

Vyacheslav Pivovarchik

**RECOVERING A PART OF POTENTIAL
BY PARTIAL INFORMATION
ON SPECTRA OF BOUNDARY PROBLEMS**

Abstract. Under additional conditions uniqueness of the solution is proved for the following problem. Given 1) the spectrum of the Dirichlet problem for the Sturm–Liouville equation on $[0, a]$ with real potential $q(x) \in L_2(0, a)$, 2) a certain part of the spectrum of the Dirichlet problem for the same equation on $[\frac{a}{3}, a]$ and 3) the potential on $[0, \frac{a}{3}]$. The aim is to find the potential on $[\frac{a}{3}, a]$.

Keywords: sine-type function, Lagrange interpolation series, Dirichlet boundary value problem, Dirichlet–Neumann boundary value problem.

Mathematics Subject Classification: 34B24, 34A55, 34B10, 73K03.

1. INTRODUCTION

The first result on uniqueness of the potential of the Sturm–Liouville equation producing the prescribed spectrum of a corresponding boundary problem was obtained in [1]. In this paper it was shown that if the spectrum of the problem with the Neumann boundary conditions coincides with the set $\{k^2\}$ where $k \in \{0\} \cup \mathbb{N}$, then $q(x) \stackrel{a.e.}{=} 0$. In [2] it was proved that in most cases two spectra of corresponding boundary problems uniquely determine the potential. Leaving aside the history of other aspects of Sturm–Liouville inverse theory (see [3]–[5]) we should mention that important step was done in [6] where it was shown that the spectrum of one boundary problem on $[0, a]$ and the potential on $[0, \frac{a}{2}]$ uniquely determine the potential on $[\frac{a}{2}, a]$. It was shown in [7], that a half of the spectrum of a boundary problem (for example Dirichlet boundary problem) on $[0, a]$ and the potential on $[0, \frac{3}{4}a]$ uniquely determine the potential on $[\frac{3}{4}a, a]$. In [8] the authors showed that the spectrum of a boundary problem (for example Dirichlet problem), a half of the spectrum of another boundary

problem (for example, Dirichlet–Neumann one) and the potential on $[0, \frac{a}{4}]$ uniquely determine the potential on $[\frac{a}{4}, a]$.

Three spectral problems were considered in [9], [10], where it was proved that the spectra of three Dirichlet boundary problems on the intervals $[0, a]$, $[0, \frac{a}{2}]$ and $[\frac{a}{2}, a]$ generated by the same potential uniquely determine the potential if these three spectra do not intersect.

In the present paper the potential is supposed to be known on the interval $[0, \frac{a}{3}]$ as well as the spectrum of the Dirichlet problem on the whole interval $[0, a]$ and a certain part of the spectrum of the Dirichlet problem on $[\frac{a}{3}, a]$. It is proven that under some additional conditions these data uniquely determine the potential on $[0, a]$.

2. MAIN RESULT

Let us consider the following Sturm–Liouville problems with the Dirichlet boundary conditions and common real potential $q(x) \in L_2(0, a)$.

$$y'' + \lambda^2 y - q(x)y = 0, \quad (1)$$

$$y(0) = y(a) = 0, \quad (2)$$

$$y'' + \lambda^2 y - q(x)y = 0, \\ y(0) = y\left(\frac{a}{3}\right) = 0, \quad (3)$$

$$y'' + \lambda^2 y - q(x)y = 0, \\ y(0) = y'\left(\frac{a}{3}\right) = 0, \quad (4)$$

$$y'' + \lambda^2 y - q(x)y = 0, \\ y\left(\frac{a}{3}\right) = y(a) = 0. \quad (5)$$

We denote by $\{\lambda_k\}_{-\infty, k \neq 0}^{\infty}$ the spectrum of problem (1), (2), by $\{\nu_k\}_{-\infty, k \neq 0}^{\infty}$ the spectrum of (1), (3), by $\{\mu_k\}_{-\infty, k \neq 0}^{\infty}$ the spectrum of (1), (4) and by $\{\nu_k^{(1)}\}_{-\infty, k \neq 0}^{\infty}$ the spectrum of (1), (5). For the sake of simplicity we assume $q(x)$ to be positive almost everywhere on $[0, a]$. Then the four above mentioned spectra are real. It is well known that the eigenvalues of these spectra are simple. We enumerate them such that $\lambda_{-k} = -\lambda_k$, $\lambda_{k+1} > \lambda_k$ for all $k \in \mathbb{N}$ and so on for each sequence of eigenvalues.

In the sequel we suppose the following condition to be satisfied:

Condition 2.1. $\{\lambda_k\}_{-\infty, k \neq 0}^{\infty} \cap \{\nu_k\}_{-\infty, k \neq 0}^{\infty} = \emptyset$ and $\{\lambda_k\}_{-\infty, k \neq 0}^{\infty} \cap \{\nu_k^{(1)}\}_{-\infty, k \neq 0}^{\infty} = \emptyset$.

Let us denote by $s(\lambda, x)$ the solution of equation (1) which satisfies the conditions $s(\lambda, 0) = s'(\lambda, 0) - 1 = 0$, by $s_1(\lambda, x)$ the solution of (1) which satisfies the conditions

$$s_1\left(\lambda, \frac{a}{3}\right) = s_1'\left(\lambda, \frac{a}{3}\right) - 1 = 0 \quad (6)$$

and by $c_1(\lambda, x)$ the solution of (1) which satisfies the conditions

$$c_1\left(\lambda, \frac{a}{3}\right) - 1 = c_1'\left(\lambda, \frac{a}{3}\right) = 0. \tag{7}$$

It is easy to check up that

$$s(\lambda, a) = s'\left(\lambda, \frac{a}{3}\right) s_1(\lambda, a) + s\left(\lambda, \frac{a}{3}\right) c_1(\lambda, a). \tag{8}$$

Relation (8) implies that if $\nu_k^{(1)} = \nu_p$ for some k and p then $\nu_p = \lambda_s$ for some s . That means that Condition 1 implies $\{\nu_k\}_{-\infty, k \neq 0}^\infty \cap \{\nu_k^{(1)}\}_{-\infty, k \neq 0}^\infty = \emptyset$.

The spectrum $\{\lambda_k\}_{-\infty, k \neq 0}^\infty$ possesses the following asymptotics (see [11])

$$\lambda_k \underset{k \rightarrow \infty}{=} \frac{\pi k}{a} + \frac{B_0}{k} + \frac{\alpha_k}{k}, \tag{9}$$

where

$$B_0 = \frac{1}{2\pi} \int_0^a q(x) dx, \quad \{\alpha_k\}_{-\infty, k \neq 0}^\infty \in l_2.$$

Applying the same results of [11] to the subintervals $[0, \frac{a}{3}]$ and $[\frac{a}{3}, a]$ we obtain

$$\nu_k \underset{k \rightarrow \infty}{=} \frac{3\pi k}{a} + \frac{B}{k} + \frac{\beta_k}{k}, \tag{10}$$

and

$$\nu_k^{(1)} \underset{k \rightarrow \infty}{=} \frac{3\pi k}{2a} + \frac{B_1}{k} + \frac{\beta_k^{(1)}}{k}, \tag{11}$$

where

$$B = \frac{1}{2\pi} \int_0^{\frac{a}{3}} q(x) dx, \quad \{\beta_k\}_{-\infty, k \neq 0}^\infty \in l_2,$$

$$B_1 = \frac{1}{2\pi} \int_{\frac{a}{3}}^a q(x) dx, \quad \{\beta_k^{(1)}\}_{-\infty, k \neq 0}^\infty \in l_2.$$

We call *fitting* any subsequence $\{\nu_{k_p}^{(1)}\}_{-\infty, p \neq 0}^\infty$ such that $\nu_{k-p}^{(1)} = -\nu_{k-p}^{(1)}$,

$$\nu_{k_p}^{(1)} \underset{p \rightarrow \infty}{=} \frac{3\pi}{2a}(2p-1) + \frac{B_1}{2p-1} + \frac{\beta_p}{p}, \tag{12}$$

where $\{\beta_p\}_{-\infty, p \neq 0}^\infty \in l_2$ and $\{\nu_{k_p}^{(1)}\}_{-\infty, p \neq 0}^\infty \cap \{\nu_k\}_{-\infty, k \neq 0}^\infty = \emptyset$.

Theorem 2.1. *Let the following data be given:*

- 1) *the spectrum $\{\lambda_k\}_{-\infty, k \neq 0}^{\infty}$;*
- 2) *the real potential $q(x) \in L_2(0, \frac{a}{3})$ on the interval $[0, \frac{a}{3}]$ (almost everywhere);*
- 3) *any fitting subsequence $\{\nu_{k_p}^{(1)}\}_{-\infty, p \neq 0}^{\infty}$.*

Then these data uniquely determine the potential $q(x)$ almost everywhere on $[0, a]$.

Proof. Knowing $q(x)$ on $[0, \frac{a}{3}]$ we can find $s(\lambda, x)$ and $s'(\lambda, x)$ on $[0, \frac{a}{3}]$ solving equation (1) with the conditions $s(\lambda, 0) = s'(\lambda, 0) - 1 = 0$. Therefore, we can find $s(\lambda, \frac{a}{3})$ and $s'(\lambda, \frac{a}{3})$. Then we find the set $\{\nu_k\}_{-\infty, p \neq 0}^{\infty}$ of zeros of $s(\lambda, \frac{a}{3})$ and the set of values $s'(\nu_k, \frac{a}{3})$. \square

Let us consider the union $\{\zeta_k\}_{-\infty}^{\infty} = \{\nu_k\}_{-\infty, k \neq 0}^{\infty} \cup \{\nu_{k_p}^{(1)}\}_{-\infty, p \neq 0}^{\infty} \cup \{\zeta_0\}$. Here we have set $\zeta_0 = 0$ and changed the enumeration to have $\zeta_{-k} = -\zeta_k$ and $\zeta_k < \zeta_{k+1}$ for all k . Due to (10) and (12) we obtain

$$\zeta_k \underset{k \rightarrow \infty}{=} \frac{3\pi}{2a}k + O\left(\frac{1}{k}\right). \quad (13)$$

The following definition is due to [14]:

Definition 2.1. *An entire function $\omega(\lambda)$ of exponential type $\sigma > 0$ is said to be a function of sine-type if:*

- 1) *all the zeros of $\omega(\lambda)$ lie in a strip $|\operatorname{Im}\lambda| < h < \infty$;*
- 2) *for some h_1 and all $\lambda \in \{\lambda: \operatorname{Im}\lambda = h_1\}$ the following inequalities hold: $0 < m \leq |\omega(\lambda)| \leq M < \infty$;*
- 3) *the type of $\omega(\lambda)$ in the lower half-plane coincides with that in the upper half-plane.*

Thus, using Corollary after Lemma 4 in [12] we conclude that $\{\zeta_k\}_{-\infty, k \neq 0}^{\infty}$ is the set of zeros of a sine-type function. This function can be given as

$$\varphi(\lambda) \underset{n \rightarrow \infty}{=} \prod_{-n, k \neq 0}^n C \left(1 - \frac{\lambda}{\zeta_k}\right). \quad (14)$$

Now our aim is to construct $s_1(\lambda, a)$. We know the part $\{\nu_p^{(1)}\}_{-\infty, p \neq 0}^{\infty}$ of the set of zeros of this function. From (8) we obtain

$$s_1(\nu_k, a) = \frac{s(\nu_k, a)}{s'(\nu_k, \frac{a}{3})} \quad (15)$$

We have already shown how to find all the values $s'(\nu_k, \frac{a}{3})$. Now we find (see [11])

$$s(\lambda, a) = a \prod_{k=1}^{\infty} \left(\frac{a}{\pi k}\right)^2 (\lambda_k^2 - \lambda^2).$$

Using this we find $s_1(\nu_k, a)$ for all k via (15). Also we know from [11] that

$$s_1(\lambda, a) = \frac{\sin \frac{2}{3}\lambda a}{\lambda} - \frac{\pi B_1}{\lambda^2} \cos \frac{2}{3}\lambda a + \frac{\psi(\lambda)}{\lambda^2}, \tag{16}$$

where $\psi(\lambda) \in \mathcal{L}_{\frac{2}{3}a}$ (\mathcal{L}_a stands for the set of entire functions of exponential type $\leq a$ which belong to $L_2(-\infty, \infty)$ for real λ). Using (15) and (16) we obtain

$$\psi(\nu_k) = \nu_k^2 \left(\frac{s(\nu_k, a)}{s'(\nu_k, \frac{a}{3})} - \frac{1}{\nu_k} \sin \frac{2}{3}\nu_k a + \frac{\pi B_1}{\nu_k^2} \cos \frac{2}{3}\nu_k a \right).$$

According to Lemma 1.4.3 in [11] $\{\psi(\nu_k)\}_{-\infty, k \neq 0}^{\infty} \in l_2$.

Let us associate $a_p = 0$ with every $\zeta_k = \nu_{k_p}^{(1)}$ and with ζ_0 . Each ζ_k which does not coincide with any of $\nu_{k_p}^{(1)}$ coincide with one of ν_k . Let it be ν_{k_1} , then we associate $a_{k_1} = \psi(\nu_{k_1})$ with this ζ_k . Thus we obtain the sequence $\{a_k\}_{-\infty}^{\infty} \in l_2$. From the other hand $\{\zeta_k\}$ is the set of zeros fo a sine-type function $\varphi(\lambda)$ defined by (14). Solving the interpolation problem we obtain

$$\psi(\lambda) = \varphi(\lambda) \sum_{-\infty}^{\infty} \frac{a_k}{\varphi'(\lambda)|_{\lambda=\zeta_k}(\lambda - \zeta_k)}. \tag{17}$$

This Lagrange series converges uniformly on any compact of complex plane and in the norm of $L_2(-\infty, \infty)$ for real λ (see [13]). Here $\psi(\lambda) \in \mathcal{L}_{\frac{2}{3}a}$. Substituting obtained $\psi(\lambda)$ into (16) we find $s_1(\lambda, a)$. Using (8) we find $c_1(\lambda, a)$:

$$c_1(\lambda, a) = \frac{s(\lambda, a) - s'(\lambda, \frac{a}{3}) s_1(\lambda, a)}{s(\lambda, \frac{a}{3})}.$$

Now knowing $s_1(\lambda, a)$ and $c_1(\lambda, a)$ we construct the potential $q(x)$ via the procedure presented in [11] which we describe below.

First of all we introduce the function

$$e(\lambda) = e^{-i\lambda a} (c(\lambda, a) + i\lambda s(\lambda, a))$$

which is so-called Jost function of the prolonged Sturm–Liouville problem on the semiaxis

$$\begin{aligned} y'' + \lambda^2 y - \tilde{q}(x)y &= 0, \\ y_j(\lambda, 0) &= 0, \end{aligned}$$

where

$$\tilde{q}(x) \stackrel{a.e.}{=} \begin{cases} q(x), & \text{if } x \in [0, \frac{2}{3}a] \\ 0, & \text{if } x \in (\frac{2}{3}a, \infty). \end{cases}$$

For the sake of simplicity let $\mu_1^2 > 0$ (otherwise we may shift). Then the Jost function $e(\lambda)$ has no zeros in the closed lower half-plane. Introduce so-called “S-matrix”

$$S(\lambda) = \frac{e(\lambda)}{e(-\lambda)}$$

and the function

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - S(\lambda)) e^{i\lambda x} d\lambda.$$

The Marchenko integral equation

$$K(x, t) + F(x + t) + \int_x^{\infty} K(x, s) F(s + t) ds = 0$$

possesses unique solution $K_j(x, t)$, and the potential

$$\tilde{q}(x) = -2 \frac{dK(x, x)}{dx} \quad (18)$$

is real and belongs to $L_2(0, \infty)$ and $\tilde{q}_j(x) = 0$ for $x \in (\frac{2}{3}a, \infty)$. The shift $q(x + \frac{a}{3})$ of the projection $g(x)$ of $\tilde{q}(x)$ onto the interval $[0, \frac{2}{3}a]$ gives the unknown part of the potential we are looking for. Now the uniqueness of the procedure of recovering follows from the uniqueness of the recovering procedure in [11]. Theorem is proved.

REFERENCES

- [1] Ambarzumian V.: *Über eine Frage der Eigenwerttheorie*. Zeitschrift für Physik **53** (1929), 690–695.
- [2] Borg G.: *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte*. Acta Math. **78** (1946), 1–96.
- [3] Marchenko V. A.: *Some questions of the theory of one-dimensional linear differential operators of second order*. Trudy Moskovskogo matematicheskogo obschestva **1** (1952), 327–420 (Russian).
- [4] Krein M. G.: *Solution of inverse Sturm-Liouville problem*. Doklady AN SSSR **76** (1951) 3, 345–348.
- [5] Levitan B. M., Gasymov M. G.: *Determination of differential equation by two spectra*. Uspechi Math. Nauk **19** (1964), N 2/116, 3–63 (Russian).
- [6] Hochstadt H., Lieberman B.: *An inverse Sturm-Liouville problem with mixed given data*. SIAM J. Appl. Math. **34** (1978), 676–680.

- [7] Gesztesy F., Simon B.: *Inverse spectral analysis with partial information on the potential. II: The case of discrete spectrum*. Trans. Amer. Math. Soc. **352** (1999), 2765–2787.
- [8] del Rio R., Gesztesy F., Simon B.: *Inverse spectral analysis with partial information on the potential. III: Updating boundary conditions*. Internat. Math. Res. Notices **15** (1997), 751–758.
- [9] Pivovarchik V.: *An Inverse Sturm-Liouville Problem By Three Spectra*. Integral Equations and Operator Theory **34** (1999), 234–243.
- [10] Gesztesy F., Simon B.: *On the determination of a potential from three spectra*. In: Buslaev V., Solomyak M. (Eds), *Advances in Mathematical Sciences*, Amer. Math. Soc. Transl. Ser. 2, **189** (1999), AMS, Providence, RI, 85–92.
- [11] Marchenko V.A.: *Sturm–Liouville operators and applications*. Birkhauser, OT **22** (1986), 367.
- [12] Levin B. Ja., Ostrovsky I.V.: *On small perturbations of sets of roots of sinus-type functions*. Izvestiya Akad. Nauk USSR, ser. mathem. **43** (1979)1, 87–110 (Russian).
- [13] Levin B. Ja., Lyubarskii Yu. I.: *Interpolation by entire functions of special classes and related expansions in series of exponents*. Izv. Acad. Sci. USSR, ser. Mat. **39** (1975)3, 657–702 (Russian).
- [14] Levin B. Ja.: *Lectures on Entire Functions*. AMS, Transl. Math. Monographs, vol. 150, 1996.

Vyacheslav Pivovarchik
v.pivovarcik@paco.net

South-Ukrainian State Pedagogical University
Department of Applied Mathematics and Informatics
Staropotofrankovskaya 26, Odessa, Ukraine 65029

Received: September 22, 2004.