,a],\\ g(t,Mx/\|x\|), & \text{if } \|x\| \ge M \text{ and } t \in [0,a], \end{cases}$$

where $M = (\|x_0\| + \gamma)\exp(\gamma)$.

Proposition 5.5. *If g is a Carathéodory map with linear growth, then:*

1. *g_0 is Carathéodory and integrably bounded; and*
2. *$S(g_0,0,x_0) = S(g,0,x_0)$.*

The proof of Proposition 5.5 is straightforward. Now, from Theorem 5.4 and Propostion 5.5 we obtain immediately:

Corollary 5.6. *If $g\colon [0,a]\times\mathbb{R}^n\to\mathbb{R}^n$ is a Carathéodory map and has linear growth then $S(g,0,x_0)$ is a R_δ-set.*

Finally, let us recall the following classical result from the theory of ordinary differential equations.

Theorem 5.7. *If $f\colon [0,a]\times\mathbb{R}^n\to\mathbb{R}^n$ is an integrably bounded, locally-measurable Lipschitz map, then the set $S(f,0,x_0)$ is a singleton for every $x_0\in\mathbb{R}^n$.*

Later we shall make use of the following:

Theorem 5.8. *Let E be a normed space, X a metric space and $F\colon E\times X\to E$ a continuous (singlevalued) map such that for any compact subset $A\subset X$ the closure $\overline{F(E\times A)}$ of $F(E\times A)$ is a compact subset of E. Then the (multivalued) map $\varphi\colon X\multimap E$ defined as follows*

$$\varphi(x)=S\big(F(\cdot,x)\big)$$

is an u.s.c. mapping.

Proof. It follows from the Schauder Fixed Point Theorem that the set $\varphi(x)$ is compact and nonempty for every $x\in X$.

Let $x_0\in X$ and let U be an open neighbourhood of $\varphi(x_0)$ in E. It is enough to prove that there exists $r>0$ such that for every $x\in B(x_0,r)$ we have $\varphi(x)\subset U$. Assume to the contrary that for every $n=1,2,\dots$ there exists $x_n\in B(x_0,1/n)$ and $y_n\in S\big(F(\cdot,x_n)\big)$ such that $y_n\notin U$.

We let $A=\overline{\{x_n\}}$. So, $A=\{x_n\}\cup\{x_0\}$. Consequently, in view of our assertion, we can assume that $\lim_{n\to\infty}y_n=y_0$. Then $y_n\notin U$ and $y_0\in S\big(F(\cdot,x_0)\big)=\varphi(x_0)\subset U$, so we obtain a contradiction. □

5.2. THE POINCARÉ TRANSLATION OPERATOR

In this section, we shall define the Poincaré translation operator along trajectories of a differential equations ([M-2]).

This operator is useful in the qualitative study of both differential equations and differential inclusions (comp. [M-2]).

We shall restrict our considerations to the periodic problem for ordinary differential equations.

Let $f\colon [0,a]\times\mathbb{R}^n\to\mathbb{R}^n$ be a continuous bounded function. We shall consider the following periodic problem

$$\begin{cases} x'(t)=f(t,x(t)),\\ x(0)=x(a). \end{cases} \tag{5.5}$$

Let $C([0,a],\mathbb{R}^n)$ be the Banach space of continuous functions with the usual maximum norm. For the Cauchy problem

$$\begin{cases} x'(t)=f(t,x(t)),\\ x(0)=x_0 \end{cases} \tag{5.6}$$

by $S(f;x_0)$ we shall denote the set of all solutions of (5.6). According to the Aronszajn theorem, we know that $S(f;x_0)$ is an R_δ-set.

We define a multivalued map

$$P\colon\ \mathbb{R}^m \multimap C([0,a],\mathbb{R}^n)$$

by putting

$$P(x) = S(f;x).$$

Let $l_a\colon C([0,a],\mathbb{R}^n) \to \mathbb{R}^n$, $l_a(x) = x(a)$ be the evaluation map.

So we have

$$\mathbb{R}^n \overset{P}{\multimap} C\big([0,a],\mathbb{R}^n\big) \overset{l_a}{\multimap} \mathbb{R}^n\,.$$

Then the (multivalued) map

$$P_a\colon\ \mathbb{R}^n \multimap \mathbb{R}^n$$

defined as follows

$$P_a = l_a \circ P$$

is called the *Poincaré translation operator.*

Remark 5.9. *Observe that if* (5.6) *has exactly one solution, then P_a is single valued operator as considered in* [M-6, M-11].

In general P_a is multivalued admissible map in the sense of L. Górniewicz. Consequently (see [6, M-10]) the fixed point index for P_a is well defined for any open $U \subset \mathbb{R}^n$ such that P_a is fixed point free on the boundary ∂U of U in $\mathbb{R}^n$.

Our first result is selfevident.

Proposition 5.10. *Problem* (5.5) *has a solution iff there exists $x \in \mathbb{R}^n$ such that $x \in P_a(x)$.*

So to study the periodic problem it is sufficient to apply the fixed point index theory to P_a. Roughly speaking, if on some ball in $\mathbb{R}^n$ the fixed point index of P_a turns out to be different from zero, then the problem (5.5) has solutions. Subsequently, by using the classical Liapunov–Krasnoselskiĭ guiding potential method we are able to calculate the fixed point index of P_a.

Definition 5.11. *A C^1-map $V\colon \mathbb{R}^n \to \mathbb{R}$ is called a direct potential for the problem* (5.5) *provided the following two conditions are satisfied:*

(i) $\exists\, r_0 > 0\ \forall\, \|x\| \geq r_0\ \operatorname{grad} V(x) \neq 0$,

(ii) $\langle f(t,x), \operatorname{grad} V(x)\rangle \geq 0$,

for every $x \in \mathbb{R}^n$ such that $\|x\| \geq x_0$ and for every $t \in [0,a]$, where $\operatorname{grad} V$ *denote the gradient of V and $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^n$.*

It follows from (i) that the fixed point index $\mathrm{Ind}(\mathrm{id\text{-}grad}\, V, B(r))$ is well defined on every ball with the radius $r \geq r_0$ and is always the same (by localization property of the fixed point index).

So it enables us to define the index $\mathrm{Ind}(V)$ of V by putting:

$$\mathrm{Ind}(V) = \mathrm{Ind}(\mathrm{id} - \mathrm{grad}\, V, B(r)), \quad r \geq r_0.$$

Now, we can formulate the following (see [1, M-10, M-11]).

Theorem 5.12. *If the problem* (5.5) *has a quaiding function* V *with* $\mathrm{Ind}V \neq 0$*, then the fixed point index of* P_a *on* $B(r)$*,* $r \geq r_0$ *is different from zero; consequently* (5.5) *has a solution.*

For more details and another generalizations see [M-2, M-10].

5.3. IMPLICIT DIFFERENTIAL EQUATIONS

The aim of this part is to show that, using the fixed point index theory as a tool, many types of implicit differential equations can be reduced very easily to differential inclusions without implicity. We shall show how to apply our method to:

(i) ordinary differential equations of first or higher order (e.g., the satellite equation),

(ii) hyperbolic differential equations,

(iii) elliptic differential equations.

First, we shall prepare the topological material needed in our applications. We have the following (see: [M-2, M-10]).

Proposition 5.13. *Let* $X \in \mathrm{ANR}$ *and* $g\colon X \to X$ *be a* CAC *map. Assume further that the following conditions are satisfied:*

1. *the topological dimension* $\dim S(g)$ *of the solution set* $S(g)$ *of* g *is equal to zero,*
2. *there exists an open* $U \subset X$ *such that* $\partial U \cap S(g) = \emptyset$ *and* $\mathrm{Ind}(g, U) \neq 0$.

Then there exists a point $z \in S(g)$ *for which we have:*

3. *for every open neighbourhood* U_z *of* z *in* X *there exists an open neighbourhood* V_z *of* z *in* X *such that* $V_z \subset U_z$, $\partial V_z \cap S(g) = \emptyset$ *and* $\mathrm{Ind}(g, V_z) \neq 0$.

Now, we are going to consider a more general situation. Namely, let Y be a space such that for every $y \in Y$ and for every open neighbourhood U_y of y in Y there exists an open arcwise connected $W \subset Y$ such that $y \in W \subset U_y$, $X \in \mathrm{ANR}$ and let $f\colon Y \times X \to X$ be a compact map. In what follows we shall assume that f satisfies the following condition:

$$\forall y \in Y \ \exists U_y\colon U_y \text{ is open in } X \text{ and } \mathrm{Ind}(f_y, U_y) \neq 0, \tag{5.7}$$

where $f_y\colon X \to X$ is given by the formula $f_y(x) = f(y,x)$ for every $x \in X$. Observe that in particular, if X is an absolute retract, then (5.7) holds automatically. We associate with a map $f\colon Y \times X \to X$ satisfying the above conditions the following multivalued map:

$$\varphi_f\colon Y \multimap X, \quad \varphi_f(y) = S(f_y).$$

Then from (5.7) follows that φ_f is well defined. Moreover, we get:

Proposition 5.14. *Under all of the above assumptions the map $\varphi_f\colon Y \multimap X$ is u.s.c.*

Let us remark that, in general, φ_f is not a l.s.c. map.[2)] Below we would like to formulate a sufficient condition which guarantees that φ_f has a l.s.c. selector. To get it we shall add one more assumption. Namely, we assume that f satisfies the following condition

$$\forall y \in Y \dim S(f_y) = 0. \tag{5.8}$$

Now, in view of (5.7) and (5.8), we are able to define the map $\psi_f\colon Y \to X$ by putting $\psi_f(y) = \mathrm{cl}\{z \in S(f_y) \mid \text{for } z \text{ from condition 5.13 (1) is satisfied}\}$, for every $y \in Y$.

We prove the following:

Theorem 5.15. *Under all of the above assumptions we have:*

1. *ψ_f is a selector of φ_f, i.e. $\psi_f(x) \subset \varphi_f(x)$, for every x,*
2. *ψ_f is a l.s.c. map.*

Proof. Since 5.15 (1) follows immediately from the definition we shall prove 5.15 (2). To do it we let:

$$\eta_f\colon Y \multimap X, \quad \eta_f(y) = \{x \in S(f_y) \mid x \text{ satisfies 5.13 (3)}\}.$$

For the proof it is sufficient to show that η_f is l.s.c. Let U be an open subset of X and let $y_0 \in Y$ be a point such that $\eta_f(y_0) \cap U \neq \emptyset$. Assume further that $x_0 \in \eta_f(y_0) \cap U$. Then there exists an open neighbourhood V of x_0 in X such that $V \subset U$ and $\mathrm{Ind}(f_{y_0}, V) \neq 0$. Since φ_f is an u.s.c. map and Y satisfies our assumptions, we can find an open arcwise connected W in Y such that $y_0 \in W$ and for every $y \in W$ we have

$$S(f_y) \cap \partial V = \emptyset. \tag{$**$}$$

Let $y \in W$ and let $\sigma\colon [0,1] \to W$ be an arc joining y_0 with y, i.e., $\sigma(0) = y_0$ and $\sigma(1) = y$. We define a homotopy $h\colon [0,1] \times V \to X$ by putting: $h(t,x) = f(\sigma(t), x)$. Then it follows form $(**)$ that h is a well defined homotopy joining f_{y_0} with f_y and hence we get: $\mathrm{Ind}(f_{y_0}, V) \neq 0$; so $S(f_y) \cap V \neq \emptyset$ and our assertion follows from Proposition 5.13. □

[2)] A multivalued mapping $\varphi\colon X \multimap Y$ is called a *lower semi continuous* (l.s.c.) provided for every open U in Y the set $\varphi^{-1}(U) = \{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$ is open.

Remark 5.16. *Let us remark that the above results remain true for admissible multivalued maps* (*cf.* (5.8))*; proofs are completely analogous.*

Observe that condition (5.8) is quite restrictive. Therefore it is interesting to characterize the topological structure of all mappings satisfying (5.3.4). We shall do it in the case of Euclidean spaces (which+ is sufficient from the point of view of our applications), but in fact it is possible for arbitrary smooth manifolds. Let A be a closed subset of the Euclidean space $\mathbb{R}^m$. By $C(A \times \mathbb{R}^n, \mathbb{R}^n)$ we shall denote the Banach space of all compact (singlevalued) maps from $A \times \mathbb{R}^n$ into $\mathbb{R}^n$ with the usual supremum norm.

Let

$$D = \{f \in C(A \times \mathbb{R}^n, \mathbb{R}^n) \mid f \text{ satisfies (5.8) for } Y = A \text{ and } X = \mathbb{R}^n\}.$$

We have:

Theorem 5.17. *The set D is dense in $C(A \times \mathbb{R}^n, \mathbb{R}^n)$.*

Remark 5.18. *In fact, one can easily prove that set D is residual in $C(A \times \mathbb{R}^n, \mathbb{R}^n)$.*

Now, we shall show how to apply the results formulated

Ordinary differential equations of first order 5.19. We let $Y = [0,1] \times \mathbb{R}^n$, $X = \mathbb{R}^n$ and let $f\colon Y \times X \to X$ be a compact map. Then f satisfies condition (5.7) automatically so we shall assume only (5.8). Let us consider the following equation:

$$x'(t) = f(t, x(t), x'(t)), \tag{5.9}$$

where the solution is understood in the sense of almost everywhere $t \in [0,1]$ (a.e., $t \in [0,1]$).

We shall associate with (5.9) the following two differential inclusions:

$$x'(t) \in \varphi_f(t, x(t)) \tag{5.10}$$

and

$$x'(t) \in \psi_f(t, x(t)), \tag{5.11}$$

where φ_f and ψ_f are defined earlier for f and by solution of (5.10) or (5.11) we mean an absolutely continuous function which satisfies (5.10) (resp. (5.11)) in the sense of a.e., $t \in [0,1]$).

Denote by $S(f)$, $S(\varphi_f)$ and $S(\psi_f)$ the set of all solutions of (5.9), (5.10) and (5.11) respectively. Then we get:

$$S(\psi_f) \subset S(f) = S(\varphi_f). \tag{5.12}$$

But in view of Theorem 5.15 the map φ_f is l.s.c., so (see [4] or [9]) we obtain

$$S(\psi_f) \neq \emptyset. \tag{5.13}$$

Thus we have proved:

$$\emptyset \neq S(\psi_f) \subset S(\varphi_f) = S(f), \tag{5.14}$$

Observe that in (5.10) and (5.11) the right side doesn't depend on derivative.

Ordinary differential equations of higher order 5.20. We let $Y = [0,1] \times \mathbb{R}^{kn}$, $X = \mathbb{R}^n$ and let $f\colon Y \times X \to X$ be a compact map. Then, similarily as in 5.19 f satisfies (5.7) so we shall assume only (5.8). To study the existence problem for the following equation:

$$x^{(k)}(t) = f(t, x(t), x'(t), \ldots, x^{(k)}(t)) \tag{5.15}$$

we consider the following two differential inclusions (cf. [1] or [13]):

$$x^{(k)}(t) \in \varphi_f(t, x(t), x'(t), \ldots, x^{(k-1)}(t)) \tag{5.16}$$

and

$$x^{(k)}(t) \in \psi_f(t, x(t), x'(t), \ldots, x^{(k-1)}(t)). \tag{5.17}$$

Then the existence problem for (5.15) can be reduced very easily to (5.16) or (5.17).

Hyperbolic equations 5.21. Now, let $Y = [0,1] \times [0,1] \times \mathbb{R}^{3n}$, $X = \mathbb{R}^n$ and let $f\colon Y \times X \to X$ be a compact map. Again it is easy to see that f satisfies (5.7) so we shall assume (5.8). Now, let us consider the following hyperbolic equation

$$u_{ts}(t,s) = f(t, s, u(t,s), u_t(t,s), u_s(t,s), u_{ts}(t,s)), \tag{5.18}$$

where the solution $u\colon [0,1] \times [0,1] \to \mathbb{R}^n$ is understood in the sense of a.e., $(t,s) \in [0,1] \times [0,1]$.

As above we associate with (5.18) the following two differential inclusions:

$$u_{ts}(t,s) \in \varphi_f(t, s, u(t,s), u_t(t,s), u_s(t,s)) \tag{5.19}$$

and

$$u_{ts}(t,s) \in \psi_f(t, s, u(t,s), u_t(t,s), u_s(t,s)). \tag{5.20}$$

Then it is evident that the set of all solutions of (5.18) is equal to the set of all solutions of (5.19) and every solution of (5.20) is a solution of (5.19). So inclusions (5.19) and (5.20) give us full information about (5.18).

Elliptic differential equations 5.22. Let $K(0,r)$ denotes the closed ball in $\mathbb{R}^n$ with center at 0 and radius r. Now, we put $Y = K(0,r) \times \mathbb{R}^{2n}$, $X = \mathbb{R}^n$ and let $f\colon Y \times X \to X$ be a compact map. Since (5.7) is satisfied we assume only (5.8). We consider the following elliptic equation:

$$\Delta(u)(z) = f(z, u(z), D(u)(z)), \quad \text{a.e., } z \in K(0,r), \tag{5.21}$$

where Δ denotes the Laplace operator and $D(u)(z) = u_{z_1}(z) + \ldots + u_{z_n}(z)$; $z = \{z_1, \ldots, z_n)$.

Then we can consider the following two differential inclusions

$$\Delta(u)(z) \in \varphi_f(z, u(z), D(u)(z), \Delta(u)(z)) \tag{5.22}$$

$$\Delta(u)(z) \in \psi_f(z, u(z), D(u)(z)). \tag{5.23}$$

and we have exactly the same situation as in 5.21 or 5.20.

We shall end our applications by making the following three remarks.

Remark 5.23. *Observe that all results of this section, except* (5.8)*, remain true if we replace the Euclidean space* $\mathbb{R}^n$ *by an arbitrary Banach space.*

Remark 5.24. *Let us observe if we replace* (5.9)*,* (5.15)*,* (5.18) *and* (5.21) *by the respective differential inclusions then we get all results of this section without any change.*

Remark 5.25. *We recommend* [M-6] *and* [M-2] *for some other applications of our approach. In particular, for the case when* f *is not a compact map.*

Acknowledgments

The author wishes to express his gratitude to the referee for helpful remarks and improvements.

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Received: December 17, 2004.