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CONVEX COMPACT FAMILY OF POLYNOMIALS AND ITS STABILITY

Abstract. Let P be a set of real polynomials of degree n. Set P can be identified with some subset $\mathcal{P}$ of $\mathbb{R}^n$ consists of vectors of coefficients of P. If $\mathcal{P}$ is a polytope, then to ascertain whether the entire family of polynomials P is stable or not, it suffices to examine the stability of the one–dimensional boundary sets of $\mathcal{P}$. In present paper, we extend this result to convex compact polynomial families. Examples are presented to illustrate the results.

1. REGULAR SETS

Consider a whole family W_n of real polynomials of degree n with typical element

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0.$$

It can be easily seen that the set W_n is a n-dimensional affine space. Because there exists a bijection Φ between W_n and $\mathbb{R}^n$

$$\Phi\colon W_n \ni \left\{ x \to x^n + a_{n-1}x^{n-1} + \ldots + a_0 \right\} \to [a_{n-1}, \ldots, a_0]^T \in \mathbb{R}^n, \qquad (1.1)$$

elements of W_n can be identified with vectors of $\mathbb{R}^n$.

In view of (1.1), let $P \subset \mathbb{R}^n$ denote a family of polynomials which the stability we want to examine. The stability of a family of polynomials P means that every polynomial contained in P has its roots in the unit disc (Shur stability), in the open left half-plane (Hurwitz stability) or generally speaking (as in this paper) in some given open set.

Now, let us formulate some notions that are useful in proving the principal result of this paper. A root space and a domain are defined just like in [1].

Definition 1.1. *Suppose we are given a set $P \subset \mathbb{R}^n$. A set $\mathcal{R}(P)$ called a root space of P is defined as*

$$\mathcal{R}(P) \stackrel{df}{=} \{z \in \mathbb{C} \colon \exists f \in P \colon f(z) = 0\}.$$

Definition 1.2. *An open set $U \subset \mathbb{C}$ is called a domain if every closed curve without self-crossings contained in U encloses only points of the set U.*

The most interesting examples of the domain are the open left-half plane, the unit disc, the plane without the half line $\mathbb{R}_- \times \{0\}$. These sets appear universally in variety of applications.

Definition 1.3. *Suppose we are given a set $P \subset \mathbb{R}^n$. P is said to be regular if for every domain U the following implication is fulfilled*

$$\mathcal{R}(\partial P) \subset U \Rightarrow \mathcal{R}(P) \subset U, \tag{1.2}$$

where ∂P denotes the boundary of P.

In stability theory, the regularity of a set appears to be a very useful property. In view of (1.2), if we know that some set P is regular then to check its stability it suffices to check the stability of its boundary. In fact, the boundary of a set is often a set of the lower dimension.

Following [1], we define the m-dimensional polytope as the m-dimensional convex hull of a finite number of points called vertices. The dimension of a set contained in $\mathbb{R}^n$ is defined in the usual sense, i.e. it is the dimension of the affine subspace of $\mathbb{R}^n$ generated by the set. Basing on the notion of the regularity of the set, the main result obtained in [1] we are able to note deftly as the following theorem.

Theorem 1.4. *Every at least two-dimensional polytope is regular.*

We now attempt to extend Theorem 1.4 to any convex compact set.

2. REGULARITY OF CONVEX SETS

Let $A \subset \mathbb{R}^n$ be an arbitrary compact set of polynomials. For a given $\varepsilon > 0$, let A_ε denote the set defined by

$$A_\varepsilon \stackrel{df}{=} \bigcup_{x \in A} \{y \in \mathbb{R}^n \colon d(x, y) < \varepsilon\},$$

where d is the natural distance function in the n-dimensional euclidean space $\mathbb{R}^n$. Obviously, A_ε is an open subset of $\mathbb{R}^n$ including A. Moreover, if A is convex then A_ε is also convex.

Theorem 2.1. *Every at least two-dimensional convex compact set is regular.*

Proof. To see this, let U be an arbitrary domain, let P be at least two-dimensional convex compact set and suppose that $R(\partial P) \subset U$. It follows from the continuous dependence the roots of polynomial on its coefficients, that for $\varepsilon > 0$ sufficiently small, we have

$$R((\partial P)_\varepsilon) \subset U. \tag{2.1}$$

Using the well known fact that every open set contained in $\mathbb{R}^n$ is the countable union of the arbitrary small sets of the form $[a_1, b_1) \times \ldots \times [a_n, b_n)$, where $a_i, b_i \in \mathbb{R}$ for $i = 1, \ldots, n$, we can conclude, that there exists, for chosen ε, at least two-dimensional polytope W^ε such that

$$\partial W^\varepsilon \subset (\partial P)_\varepsilon \tag{2.2}$$

and

$$P \subset W^\varepsilon. \tag{2.3}$$

By applying Theorem 1.4, it readily follows from (2.1), (2.2) and (2.3) that $\mathcal{R}(P) \subset U$. $\qquad \square$

It must be notice that one–dimensional convex compact set do not have to be regular. The required example given by Bialas and Garloff can be found in [2].

Basing on the notion of the stability of the set, Theorem 2.1 can be write as the following

Theorem 2.2. *Every at least two-dimensional convex compact family of real polynomials of degree n is stable, if and only if the boundary polynomials are stable.*

3. THE OTHER CLASS OF THE REGULAR SETS

We end this paper with an extension of Theorem 2.1. For any $A \subset \mathbb{R}^n$, denote the smallest convex subset of $\mathbb{R}^n$ including A by $\text{conv}(A)$.

Theorem 3.1. *Suppose we are given at least two–dimensional compact set $A \subset \mathbb{R}^n$ satisfying*

$$\partial \text{conv}(A) \subset \partial A. \tag{3.1}$$

Then A is regular.

Before we proof Theorem 3.1, it must be notice, that for every convex set the condition (3.1) holds. On the other hand, (3.1) does not guarantee the convexity of the set. To show an appropriate example, consider a compact convex set A. Then the set ∂A satisfies (3.1) and it is not convex.

Proof. Let $P \subset \mathbb{R}^n$ be at least two-dimensional compact set for which (3.1) holds. In view of (3.1), if for any domain U it holds $\mathcal{R}(\partial P) \subset U$ then $\mathcal{R}(\partial \text{conv}(P)) \subset U$. By applying Theorem 2.1, if follows that $\mathcal{R}(\text{conv}(P)) \subset U$. Because $P \subset \text{conv}(P)$, we have $\mathcal{R}(P) \subset U$. $\qquad \square$

4. EXAMPLES

This is a good place to visualize the results obtained in this paper.

Example 4.1. *To illustrate Theorem 2.2, consider two-dimensional compact convex polynomials family P described by*

$$P = \left\{ (x, y) \in \mathbb{R}^2 \colon x = 2r \cos \varphi, \ y = r \sin \varphi, \ (r, \varphi) \in [0, 1] \times [0, 2\pi) \right\}$$

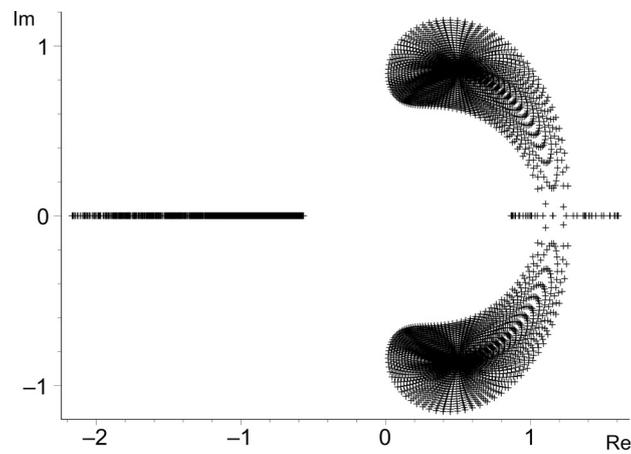*and determine the root spaces of the set P and its boundary ∂P. On the Figure 1 and Figure 2 these sets are shown.*

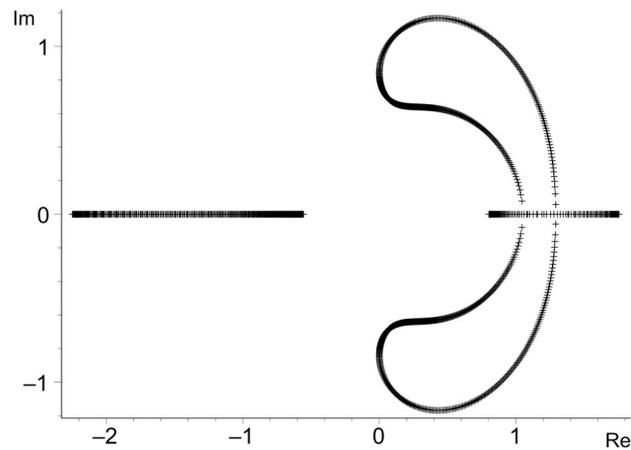**Fig. 1.** Root space of an entire set P

Fig. 2. Root space of the boundary of P

Example 4.2. *In the following example we illustrate Theorem 3.1. Let us consider polynomials family P describe by*

$$P = \left\{ (x, y) \in \mathbb{R}^2 \colon \ 4x^2 + 9y^2 \leq 36 \ \land \ 9x^2 + 4y^2 \geq 9 \right\}.$$

Obviously, the set P satisfies (3.1). *It is easy to see that to examine the stability of the set P, it suffices to examine the stability of* conv(P). *By Theorem 3.1, the stability of* conv(P) *is equivalent to the stability of* ∂ conv(P). *This fact is illustrated on the Figure 3 and Figure 4.*
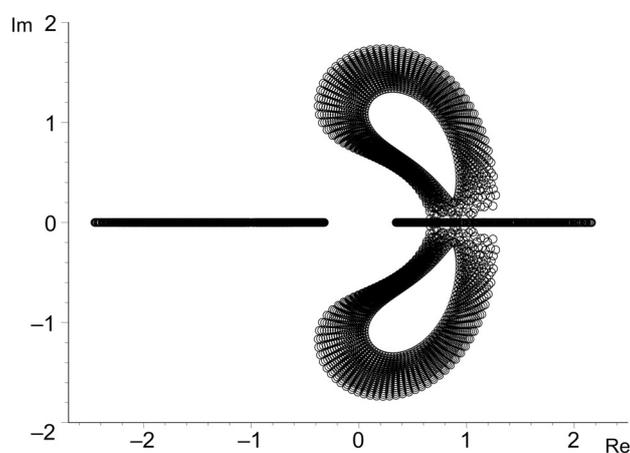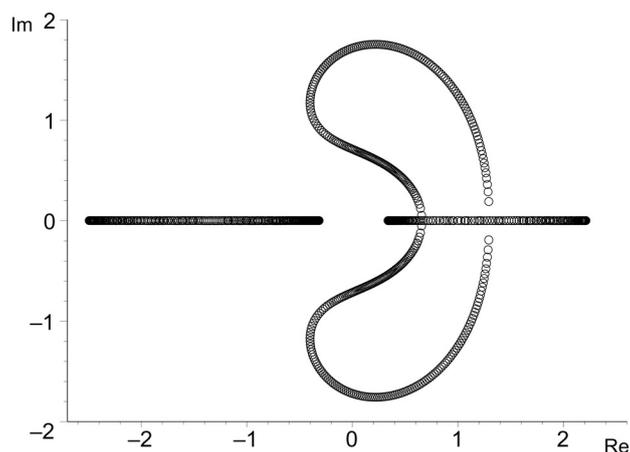

Fig. 3. Root space of an entire set P

Fig. 4. Root space of the boundary of conv(A)

5. CONCLUSIONS

The regularity of the set of polynomials appears to be a very useful propriety. As was shown in present paper, the regularity of a set makes possible to decrease an amount of operations which have to be done in order to determine the root locations of polynomials contained in some convex sets. Moreover, it seems to us that it is not very difficult to formulate some additional conditions which will guarantee the regularity of the others families of polynomials.

REFERENCES

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