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NUMERICAL APPROXIMATIONS OF DIFFERENCE FUNCTIONAL EQUATIONS AND APPLICATIONS

Abstract. We give a theorem on the error estimate of approximate solutions for difference functional equations of the Volterra type. We apply this general result in the investigation of the stability of difference schemes generated by nonlinear first order partial differential functional equations and by parabolic problems. We show that all known results on difference methods for initial or initial boundary value problems can be obtained as particular cases of this general and simple result.

We assume that the right hand sides of equations satisfy nonlinear estimates of the Perron type with respect to functional variables.

Keywords: functional differential equations, stability and convergence, interpolating operators, nonlinear estimates of the Perron type.

Mathematics Subject Classification: 35R10, 65M12.

1. INTRODUCTION

Difference methods for nonlinear first order partial functional differential equations and for nonlinear parabolic problems were considered by may authors and under various assumptions. It is easy to construct an explicit Euler's type difference method which satisfies the consistency conditions on all sufficiently regular solutions of the above problems. The main task in these investigations is to find a finite difference scheme which is stable. The method of difference inequalities and simple theorems on recurrent inequalities are used in the investigations of the stability of nonlinear difference or functional difference equations.

It is not our aim to show a full review of papers concerning difference methods for partial functional differential equations. We mention only those which contain such reviews. They are [1, 4, 5, 7, 8]. The monograph [3] contains an exposition of the numerical methods for nonlinear hyperbolic functional differential problems.

In the paper we present a general method for the investigation of the stability of difference or functional difference problems generated by initial or initial boundary value problems for nonlinear first order partial functional differential equations and for parabolic problems. We prove a simple theorem on the error estimates of approximate solutions for difference functional equations of the Volterra type with unknown function of several variables. The error of an approximate solution is estimated by a solution of an initial problem for a nonlinear difference equation. We will apply this general and simple idea in the investigation of the stability of difference schemes generated by various problems.

It is essential fact in our considerations that the right hand sides of functional differential equations satisfy the nonlinear estimates of the Perron type with respect to unknown functions. They are identic with assumptions which guarantee the uniqueness of classical solution of initial or initial boundary value problems.

We use in the paper these general ideas for finite difference equations which were introduced in [2, 6].

All problems considered in the paper have the following property: the unknown function is the functional variable in equations. The partial derivatives appear in a classical sense.

2. DIFFERENCE FUNCTIONAL EQUATIONS

For any two sets X and Y we denote by $\mathbb{F}(X,Y)$ the class of all functions defined on X and taking values in Y. We will denote by $\mathbb{N}$ and $\mathbb{Z}$ the sets of natural numbers and integers respectively. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. For $x,\ y \in \mathbb{R}^n$, $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$ we write $x \diamond y = (x_1y_1,\dots,x_ny_n)$ and $\|x\| = |x_1| + \ldots + |x_n|$. Let $a_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0,+\infty)$, and $a > 0$ be given. Write

$$\Sigma = [0,a] \times \mathbb{R}^n \quad \text{and} \quad \Sigma_0 = [-a_0,0] \times \mathbb{R}^n.$$

We define a mesh on $\Sigma_0 \cup \Sigma$ in the following way. Suppose that $(h_0,h') = h$, $h' = (h_1,\dots,h_n)$, stand for steps of the mesh. For $(r,m) \in \mathbb{Z}^{1+n}$ where $m = (m_1,\dots,m_n)$ we define nodal points as follows:

$$t^{(r)} = rh_0, \quad x^{(m)} = m \diamond h', \quad x^{(m)} = \left(x_1^{(m_1)},\dots,x_n^{(m_n)}\right).$$

We assume that there is $K_0 \in \mathbb{Z}$ such that $K_0h_0 = a_0$. Let $K \in \mathbb{N}$ be defined by the relations $Kh_0 \le a < (K+1)h_0$. We will denote by $H \subset \mathbb{R}^n_+$ the set of all steps $h = (h_0,h')$ of the meshes. We assume that there is a sequence

$$\left\{h^{(k)}\right\}_{k=0}^{\infty}, \quad h^{(k)} \in H,$$

such that $\lim_{k\to\infty} h^{(k)} = 0$. In the paper we formulate further assumptions on H. For $h \in H$ we put

$$\mathbb{R}^{1+n}_h = \left\{(t^{(r)},x^{(m)})\colon (r,m) \in \mathbb{Z}^{1+n}\right\}$$

and

$$\Sigma_{0.h} = \Sigma_0 \cap \mathbb{R}_h^{1+n}, \quad \Sigma_h = \Sigma \cap \mathbb{R}_h^{1+n}.$$

We assume that E_h, $\partial_0 E_h \subset \Sigma_h$, $E_{0.h} \subset \Sigma_{0.h}$ are given sets where $h \in H$. Write

$$\Omega_h = E_{0.h} \cup E_h \cup \partial_0 E_h$$

and

$$\Omega_{h.i} = \Omega_h \cap \left(\left[-a_0, t^{(i)} \right] \times \mathbb{R}^n \right), \quad 0 \le i \le K.$$

For a function $z\colon \Omega_h \to \mathbb{R}$ and for a point $(t^{(r)}, x^{(m)}) \in \Omega_h$ we put $z^{(r,m)} = z(t^{(r)}, x^{(m)})$ and

$$\|z\|_{h.r} = \max \left\{ \left| z^{(i,m)} \right| : (t^{(i)}, x^{(m)}) \in \Omega_{h.r} \right\}.$$

Set

$$E'_h = \left\{ (t^{(r)}, x^{(m)}) \in E_h \colon (t^{(r+1)}, x^{(m)}) \in E_h \right\}$$

and

$$J_h = \left\{ t^{(r)} \colon 0 \le r \le K \right\}, \quad J'_h = J_h \setminus \left\{ x^{(K)} \right\}.$$

For a function $\alpha\colon J_h \to \mathbb{R}$ we write $\alpha^{(r)} = \alpha(t^{(r)})$, $0 \le r \le K$.

Assumption H $[\Omega_h]$. *Suppose that the sets $E_{0.h}$, E_h, $\partial_0 E_h$, $h \in H$, satisfy the conditions:*

1) $E_h \ne \emptyset$, $E_{0.h} \ne \emptyset$ *and* $E_h \cap \partial_0 E_h = \emptyset$,
2) *if* $(t^{(r+1)}, x^{(m)}) \in E_h$ *and* $0 \le r \le K-1$ *then* $(t^{(r)}, x^{(m)}) \in E_h$,
3) *the set* Ω_h *is bounded.*

Suppose that $F_h\colon E'_h \times \mathbb{F}(\Omega_h, \mathbb{R}) \to \mathbb{R}$ is a given operator. For $(t^{(r)}, x^{(m)}, z) \in E'_h \times \mathbb{F}(\Omega_h, \mathbb{R})$ we write

$$F_h[z]^{(r,m)} = F_h(t^{(r)}, x^{(m)}, z).$$

The operator F_h is said to satisfy the Volterra condition if for each $(t^{(r)}, x^{(m)}) \in E'_h$ and for z, $\bar{z} \in \mathbb{F}(\Omega_h, \mathbb{R})$ such that $z|_{\Omega_{h.r}} = \bar{z}|_{\Omega_{h.r}}$ we have $F_h[z]^{(r,m)} = F_h[\bar{z}]^{(r,m)}$. Note that the Volterra condition states that the value of F_h at the point $(t^{(r)}, x^{(m)}, z)$ depends on $(t^{(r)}, x^{(m)})$ and on the restriction of z to the set $\Omega_{h.r}$ only.

Given $\varphi \in \mathbb{F}(E_{0.h} \cup \partial_0 E_h, \mathbb{R})$, we consider the functional difference equation

$$z^{(r+1,m)} = F_h[z]^{(r,m)} \tag{1}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h. \tag{2}$$

If F_h satisfies the Volterra condition then there exists exactly one solution $u_h\colon \Omega_h \to \mathbb{R}$ of problem (1), (2). Let $Y_h \subset \mathbb{F}(\Omega_h, \mathbb{R})$ be a fixed subset. Suppose that $v_h\colon \Omega_h \to \mathbb{R}$, $\gamma_h\colon J'_h \to \mathbb{R}_+$, $\alpha_{0.h} \in \mathbb{R}_+$ are such elements that $v_h \in Y_h$ and

$$\left| v_h^{(r+1,m)} - F_h[v_h]^{(r,m)} \right| \le \gamma_h^{(r)} \quad \text{on } E'_h \tag{3}$$

and

$$\left| v_h^{(r,m)} - \varphi_h^{(r,m)} \right| \le \alpha_{0.h} \quad \text{on } E_{0.h} \cup \partial_0 E_h. \tag{4}$$

The function v_h satisfying the above relations is considered as an approximate solution of problem (1), (2). We give a theorem on the estimate of the difference between the exact and approximate solutions of (1), (2). We look for approximate solutions of (1), (2) in the space Y_h.

Assumption H $[F_h, \sigma_h]$. *Suppose that the operator $F_h\colon E'_h \times \mathbb{F}(\Omega_h, \mathbb{R}) \to \mathbb{R}$, $h \in H$, satisfies the Volterra condition and there is a function $\sigma\colon J'_h \times \mathbb{R}_+ \to \mathbb{R}_+$ such that σ_h is nondecreasing with respect to the second variable and the estimate*

$$\left| F_h[z]^{(r,m)} - F_h[\bar{z}]^{(r,m)} \right| \le \sigma_h \left(t^{(r)}, \|z - \bar{z}\|_{h.r} \right)$$

is satisfied for $(t^{(r)}, x^{(m)}) \in E'_h$, $z \in \mathbb{F}(\Omega_h, \mathbb{R})$, $\bar{z} \in Y_h$.

Theorem 2.1. *Suppose that Assumptions H $[\Omega_h]$ and H $[F_h, \sigma_h]$ are satisfied and:*

1) *$\varphi \in \mathbb{F}(E_{0.h} \cup \partial_0 E_h, \mathbb{R})$ and $u_h\colon \Omega_h \to \mathbb{R}$ is a solution of problem (1), (2);*
2) *the functions $v_h\colon \Omega_h \to \mathbb{R}$, $\gamma_h\colon J'_h \to \mathbb{R}_+$ and the constant $\alpha_{0.h} \in \mathbb{R}_+$ satisfy relations (3), (4) and $v_h \in Y_h$;*
3) *the function $\beta_h\colon J_h \to \mathbb{R}_+$ is nondecreasing and it satisfies the recurrent inequality*

$$\beta_h^{(r+1)} \ge \sigma_h(t^{(r)}, \beta_h^{(r)}) + \gamma_h^{(r)}, \quad 0 \le r \le K-1, \tag{5}$$

and $\beta^{(0)} \ge \alpha_{0.h}$.

Under these assumptions we have

$$\|u_h - v_h\|_{h.r} \le \beta_h^{(r)}, \quad 0 \le r \le K. \tag{6}$$

Proof. We prove the assertion (6) by induction. It follows from the initial boundary estimate (4) that inequality (6) is satisfied for $r = 0$. Suppose that $\|u_h - v_h\|_{h.i} \le \beta_h^{(i)}$ with fixed i, $0 \le i \le K-1$, and $(t^{(i+1)}, x^{(m)}) \in E_h$. Then we have

$$\begin{aligned} \left| u_h^{(i+1,m)} - v_h^{(i+1,m)} \right| &\le \left| F_h[u_h]^{(i,m)} - F_h[v_h]^{(i,m)} \right| + \left| F_h[v_h]^{(i,m)} - v_h^{(i+1,m)} \right| \le \\ &\le \sigma_h \left(t^{(i)}, \|u_h - v_h\|_{h.i} \right) + \gamma_h^{(i)} \le \beta_h^{(i+1)}. \end{aligned}$$

We conclude from the above estimate and from (4) that $\|u_h - v_h\|_{h.i+1} \le \beta_h^{(i+1)}$. Hence, the proof is completed by induction. □

Remark 2.1. *Suppose that:*

1) *there is* $L > 0$ *such that*

$$\sigma_h(t,p) = (1 + Lh_0)p, \quad (t,p) \in J'_h \times \mathbb{R}_+;$$

2) *there is* $\tilde{\gamma}_h \in \mathbb{R}_+$ *such that* $\gamma_h^{(r)} \le h_0 \tilde{\gamma}_h$ *for* $0 \le r \le K - 1$.

Then Assumption $H\,[F_h, \sigma_h]$ *states that* F_h *satisfies the Lipschitz condition with respect to the functional variable and* $1 + Lh_0$ *is the Lipschitz constant. In this case we have the estimates*

$$\|u_h - v_h\|_{h.r} \le (1 + Lh_0)^r \, \alpha_{0.h} + \tilde{\gamma}_h \frac{(1 + Lh_0)^r - 1}{L} \le$$
$$\le \exp\,[La]\;\; \alpha_{0.h} + \tilde{\gamma}_h \, \frac{\exp[La] - 1}{L} \quad \text{for} \quad 0 \le r \le K.$$

The above example is important in applications.

3. INITIAL PROBLEMS FOR FIRST ORDER PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

For any two metric spaces X and Y we denote by $C(X, Y)$ the class of all continuous functions from X into Y. Let E be the Haar pyramid

$$E = \big\{(t,x) \in \mathbb{R}^{1+n} \colon t \in [0,a], \quad -b + Mt \le x \le b - Mt\big\}$$

where $a > 0$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, $M = (M_1, \ldots, M_n) \in \mathbb{R}^n_+$ and $b > Ma$. Write $E_0 = [-a_0, 0] \times [-b, b]$, $\Omega = E \cup E_0$ and suppose that $f \colon E \times C(\Omega, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$, $\varphi \colon E_0 \to \mathbb{R}$ are given functions. We consider the Cauchy problem

$$\partial_t z(t,x) = f(t, x, z, \partial_x z(t,x)) \tag{7}$$

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0. \tag{8}$$

Write $\Omega_t = \Omega \cap ([-a_0, t] \times \mathbb{R}^n)$, $0 \le t \le a$. The maximum norm in the space $C(\Omega_t, \mathbb{R})$ is denoted by $\|\cdot\|_t$. We assume that f satisfies the following Volterra condition: if $(t, x, q) \in E \times \mathbb{R}^n$, $z, \bar{z} \in C(\Omega, \mathbb{R})$ and $z|_{\Omega_t} = \bar{z}|_{\Omega_t}$ then $f(t, x, z, q) = f(t, x, \bar{z}, q)$.

Now we formulate a class of difference equations corresponding to (7), (8). Let us denote by H the set of all $h = (h_0, h')$ satisfying the conditions:

(i) there are $K_0 \in \mathbb{Z}$ and $(N_1, \ldots, N_n) \in \mathbb{N}^n$ such that $K_0 h_0 = a_0$ and $N \diamond h' = b$;

(ii) $h' \le M h_0$ and there is $c_0 > 0$ such that $h_i h_j^{-1} \le c_0$ for $1 \le i, j \le n$.

Let $K \in \mathbb{N}$ be the constant defined in Section 1. For $h \in H$ we put

$$E_{0.h} = E_0 \cap \mathbb{R}^{1+n}_h, \quad E_h = E \cap \mathbb{R}^{1+n}_h, \quad \partial_0 E_h = \emptyset$$

and

$$E'_h = \left\{ \left(t^{(r)}, x^{(m)}\right) \in E_h \colon \left(t^{(r+1)}, x^{(m)}\right) \in E_h \right\}, \quad \Omega_h = E_{0.h} \cup E_h.$$

Write $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ with 1 standing on the i-th place, $1 \le i \le n$. Suppose that $z \colon \Omega_h \to \mathbb{R}$ and $(t^{(r)}, x^{(m)}) \in E'_h$. We define the difference operators δ_0 and $\delta = (\delta_1, \dots, \delta_n)$ in the following way:

$$\delta_0 z^{(r,m)} = \frac{1}{h_0} \left[z^{(r+1,m)} - A z^{(r,m)} \right], \tag{9}$$

where

$$A z^{(r,m)} = \frac{1}{2n} \sum_{i=1}^{n} \left[z^{(r,m+e_j)} + z^{(r,m-e_j)} \right]$$

and

$$\delta_j z^{(r,m)} = \frac{1}{2h_j} \left[z^{(r,m+e_j)} - z^{(r,m-e_j)} \right], \quad 1 \le j \le n. \tag{10}$$

Approximate solutions of (7), (8) are functions defined on Ω_h. Moreover, because equation (7) contains the functional variable z which is an element of the space $C(\Omega, \mathbb{R})$, we need an interpolating operator $T_h \colon \mathbb{F}(\Omega_h, \mathbb{R}) \to C(\Omega, \mathbb{R})$.

Assumption H $[T_h]$. *Suppose that the operator $T_h \colon \mathbb{F}(\Omega_h, \mathbb{R}) \to C(\Omega, \mathbb{R})$ satisfies the conditions:*

1) *for $z, \bar{z} \in \mathbb{F}(\Omega_h, \mathbb{R})$ we have*

$$\|T_h[z] - T_h[\bar{z}]\|_{t^{(r)}} \le \|z - \bar{z}\|_{h.r}, \quad 0 \le r \le K;$$

2) *for each function $z \colon \Omega \to \mathbb{R}$ which is of class C^1 there is $\tilde{C} \in \mathbb{R}_+$ such that*

$$\|z - T_h[z_h]\|_t \le \tilde{C} \|h\|, \quad 0 \le t \le a,$$

where z_h is the restriction of z to the set Ω_h and $\|h\| = h_0 + \|h'\|$.

Remark 3.1. *Condition* 1) *of Assumption H $[T_h]$ states that T_h satisfies the Lipschitz condition with a constant $L = 1$ and that it satisfies the Volterra condition. Assumption* 2) *implies that the function z is approximated by $T_h[z_h]$ and the error of this approximation is estimated by $\tilde{C}(\|h\|$.*

An example of T_h satisfying Assumption H $[T_h]$ can be found in [3], Chapter III.

We will approximate classical solutions of (7), (8) by means of solutions of the difference functional equation

$$\delta_0 z^{(r,m)} = f\left(t^{(r)}, x^{(m)}, T_h[z], \delta z^{(r,m)}\right) \tag{11}$$

with the initial condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on} \quad E_{0.h} \tag{12}$$

where $\varphi_h \colon E_{0.h} \to \mathbb{R}$ is a given function.

Problem (11), (12) is called the Lax scheme for (7), (8).

Assumption H $[f, \sigma]$. *Suppose that the function* $f\colon E \times C(\Omega, \mathbb{R}) \times R^n \to \mathbb{R}$ *of the variables* (t, x, z, q), $q = (q_1, \dots, q_n)$, *is continuous and satisfies the Volterra condition and there is* $\sigma\colon [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that:*

1) σ *is continuous and it is nondecreasing with respect to both variables;*

2) *for each* $\varepsilon \in \mathbb{R}_+$ *the maximal solution* $\omega(\cdot, \varepsilon)$ *of the Cauchy problem*

$$\omega'(t) = \sigma(t, \omega(t)) + \varepsilon, \quad \omega(0) = \varepsilon,$$

is defined on $[0, a]$ *and* $\omega(t, 0) = 0$ *for* $t \in [0, a]$;

3) *the estimate*

$$|f(t, x, z, q) - f(t, x, \bar{z}, q)| \le \sigma(t, \|z - \bar{z}\|_t)$$

is satisfied on $E \times C(\Omega, \mathbb{R}) \times \mathbb{R}^n$.

Theorem 3.1. *Suppose that Assumptions* $H\,[T_h]$, $H\,[f, \sigma]$ *are satisfied and:*

1) *the partial derivatives*

$$(\partial_{q_1} f, \dots, \partial_{q_n} f) = \partial_q f$$

exist on $E \times C(\Omega, \mathbb{R}) \times \mathbb{R}^n$ *and* $\partial_q f(t, x, z, \cdot\,) \in C(\mathbb{R}^n, \mathbb{R}^n)$ *and*

$$\frac{1}{n} - \frac{h_0}{h_i} |\partial_{q_i} f(t, x, z, q)| \ge 0, \quad 1 \le i \le n, \tag{13}$$

where $(t, x, z, q) \in E \times C(\Omega, \mathbb{R}) \times \mathbb{R}^n$ *and* $h \in H$;

2) $v\colon \Omega \to \mathbb{R}$ *is a solution of* (7), (8) *and* v *is of class* C^1;

3) $u_h\colon \Omega_h \to \mathbb{R}$ *is a solution of* (11), (12) *and there is* $\alpha_0\colon H \to \mathbb{R}_+$ *such that*

$$\left|\varphi^{(r,m)]} - \varphi_h^{(r,m)}\right| \le \alpha_0(h) \quad \text{on} \quad E_{0.h} \quad \text{and} \quad \lim_{h \to 0} \alpha_0(h) = 0.$$

Then there is $\alpha\colon H \to \mathbb{R}_+$ *such that*

$$\|v_h - u_h\|_{h.r} \le \alpha(h) \quad \text{for} \quad 0 \le r \le K \quad \text{and} \quad \lim_{h \to 0} \alpha(h) = 0.$$

Proof. Consider the operator $F_h\colon E'_h \times \mathbb{F}(\Omega_h, \mathbb{R}) \to \mathbb{R}$ defined by

$$F_h[z]^{(r,m)} = Az^{(r,m)} + h_0\, f\left(t^{(r)}, x^{(m)}, T_h[z], \delta z^{(r,m)}\right).$$

Then u_h satisfies (1), (2) and there is $\gamma\colon H \to \mathbb{R}_+$ such that

$$\left|v_h^{(r+1,m)} - F_h[v_h]^{(r,m)}\right| \le h_0 \gamma(h) \quad \text{on} \quad E'_h \quad \text{and} \quad \lim_{h \to 0} \gamma(h) = 0.$$

Write $Y_h = \mathbb{F}(\Omega_h, \mathbb{R})$. It follows from Assumptions H $[T_h]$ and H $[f, \sigma]$ and from (13) that

$$\begin{aligned}
&\left| F_h[z]^{(r,m)} - F_h[\bar{z}]^{(r,m)} \right| \le \\
&\quad \le \left| A(z-\bar{z})^{(r,m)} + h_0 \left[f(t^{(r)}, x^{(m)}, T_h[z], \delta z^{(r,m)}) - f(t^{(r)}, x^{(m)}, T_h[z], \delta \bar{z}^{(r,m)}) \right] \right| + \\
&\quad\quad + h_0 \left| f(t^{(r)}, x^{(m)}, T_h[z], \delta \bar{z}^{(r,m)}) - f(t^{(r)}, x^{(m)}, T_h[\bar{z}], \delta \bar{z}^{(r,m)}) \right| \le \\
&\quad \le \frac{1}{2} \sum_{i=1}^{n} \left[\left| (z-\bar{z})^{(r,m+e_i)} \right| \left(\frac{1}{n} + \frac{h_0}{h_i} \partial_{q_i} f(Q) \right) \right] + \\
&\quad\quad + \frac{1}{2} \sum_{i=1}^{n} \left[\left| (z-\bar{z})^{(r,m-e_i)} \right| \left(\frac{1}{n} - \frac{h_0}{h_i} \partial_{q_i} f(Q) \right) \right] + \\
&\quad\quad + h_0 \sigma \left(t^{(r)}, \| T_h[z] - T_h[\bar{z}] \|_{t^{(r)}} \right)
\end{aligned}$$

where $Q \in E \times C(\Omega, \mathbb{R}) \times \mathbb{R}^n$ is an intermediate point. The result is

$$\left| F_h[z]^{(r,m)} - F_h[\bar{z}]^{(r,m)} \right| \le \|z-\bar{z}\|_{h.r} + h_0\, \sigma \left(t^{(r)}, \|z-\bar{z}\|_{h.r} \right)$$

on $E'_h \times \mathbb{F}(\Omega_h, \mathbb{R})$. Let $\eta_h \colon J_h \to \mathbb{R}_+$ be the solution of the difference problem

$$\begin{aligned}
\eta^{(r+1)} &= \eta^{(r)} + h_0\, \sigma \left(t^{(r)}, \eta^{(r)} \right) + h_0 \gamma(h), \quad 0 \le r \le K-1, \\
\eta^{(0)} &= \alpha_0(h).
\end{aligned}$$

It follows from Theorem 2.1 that

$$\|v_h - u_h\|_{h.r} \le \eta_h^{(r)}, \quad 0 \le r \le K.$$

Consider the Cauchy problem

$$\omega'(t) = \sigma\,(t, \omega(t)) + \gamma(h), \quad \omega(0) = \alpha_0(h), \tag{14}$$

and its maximal solution $\omega(\,\cdot\,, h) \colon [0,a] \to \mathbb{R}_+$. It follows easily that $\lim_{h \to 0} \omega(t,h) = 0$ uniformly on $[0,a]$ and that the function $\omega(\,\cdot\,, h)$ satisfies the recurrent inequality

$$\omega\left(t^{(r+1)}, h\right) \ge \omega\left(t^{(r)}, h\right) + h_0\, \sigma \left(t^{(r)}, \omega(t^{(r)}, h) \right) + h_0 \gamma(h), \quad 0 \le r \le K-1.$$

We thus get $\eta_h^{(r)} \le \omega\left(t^{(r)}, h\right)$ for $0 \le r \le K$ and consequently

$$\|v_h - u_h\|_{h.r} \le \omega\left(t^{(r)}, h\right) \le \omega(a,h) \quad \text{for} \quad 0 \le r \le K.$$

This completes the proof. □

Now we consider functional difference problem (11), (12) with δ_0 and $\delta = (\delta_1, \ldots, \delta_n)$ defined in the following way:

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\left[z^{(r+1,m)} - z^{(r,m)}\right], \tag{15}$$

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\left[z^{(r,m+e_i)} - z^{(r,m)}\right] \quad \text{for} \quad 1 \le i \le \kappa_0, \tag{16}$$

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\left[z^{(r,m)} - z^{(r,m-e_i)}\right] \quad \text{for} \quad \kappa_0 + 1 \le i \le n, \tag{17}$$

where $0 \le \kappa_0 \le n$ is fixed and $\left(t^{(r)}, x^{(m)}\right) \in E'_h$. If $\kappa_0 = 0$ then $\delta z^{(r,m)}$ is defined by (17), if $\kappa_0 = n$ then $\delta z^{(r,m)}$ is given by (16).

Numerical scheme (11), (12) with δ_0 and δ defined by (15), (17) is called the Euler method for (7), (8).

Theorem 3.2. *Suppose that Assumptions $H\,[T_h]$, $H\,[f, \sigma]$ are satisfied and:*

1) *the partial derivatives*
$$(\partial_{q_1} f, \ldots, \partial_{q_n} f) = \partial_q f$$
exist on $E \times C(\Omega, \mathbb{R}) \times \mathbb{R}^n$ and $\partial_q f(t, x, z, \cdot\,) \in C(\mathbb{R}^n, \mathbb{R}^n)$ and
$$\partial_{q_i} f(t, x, z, q) \ge 0 \quad \textit{for} \quad 1 \le i \le \kappa_0,$$
$$\partial_{q_i} f(t, x, z, q) \le 0 \quad \textit{for} \quad \kappa_0 + 1 \le i \le n,$$
where $(t, x, z, q) \in E \times C(\Omega, \mathbb{R}) \times \mathbb{R}^n$;
2) *the estimate*
$$1 - h_0 \sum_{i=1}^{n} \frac{1}{h_i} |\partial_{q_i} f(t, x, z, q)) \ge 0$$
is satisfied for $(t, x, z, q) \in E \times C(\Omega, \mathbb{R}) \times \mathbb{R}^n$ and $h \in H$;
3) *$v\colon \Omega \to \mathbb{R}$ is a solution of (7), (8) and v is of class C^1;*
4) *$u_h\colon \Omega_h \to \mathbb{R}$ is a solution of (11), (12) with δ_0 and δ defined by (15)–(17) and there is $\alpha_0\colon H \to \mathbb{R}_+$ such that $\left|\varphi^{(r,m)} - \varphi_h^{(r,m)}\right| \le \alpha_0(h)$ on E'_h and $\lim_{h\to 0} \alpha_0(h) = 0$.*

Then there is $\alpha\colon H \to \mathbb{R}_+$ such that
$$\|v_h - u_h\|_{h.r} \le \alpha(h) \quad \textit{for} \quad 0 \le r \le K \quad \textit{and} \quad \lim_{h\to 0} \alpha(h) = 0,$$
where v_h is the restriction of v to the set Ω_h.

The proof of the above theorem is similar to the proof of Theorem 3.1. Details are omitted.

4. INITIAL PROBLEMS FOR QUASILINEAR FUNCTIONAL DIFFERENTIAL PROBLEMS

Let $E, E_0, \Omega, E_h, E_{0.h}, \Omega_h$ and H be the sets defined in Section 3. Assume that

$$f\colon E \times C(\Omega, \mathbb{R}) \to \mathbb{R}^n \quad f = (f_1, \ldots, f_n),$$
$$g\colon E \times C(\Omega, \mathbb{R}) \to \mathbb{R}, \quad \varphi\colon E_0 \to \mathbb{R}$$

are given functions. We consider the quasilinear functional differential; equation

$$\partial_t z(t,x) = \sum_{i=1}^{n} f_i(t,x,z)\, \partial_{x_i} z(t,x) + g(t,x,z) \tag{18}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0. \tag{19}$$

The functions f and g is said to satisfy the Volterra condition if for each $(t,x) \in E$ and for $z, \bar{z} \in C(\Omega, \mathbb{R})$ such that $z|_{\Omega_t} = \bar{z}|_{\Omega_t}$ we have $f(t,x,z) = f(t,x,\bar{z})$ and $g(t,x,z) = g(t,x,\bar{z})$.

The results given in Section 3 for nonlinear functional differential problems are not applicable to quasilinear equation (18). We prove that there is a class of difference methods of the Euler type for (18), (19) which is convergent.

Suppose that the interpolation operator $T_h\colon \mathbb{F}(\Omega_h, \mathbb{R}) \to C(\Omega, \mathbb{R})$ is given. We consider the difference functional equation

$$\delta_0 z^{(r,m)} = \sum_{i=1}^{n} f_i\left(t^{(r)}, x^{(m)}, T_h[z]\right) \delta_i z^{(r,m)} + g\left(t^{(r)}, x^{(m)}, T_h[z]\right) \tag{20}$$

with the initial condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on} \quad E_{0.h}, \tag{21}$$

where $\varphi_h\colon E_{0.h} \to \mathbb{R}$ is a given function. The operator δ_0 is defined by (15) and the operators $(\delta_1, \ldots, \delta_n) = \delta$ are calculated in the following way

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\left[z^{(r,m+e_i)} - z^{(r,m)}\right] \quad \text{if} \quad f_i\left(t^{(r)} x^{(m)}, T_h[z]\right) \geq 0, \tag{22}$$

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\left[z^{(r,m)} - z^{(r,m-e_i)}\right] \quad \text{if} \quad f_i\left(t^{(r)} x^{(m)}, T_h[z]\right) < 0. \tag{23}$$

It is easily seen that if f and g satisfy the Volterra condition and Assumption H $[T_h]$ is satisfied then for each $h \in H$ there exists exactly one solution $u_h\colon \Omega_h \to \mathbb{R}$ of problem (20), (21) with δ_0 and δ given by (15) and (22), (23).

Assumption H $[f, g, \sigma]$. *Suppose that:*

1) *the functions*

$$f\colon E \times C(\Omega, \mathbb{R}) \to \mathbb{R}^n \quad \text{and} \quad g\colon E \times C(\Omega, \mathbb{R}) \to \mathbb{R}$$

are continuous and they satisfy the Volterra condition;

2) *there is* $\sigma\colon [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that:*

(i) σ *is continuous and it is nondecreasing with respect to both variables;*

(ii) *for each* $c \geq 1$ *and* $\varepsilon \in \mathbb{R}_+$ *the maximal solution of the Cauchy problem*

$$\omega'(t) = c\sigma(t, \omega(t)) + \varepsilon, \quad \omega(0) = \varepsilon, \tag{24}$$

is defined on $[0, a]$ *and the function* $\tilde{\omega}(t) = 0$ *for* $t \in [0, a]$ *is the maximal solution of* (24) *for* $\varepsilon = 0$;

(iii) *the estimates*

$$\|f(t, x, z) - f(t, x, \bar{z})\| \leq \sigma(t, \|z - \bar{z}\|_t),$$
$$|g(t, x, z) - g(t, x, \bar{z})| \leq \sigma(t, \|z - \bar{z}\|_t)$$

are satisfied on $E \times C(\Omega, \mathbb{R})$.

Theorem 4.1. *Suppose that Assumptions* $H[T_h]$ *and* $H[f, g, \sigma]$ *are satisfied and:*

1) *for* $h \in H$ *we have*

$$1 - h_0 \sum_{1}^{n} \frac{1}{h_i} |f_i(t, x, z)| \geq 0 \quad \text{on} \quad E \times C(\Omega, \mathbb{R});$$

2) $u_h\colon \Omega_h \to \mathbb{R}$ *is a solution of problem* (20), (21) *with* δ_0 *and* δ *given by* (15), (22), (23) *and there is* $\alpha_0\colon H \to \mathbb{R}_+$ *such that*

$$\left|\varphi_h^{(r,m)} - \varphi^{(r,m)}\right| \leq \alpha_0(h) \quad \text{on} \quad E_{0.h} \quad \text{and} \quad \lim_{h\to 0} \alpha_0(h) = 0;$$

3) $v\colon E_0 \cup E \to \mathbb{R}$ *is a solution of* (18), (19) *and* v *is of class* C^1.

Then there is $\alpha\colon H \to \mathbb{R}_+$ *such that*

$$\|v_h - u_h\|_{h.r} \leq \alpha(h) \quad \text{for} \quad 0 \leq r \leq K \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0. \tag{25}$$

Proof. We apply Theorem 2.1 to prove (25). Consider the operator $F_h\colon E'_h \times \times \mathbb{F}(\Omega_h, \mathbb{R}) \to \mathbb{R}$ defined by

$$F_h[z]^{(r,m)} = z^{(r,m)} + h_0 \sum_{i=1}^{n} f_i\left(t^{(r)}, x^{(m)}, T_h[z]\right) \delta_i z^{(r,m)} + h_0 g\left(t^{(r)}, x^{(m)}, T_h[z]\right).$$

Then u_h satisfies (1), (2) and there is $\gamma\colon H \to \mathbb{R}_+$ such that

$$\left|v_h^{(r+1,m)} - F_h[v_h]^{(r,m)}\right| \le h_0\gamma(h) \quad \text{on} \quad E'_h$$

and $\lim_{h\to 0}\gamma(h) = 0$. Let $\tilde{C}$ be such a constant that

$$|\partial_{x_i} v(t,x)| \le \tilde{C}, \quad (t,x) \in E, \quad 1 \le i \le n.$$

We will denote by Y_h the class of all functions $z\colon \Omega_h \to \mathbb{R}$ such that

$$\left|\delta_i z^{(r,m)}\right| \le \tilde{C} \quad \text{for} \quad \left(t^{(r)}, x^{(m)}\right) \in E'_h \quad 1 \le i \le n.$$

Suppose that $z \in \mathbb{F}(\Omega_h, \mathbb{R})$ and $\bar{z} \in Y_h$. We prove that

$$\left|F_h[z]^{(r,m)} - F_h[\bar{z}]^{(r,m)}\right| \le \|z - \bar{z}\|_{h.r} + (1+\tilde{C})\sigma\left(t^{(r)}, \|z-\bar{z}\|_{h.r}\right) \tag{26}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. Write

$$J_+^{(r,m)} = \{i\colon 1 \le i \le n \quad \text{and} \quad f_i\left(t^{(r)}, x^{(m)} T_h[z]\right) \ge 0,$$
$$J_-^{(r,m)} = \{1,\ldots,n\} \setminus J_+^{(r,m)}.$$

Then we have

$$\begin{aligned} F_h[z]^{(r,m)} - F_h[\bar{z}]^{(r,m)} = (z-\bar{z})^{(r,m)} &\Bigg[1 - h_0 \sum_{i\in J_+^{(r,m)}} \frac{1}{h_i} f_i\left(t^{(r)}, x^{(m)}, T_h[z]\right) + \\ &+ h_0 \sum_{i\in J_-^{(r,m)}} \frac{1}{h_i} f_i\left(t^{(r)}, x^{(m)}, T_h[z]\right)\Bigg] + \\ &+ h_0 \sum_{i\in J_+^{(r,m)}} \frac{1}{h_i} f_i\left(t^{(r)}, x^{(m)}, T_h[z]\right)(z-\bar{z})^{(r,m+e_i)} - \\ &- h_0 \sum_{i\in J_-^{(r,m)}} \frac{1}{h_i} f_i\left(t^{(r)}, x^{(m)}, T_h[z]\right)(z-\bar{z})^{(r,m-e_i)} + \\ &+ h_0 \sum_{i=1}^{n} \left[f_i\left(t^{(r)}, x^{(m)}, T_h[z]\right) - f_i\left(t^{(r)}, x^{(m)}, T_h[\bar{z}]\right)\right]\delta_i \bar{z}^{(r,m)} + \\ &+ h_0\left[g\left(t^{(r)}, x^{(m)}, T_h[z]\right) - g\left(t^{(r)}, x^{(m)}, T_h[\bar{z}]\right)\right]. \end{aligned}$$

The above relations and Assumption H $[T_h[$ and H $[f, g, \sigma]$ imply (26). It follows from Theorem 2.1 that

$$\|v_h - u_h\|_{h.r} \le \eta_h^{(r)}, \quad 0 \le r \le K,$$

where $\eta_h \colon J_h \to \mathbb{R}_+$ is a solution of the problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0(1+\tilde{C})\sigma(t^r, \eta^{(r)}) + h_0\gamma(h), \quad 0 \le r \le K-1,$$
$$\eta^{(0)} = \alpha_0(h).$$

Consider the Cauchy problem

$$\omega'(t) = (1+\tilde{C})\sigma(t, \omega(t)) + \gamma(h), \quad \omega(0) = \alpha_0(h),$$

and its maximal solution $\omega(\,\cdot\,, h)\colon [0,a] \to \mathbb{R}_+$. It follows easily that $\lim_{h\to 0}\omega(t,h) = 0$ uniformly on $[0,a]$ and $\eta_h^{(r)} \le \omega(t^{(r)}, h)$ for $0 \le r \le K$, and consequently

$$\|v_h - u_h\|_{h.r} \le \omega(t^{(r)}, h) \le \omega(a,h) \quad \text{for} \quad 0 \le r \le K.$$

This proves the theorem.

5. MIXED PROBLEMS FOR NONLINEAR PARABOLIC EQUATIONS

Write

$$E = [0,a] \times (-b,b), \quad E_0 = [-a_0, 0] \times [-b,b], \tag{27}$$
$$\partial_0 E = [0,a] \times ([-b,b] \setminus (-b,b)), \quad \Omega = E_0 \cup E \cup \partial_0 E, \tag{28}$$

where $a > 0$, $a_0 \in \mathbb{R}_+$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $b_i > 0$ for $1 \le i \le n$. Let $M_{n\times n}$ denote the class of all $n \times n$ real matrices. Set $\Xi = E \times C(\Omega, \mathbb{R}) \times M_{n\times n})$ and suppose that $f\colon \Xi \to \mathbb{R}$ and $\varphi\colon E_0 \cup \partial_0 E$ are given functions. We consider the nonlinear functional differential equation

$$\partial_t z(t,x) = f\left(t, x, z, \partial_x z(t,x), \partial_{xx} z(t,x)\right) \tag{29}$$

with the initial boundary condition

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0 \cup \partial_0 E, \tag{30}$$

where

$$\partial_x z = \left(\partial_{x_1} z, \ldots, \partial_{x_n} z\right), \partial_{xx} z = \left[\partial_{x_i x_j} z\right]_{i,j=1,\ldots,n}.$$

Set

$$\Omega_t = \Omega \cap \left([-a_0, t] \times \mathbb{R}^n\right), \quad 0 \le t \le a.$$

The maximum norm in the space $C(\Omega_t, \mathbb{R})$ is denoted by $\|\cdot\|_t$. The function $f\colon \Xi \to \mathbb{R}$ is said to satisfy the Volterra condition if for each $(t,x,q,s) \in E \times \mathbb{R}^n \times M_{n\times n}$ and for $z, \bar{z} \in C(\Omega, \mathbb{R})$ such that $z|_{\Omega_t} = \bar{z}|_{\Omega_t}$ we have $f(t,x,z,q,s) = f(t,x,\bar{z},q,s)$. Let us denote by H the set of all $h = (h_0, h')$ satisfying the conditions:

(i) there are $K_0 \in \mathbb{Z}$ and $(N_1, \ldots, N_n) = N \in \mathbb{N}^n$ such that $K_0 h_0 = a_0$ and $N \diamond h' = b$;

(ii) there is $c_0 > 0$ such that $h_i h_j^{-1} \le c_0$ for $1 \le i,\, j \le n$.

Let $K \in \mathbb{N}$ be the constant defined in Section 2. For $h \in H$ we put

$$E_{0.h} \cap \mathbb{R}_h^{1+n}, \quad E_h = E \cap \mathbb{R}_h^{1+n},$$
$$\partial_0 E_h = \partial_0 E \cap \mathbb{R}_h^{1+n}, \quad \Omega_h = E_{0.h} \cup E_h \cup \partial_0 E_h$$

and

$$E'_h = \left\{ \left(t^{(r)}, x^{(m)} \right) \in E_h \colon 0 \le r \le K-1 \right\}.$$

Write

$$U = \{(i,j,) \colon 1 \le i,j \le j, i \ne j\}$$

and suppose that we have defined the sets $U_+, U_- \subset U$ such that $U_+ \cup U_- = U$, $U_+ \cap U_- = \emptyset$ (in particular, it may happen that $U_+ = \emptyset$ or $U_- = \emptyset$). We assume that $(i,j) \in U_+$ when $(j,i) \in U_+$. Let $z \colon \Omega_h \to \mathbb{R}$ and $0 \le r \le K$, $-N < m < N$. Set

$$\delta_i^+ z^{(r,m)} = \frac{1}{h_i} \left[z^{(r,m+e_i)} - z^{(r,m)} \right], \tag{31}$$

$$\delta_i^- z^{(r,m)} = \frac{1}{h_i} \left[z^{(r,m)} - z^{(r,m-e_i)} \right], \tag{32}$$

where $1 \le i \le n$. We apply the difference operators

$$\delta_0, \ \delta = (\delta_1, \ldots, \delta_n), \ \delta^{(2)} = [\delta_{ij}]_{i,j=1,\ldots,n}$$

given by

$$\delta_0 z^{(r,m)} = \frac{1}{h_0} \left[z^{(r+1,m)} - z^{(r,m)} \right], \tag{33}$$

$$\delta_i z^{r,m)} = \frac{1}{2} \left[\delta_i^+ z^{(r,m)} + \delta_i^- z^{(r,m)} \right] \quad \text{for} \quad i = 1, \ldots, n, \tag{34}$$

$$\delta_{ii} z^{(r,m)} = \delta_i^+ \delta_i^- z^{(r,m)} \quad \text{for} \quad i = 1, \ldots, n, \tag{35}$$

$$\delta_{ij} z^{(r,m)} = \frac{1}{2} \left[\delta_i^+ \delta_j^- z^{(r,m)} + \delta_i^- \delta_j^+ z^{(r,m)} \right] \quad \text{for} \quad (i,j) \in U_-, \tag{36}$$

$$\delta_{ij} z^{(r,m)} = \frac{1}{2} \left[\delta_i^+ \delta_j^+ z^{(r,m)} + \delta_i^- \delta_j^- z^{(r,m)} \right] \quad \text{for} \quad (i,j) \in U_+. \tag{37}$$

Suppose that the interpolating operator $T_h \colon \mathbb{F}(\Omega_h, \mathbb{R}) \to C(\Omega, \mathbb{R})$ is given. We consider the difference functional equation

$$\delta_0 z^{(r,m)} = f \left(r^{(r)}, x^{(m)}, T_h[z], \delta z^{(r,m)}, \delta^{(2)} z^{(r,m)} \right) \tag{38}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h, \tag{39}$$

where $\varphi \colon E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$ is a given function. Suppose that f and T_h satisfy the Volterra condition. Then it is evident that there exists exactly one solution $u_h \colon \Omega_h \to \mathbb{R}$ of problem (38), (39).

We prove that under natural assumptions on given function the numerical method (38), (39) is convergent.

Assumption H $[\sigma, f, \partial f]$. *Suppose that:*

1) *the function* $f\colon \Xi \to \mathbb{R}$ *of the variables* (t, x, z, q, s) *is continuous and satisfies the Volterra condition;*

2) *there is* $\sigma\colon [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that:*

   (i) σ *is continuous and it is nondecreasing with respect to both variables,*

   (ii) *for each* $\varepsilon > 0$ *the maximal solution of the Cauchy problem*

$$\omega'(t) = \sigma(t, \omega(t)) + \varepsilon, \quad \omega(0) = \varepsilon \tag{40}$$

   is defined on $[0, a]$ *and the function* $\tilde{\omega}(t) = 0$ *for* $t \in [0, a]$ *is the maximal solution of* (40) *for* $\varepsilon = 0$,

   (iii) *the estimate*

$$|f(t, x, z, q, s) - f(t, x, \bar{z}, q, s)| \le \sigma(t, \|z - \bar{z}\|_t)$$

   is satisfied on Ξ*;*

3) *the partial derivatives*

$$(\partial_{q_1} f, \ldots, \partial_{q_n} f) = \partial_q f, \quad \left[\partial_{s_{ij}} f\right]_{i,j=1,\ldots,n} = \partial_s f$$

   exist on Ξ *and*

$$\partial_q f(t, x, z, \cdot\,) \in C\left(\mathbb{R}^n \times M_{n\times n}, \mathbb{R}^n\right), \ \partial_s f(t, x, z, \cdot\,) \in C\left(\mathbb{R}^n \times M_{n\times n}, M_{n\times n}\right);$$

4) *the matrix* $\partial_s f$ *is symmetric and for* $P = (t, x, z, q, s) \in \Xi$, $h \in H$ *we have*

$$\partial_{s_{ij}} f(P) \ge 0 \quad \text{for} \quad (i, j) \in U_+,$$
$$\partial_{s_{ij}} f(P) \le 0 \quad \text{for} \quad (i, j) \in U_-,$$
$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \partial_{s_{ii}} f(P) + h_0 \sum_{(i,j)\in U} \frac{1}{h_i h_j} \left|\partial_{s_{ij}} f(P)\right| \ge 0,$$

   and

$$\partial_{s_{ii}} f(P) - \frac{h_i}{2} \left|\partial_{q_i} f(P)\right| - h_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{h_j} \left|\partial_{s_{ij}} f(\,P\,)\right| \ge 0,$$

   where $1 \le i \le n$.

Theorem 5.1. *Suppose that Assumption H $[T_h]$ is satisfied with Ω given by* (27), (28) *and:*

1) *assumption H $[\sigma, f, \partial f]$ holds;*
2) $u_h \colon \Omega_h \to \mathbb{R}$, $h \in H$, *is a solution of* (38), (39) *and there is $\alpha_0 \colon H \to \mathbb{R}_+$ such that*

$$\left|\varphi^{(r,m)} - \varphi_h^{(r,m)}\right| \le \alpha_0(h) \quad on \quad E_{0.h} \cup \partial_0 E_h \quad and \quad lim_{h\to 0}\alpha_0(h) = 0$$

3) $v \colon \Omega \to \mathbb{R}$ *is a solution of* (18), (19) *and $v(t, \cdot\,)$ is of class C^2 and $v(\,\cdot\,, x)$ is of class C^1.*

Then there is $\alpha \colon H \to \mathbb{R}_+$ such that

$$\|v_h - u_h\|_{h.r} \le \alpha(h) \quad for \quad 0 \le r \le K \quad and \quad \lim_{h\to 0} \alpha(h) = 0. \tag{41}$$

where v_h is the restriction of v to the set Ω_h.

Proof. We apply Theorem 2.1 to prove (41). Consider the operator $F_h \colon E'_h \times \times \mathbb{F}(\Omega_h, \mathbb{R}) \to \mathbb{R}$ defined by

$$F_h[z]^{(r,m)} = z^{(r,m)} + h_0 f\left(t^{(r)}, x^{(m)}, T_h[z], \delta z^{(r,m)}, \delta^{(2)} z^{(r,m)}\right).$$

Then u_h satisfies (1), (2) and there is $\gamma \colon H \to \mathbb{R}_+$ such that

$$\left|v_h^{(r+1,m)} - F_h[z]^{(r,m)}\right| \le h_0 \gamma(h) \quad \text{on} \quad E'_h$$

and $\lim_{h\to 0} \gamma(h) = 0$. Write $Y_h = \mathbb{F}(\Omega_h, \mathbb{R})$. Suppose that $z, \bar{z} \in \mathbb{F}(\Omega_h, \mathbb{R})$, $(t^{(r)}, x^{(m)}) \in E'_h$. We prove that

$$\left|F_h[z]^{(r,m)} - F_h[\bar{z}]^{(r,m)}\right| \le \|z - \bar{z}\|_{h.r} + h_0\, \sigma\left(t^{(r)}, \|z - \bar{z}\|_{h.r}\right). \tag{42}$$

It follows from Assumption H $[\sigma, f, \partial f]$ that there is an intermediate point $Q \in \Xi$ such that

$$\left|F_h[z]^{(r,m)} - F_h[\bar{z}]^{(r,m)}\right| \le h_0 \sigma\left(t^{(r)}, \|T_h[z] - T_h[\bar{z}]\|_{t^{(r)}}\right) +$$
$$+ \left|(z - \bar{z})^{(r,m)} + h_0 \sum_{i=1}^{n} \partial_{q_i} f(Q) \delta_i (z - \bar{z})^{(r,m)} + h_0 \sum_{i,j=1}^{n} \partial_{s_{ij}} f(Q) \delta_{ij} (z - \bar{z})^{(r,m)}\right|. \tag{43}$$

Write

$$S^{(0)}(Q) = 1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \partial_{s_{ii}} f(Q) + h_0 \sum_{(i,j)\in U} \frac{1}{h_i h_j} \left|\partial_{s_{ij}} f(Q)\right|,$$

and

$$S_{+}^{(i)}(Q) = \frac{h_0}{2h_i}\partial_{q_i} f(Q) + \frac{h_0}{h_i^2}\partial_{s_{ii}} f(Q) - h_0 \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_i h_j} \left|\partial_{s_{ij}} f(Q)\right|,$$

$$S_{-}^{(i)}(Q) = -\frac{h_0}{2h_i}\partial_{q_i} f(Q) + \frac{h_0}{h_i^2}\partial_{s_{ii}} f(Q) - h_0 \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_i h_j} \left|\partial_{s_{ij}} f(Q)\right|,$$

where $1 \le i \le n$. It follows from (43) and from the definitions of the difference operators that

$$\begin{aligned}
\left|F_h[z]^{(r,m} - F_h[\bar{z}]^{(r,m)}\right| &\le h_0\sigma\left(t^{(r)}, \|T_h[z] - T_h[\bar{z}]\|_{t^{(r)}}\right) + \left|S^{(0)}(Q)(z-\bar{z})^{(r,m)}\right| + \\
&+ \left|\sum_{i=1}^{n} S_{+}^{(i)}(Q)(z-\bar{z})^{(r,m+e_i)}\right| + \left|\sum_{i=1}^{n} S_{-}^{(i)}(Q)(z-\bar{z})^{(r,m-e_i)}\right| + \\
&+ h_0 \sum_{(i,j)\in U_+} \frac{1}{2h_ih_j}\partial_{s_{ij}} f(Q)\left\{\left|(z-\bar{z})^{(r,m+e_i+e_j)}\right| + \left|(z-\bar{z})^{(r,m-e_i-e_j)}\right|\right\} - \\
&- h_0 \sum_{(i,j)\in U_-} \frac{1}{2h_ih_j}\partial_{s_{ij}} f(Q)\left\{\left|(z-\bar{z})^{(r,m+e_i-e_j)}\right| + \left|(z-\bar{z})^{(r,m-e_i+e_j)}\right|\right\}.
\end{aligned} \tag{44}$$

It follows from condition 4) of Assumption H $[\sigma, f, \partial f]$ that

$$S^{(0)}(Q) \ge 0 \quad \text{and} \quad S_{+}^{(i)}(Q) \ge 0, \quad S_{-}^{(i)}(Q) \ge 0 \quad \text{for} \quad 1 \le i \le n$$

and

$$\begin{aligned}
S^{(0)}(Q) &+ \sum_{i=1}^{n} S_{+}^{(i)}(Q) + \sum_{i=1}^{n} S_{-}^{(i)}(Q) + \\
&+ h_0 \sum_{(i,j)\in U_+} \frac{1}{2h_ih_j}\partial_{s_{ij}} f(Q) - h_0 \sum_{(i,j)\in U_-} \frac{1}{2h_ih_j}\partial_{s_{ij}} f(Q) = 1.
\end{aligned}$$

The above relations and (44) imply (42).

Denote by $\eta_h \colon J_h \to \mathbb{R}_+$ the solution of the difference problem

$$\begin{aligned}
\eta^{(r+1)} &= \eta^{(r)} + h_0\sigma(t^{(r)}, \eta^{(r)}) + h_0\gamma(h), \quad 0 \le r \le K-1, \\
\eta^{(0)} &= \alpha_0(h).
\end{aligned}$$

It follows from Theorem 2.1 that

$$\|v_h - u_h\|_{h.r} \le \eta_h^{(r)}, \quad 0 \le r \le K.$$

Consider the Cauchy problem (14) and its maximal solution $\omega(\,\cdot\,, h) \colon [0, a] \to \mathbb{R}_+$. It follows easily that

$$\|v_h - u_h\|_{h.r} \le \omega(t^{(r)}, h) \le \omega(a, h) \quad \text{for} \quad 0 \le r \le K,$$

and $\lim_{h\to 0} \omega(t, h) = 0$ uniformly on $[0, a]$. This completes the proof. □

6. NEUMANN TYPE PROBLEMS FOR PARABOLIC DIFFERENTIAL FUNCTIONAL EQUATIONS

Write

$$E = [0, a] \times [-b, b], \quad E_0 = [-a_0, 0] \times [-b, b], \quad \Omega = E \cup E_0 \tag{45}$$

and

$$\partial_0 E = [0, a] \times ([-b, b] \setminus (-b, b)),$$

where $a > 0$, $a_0 \in \mathbb{R}_+$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $b_i > 0$ for $1 \le i \le n$. Set

$$\Xi = E \times C(\Omega, \mathbb{R}) \times \mathbb{R}^n \times M_{n\times n} \tag{46}$$

and

$$\partial_0 E_i = \{(t, x) \in \partial_0 E \colon x_i = -b_i\} \cup \{(t, x) \in \partial_0 E \colon x_i = b_i\},$$

where $1 \le i \le n$. Suppose that

$$f \colon \Xi \to \mathbb{R}, \quad \varphi \colon E_0 \to \mathbb{R}, \quad \psi \colon \partial_0 E \to \mathbb{R}$$

are given functions. We consider the functional differential problem

$$\partial_t z(t, x) = f\,(t, x, z, \partial_x z(t, x), \partial_{xx} z(t, x)), \tag{47}$$

$$z(t, x) = \varphi(t, x) \quad \text{for} \quad (t, x) \in E_0, \tag{48}$$

and

$$\partial_{x_i} z(t, x) = \psi(t, x) \quad \text{for} \quad (t, x) \in \partial_0 E_i, \quad 1 \le i \le n. \tag{49}$$

We assume that f satisfies the Volterra condition.

Now we formulate a difference problem corresponding to (47)–(49).

Let H be the set defined in Section 5 and $K \in \mathbb{N}$ be such a constant that $Kh_0 \le a < (K + 1)h_0$. For $h \in H$ we put

$$E_{0.h} = E_0 \cap \mathbb{R}^{1+n}_h, \quad E_h \cap \mathbb{R}^{1+n}_h, \quad \partial_0 E_h = \partial_0 E \cap \mathbb{R}^{1+n}_h,$$

$$E'_h = \left\{ \left(t^{(r)}, x^{(m)}\right) \in E_h \colon 0 \le r \le K - 1 \right\}$$

and $\Omega_h = E_{0.h} \cup E_h$. Suppose that $x^{(m)} \in [-b, b] \setminus (-b, b)$. We will denote by $I[m]$ the set of all $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{Z}^n$ such that:

(i) if $-b_j < x_j^{(m_j)} < b_j$ then $\kappa_j = 0$;

(ii) if $x_j^{(m_j)} = b_j$ then $\kappa_j \in \{0, 1\}$, if $x_j^{(m_j)} = -b_j$ then $\kappa_j \in \{-1, 0\}$;

(iii) $1 \le |\kappa_1| + \ldots + |\kappa_n| \le 2$.

Define the sets

$$S_h = \left\{ \left(t^{(r)}, x^{(m+\kappa)}\right) : 0 \le r \le K,\ \left(t^{(r)}, x^{(m)}\right) \in \partial_0 E_h \quad \text{and} \quad \kappa \in I[m] \right\},$$

$$S_{0.h} = \{0\} \times \left\{ x^{(m+\kappa)} : x^{(m)} \in [-b,b] \setminus (-b,b) \quad \text{and} \quad \kappa \in I[m] \right\}$$

and

$$\tilde{E}_{0.h} = E_{0.h} \cup S_{0.h}, \quad \tilde{E}_h = E_h \cup S_h, \quad \tilde{\Omega}_h = \tilde{E}_{0.h} \cup \tilde{E}_h,$$

$$\tilde{E}'_h = \left\{ \left(t^{(r)}, x^{(m)}\right) \in \tilde{E}_h : 0 \le r \le K-1 \right\}.$$

For a function $z \colon \tilde{\Omega}_h \to \mathbb{R}$ we write

$$[|z|]_{h.r} = \max \left\{ \left| z^{(i,m)} \right| : \ \left(t^{(i)}, x^{(m)}\right) \in \tilde{\Omega}_h \quad \text{and} \quad -K_0 \le i \le r \right\}.$$

Suppose that U, U_-, U_+ are the sets defined in Section 5. Let δ_0, δ and $\delta^{(2)}$ be the difference operators defined by (31)–(37). Suppose that the interpolating operator $T_h \colon \mathbb{F}(\Omega_h, \mathbb{R}) \to C(\Omega, \mathbb{R})$ is given. We approximate classical solutions of (47)–(49) with solutions of the difference functional equation

$$\delta_0 z^{(r,m)} = f\left(t^{(r)}, x^{(m)}, T_h[z], \delta z^{(r,m)}, \delta^{(2)} z^{(r,m)}\right) \tag{50}$$

with the initial boundary conditions

$$z^{(r,m)} = \varphi_h^{(r,m)} \tag{51}$$

and

$$z^{(r,m+\kappa)} - z^{(r,m-\kappa)} = 2\psi^{(r,m)} \sum_{i=1}^{n} \kappa_i h_i \ \text{for} \ \left(t^{(r)}, x^{(m)}\right) \in \partial_0 E_h \ \text{and} \ \kappa \in I[m]. \tag{52}$$

Suppose that f and T_h satisfy the Volterra condition. Then it is evident that there exists exactly one solution $u_h \colon \tilde{\Omega}_h \to \mathbb{R}$ of problem (50)–(52).

Now we prove that the numerical method (50)–(52) is convergent under natural assumptions on given functions.

Theorem 6.1. *Suppose that Assumption $H\,[T_h]$ is satisfied with Ω given by* (45) *and:*

1) *Assumption $H\,[\sigma, f, \partial f]$ holds with Ξ defined by* (45), (46)*;*
2) *$u_h \colon \tilde{\Omega}_h \to \mathbb{R}$, $h \in H$, is a solution of* (50)*–*(52) *and there is $\alpha_0 \colon H \to \mathbb{R}_+$ such that*
$$\left| \varphi^{(r,m)} - \varphi_h^{(r,m)} \right| \le \alpha_0(h) \quad \text{on} \quad E_{0.h} \quad \text{and} \quad \lim_{h \to 0} \alpha_0(h) = 0;$$
3) *$\psi \in C(\partial_0 E, \mathbb{R})$ and there is a compact set $D \subset \mathbb{R}^{1+n}$ such that $\Omega \subset \operatorname{Int} D$ and:*

(i) *the function* $v\colon D \to \mathbb{R}$ *is a solution of* (47)–(49);

(ii) $v(\,\cdot\,, x)$ *is of class* C^1 *and* $v(t,\,\cdot\,)$ *is of class* C^3;

4) *there is* $\tilde{c} > 0$ *such that* $\|h'\|^2 \le \tilde{c} h_0$.

Then there are $\varepsilon_0 > 0$ *and* $\alpha\colon H \to \mathbb{R}_+$ *such that for* $\|h\| \le \varepsilon_0$ *we have*

$$[|v_h - u_h|]_{h.r} \le \alpha(h), \quad 0 \le r \le K, \tag{53}$$

and $\lim_{h\to 0} \alpha(h) = 0$ *where* v_h *is the restriction of* v *to the set* $\tilde{\Omega}_h$.

Proof. We apply Theorem 2.1 to prove the above statement.

Consider the operator $F_h\colon \tilde{E}'_h \times \mathbb{F}(\tilde{\Omega}_h, \mathbb{R}) \to \mathbb{R}$ defined by

$$F_h[z]^{(r,m)} = z^{(r,m)} + h_0 f\left(t^{(r)}, x^{(m)}, T_h[z], \delta z^{(r,m)}, \delta^{(2)} z^{(r,m)}\right)$$

where $(t^{(r)}, x^{(m)}) \in E'_h$, and

$$F_h[z]^{(t,m+\kappa)} = z^{(r,m-\kappa)} + 2\psi^{(r,m)} \sum_{i=1}^{n} \kappa_i h_i + \\ + h_0 f\left(t^{(r)}, x^{(m-\kappa)}, T_h[z], \delta z^{(r,m-\kappa)}, \delta^{(2)} z^{(r,m-\kappa)}\right)$$

where $(t^{(r)}, x^{(m)}) \in \partial_0 E_h$ and $\kappa \in I[m]$. Let the function $\tilde{\varphi}_h\colon \tilde{E}_{0.h} \to \mathbb{R}$ be defined by

$$\tilde{\varphi}_h^{(r,m)} = \varphi_h^{(r,m)} \quad \text{for} \quad \left(t^{(r)}, x^{(m)}\right) \in E_{0.h}$$

and

$$\tilde{\varphi}_h^{(0,m+\kappa)} = \psi^{(0,m-\kappa)} + 2\psi\left(0, x^{(m)}\right) \sum_{i=1}^{n} \kappa_i h_i$$

where $x^{(m)} \in [-b, b] \setminus (-b, b)$ and $\kappa \in I[m]$. Then we have

$$u_h^{(r+1,m)} = F_h[u_h]^{(r,m)} \quad \text{for} \quad \left(t^{(r)}, x^{(m)}\right) \in \tilde{E}'_h$$

and

$$u_h^{(r,m)} = \tilde{\varphi}_h^{(r,m)} \quad \text{on} \quad \tilde{E}_{0.h}.$$

Write $Y_h = \mathbb{F}(\tilde{\Omega}_h, \mathbb{R})$. Suppose that $z, \bar{z} \in \mathbb{F}(\tilde{\Omega}_h, \mathbb{R})$ and $\left(t^{(r)}, x^{(m)}\right) \in \tilde{E}'_h$. An easy computation shows that

$$\left| F_h[z]^{(r,m)} - F_h[\bar{z}]^{(r,m)} \right| \le [|z - \bar{z}|]_{h.r} + h_0\, \sigma\left(t^{(r)}, [|z - \bar{z}|]_{h.r}\right).$$

Let $\varepsilon_0 > 0$ be such a constant that $\Omega \subset D$ for $\|h\| \le \varepsilon_0$. It follows that there is $\gamma : H \to \mathbb{R}_+$ such that for $(t^{(r)}, x^{(m)}) \in E'_h$ we have

$$\left| v^{(r+1,m)} - F_h[v_h]^{(r,m)} \right| \le h_0 \gamma(h) \tag{54}$$

and $\lim_{h \to 0} \gamma(h) = 0$. Suppose that $(t^{(r)}, x^{(m)}) \in \partial_0 E_h$ and $\kappa \in I[m]$. Then we have

$$\left| v_h^{(r+1,m+\kappa)} - F_h[v_h]^{(r,m+\kappa)} \right| \leq \left| v_h^{(r+1,m+\kappa)} - v_h^{(r+1,m-\kappa)} - 2\psi^{(r+1,m)} \sum_{i=1}^{n} \kappa_i h_i \right| + \\ + \left| v^{(r+1,m-\kappa)} - F_h[v_h]^{(r,m-\kappa)} \right|.$$

We conclude from assumption 4) and from (54) that there is $C > 0$ such that

$$\left| v_h^{(r+1,m+\kappa)} - F_h[v_h]^{(r,m+\kappa)} \right| \leq C h_0 \sqrt{h_0} + h_0 \gamma(h). \tag{55}$$

According to (54), (55), we have

$$\left| v_h^{(r+1,m)} - F_h[v_h]^{(r,m)} \right| \leq h_0 \gamma(h) + C h_0 \sqrt{h_0}$$

where $(t^{(r)}, x^{(m)}) \in \tilde{E}'_h$. It follows that there is $\tilde{\alpha} \colon H \to \mathbb{R}_+$ such that

$$\left| (v_h - u_h)^{(r,m)} \right| \leq \tilde{\alpha}_0(h) \quad \text{on} \quad \tilde{E}_{0.h}$$

and $\lim_{h \to 0} \tilde{\alpha}_0(h) = 0$.

Denote by $\tilde{\eta}_h \colon J_h \to \mathbb{R}_+$ the solution of the difference problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0 \sigma \left(t^{(r)}, \eta^{(r)} \right) + h_0 \left(\gamma(h) + C\sqrt{h_0} \right), \quad 0 \leq r \leq K-1,$$
$$\eta^{(0)} = \tilde{\alpha}(0).$$

It follows from Theorem 2.1 that

$$[|u_h - v_h|]_{h.r} \leq \tilde{\eta}_h^{(r)}, \quad 0 \leq r \leq K.$$

Consider the Cauchy problem

$$\omega'(t) = \sigma\left(t, \omega(t)\right) + \gamma(h) + C\sqrt{h_0}, \quad \omega(0) = \tilde{\alpha}(h),$$

and its maximal solution $\tilde{\omega}(\,\cdot\,, h) \colon [0, a] \to \mathbb{R}_+$. It follows easily that

$$[|u_h - v_h|]_{h.r} \leq \tilde{\eta}_h^{(r)} \leq \tilde{\omega}(t^{(r)}, h) \leq \tilde{\omega}(a, h), \quad 0 \leq r \leq K,$$

and $\lim_{h \to 0} \tilde{\omega}(t, h) = 0$ uniformly on $[0, a]$.

This proves the theorem. □

Remark 6.1. *It is easy to see that all the results of the paper can be extended for weakly coupled functional differential systems.*

Remark 6.2. *If we assume that*

$$\sigma(t, p) = Lp, \quad (t, p) \in [0, a] \times \mathbb{R}_+,$$

then all the comparison difference problems can be solved and the errors of difference methods can be estimated, see Remark 2.1.

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