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DOMINATION PARAMETERS OF A GRAPH WITH ADDED VERTEX

Abstract. Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ is a total dominating set of G if for every vertex $y \in V$ there is a vertex $x \in D$ with $xy \in E$. A subset $D \subseteq V$ is a strong dominating set of G if for every vertex $y \in V - D$ there is a vertex $x \in D$ with $xy \in E$ and $\deg_G(x) \geq \deg_G(y)$. The total domination number $\gamma_t(G)$ (the strong domination number $\gamma_s(G)$) is defined as the minimum cardinality of a total dominating set (a strong dominating set) of G . The concept of total domination was first defined by Cockayne, Dawes and Hedetniemi in 1980 [1], while the strong domination was introduced by Sampathkumar and Pushpa Latha in 1996 [3]. By a subdivision of an edge $uv \in E$ we mean removing edge uv , adding a new vertex x , and adding edges ux and vx . A graph obtained from G by subdivision an edge $uv \in E$ is denoted by $G \oplus u_x v_x$.

The behaviour of the total domination number and the strong domination number of a graph $G \oplus u_x v_x$ is developed.

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Let G be a graph. By $V(G)$ and $E(G)$ we mean the vertex set and the edge set of a graph G , respectively. Let $uv \in E(G)$. Then, u and v *dominate* each other. Further, u *strongly dominates* v if $\deg_G(u) \geq \deg_G(v)$.

The *open neighbourhood* $N_G(u)$ is the set of all vertices dominated by u in G , whereas the *closed neighbourhood* of u in a graph G is defined as $N_G[u] = N_G(u) \cup \{u\}$. The cardinality of $N_G(u)$ is called the *degree* of a vertex u and it is denoted by $\deg_G(u)$. The number $\delta(G) \stackrel{df}{=} \min\{\deg_G(u) : u \in V(G)\}$ is the *minimum degree* of G .

By a *subdivision* of an edge uv we mean removing edge uv , adding a new vertex x , and adding edges ux and vx . A graph obtained from G by subdivision an edge $uv \in E$ is denoted by $G \oplus u_x v_x$.

A subset D of $V(G)$ is a *dominating set* of G if every vertex $v \in V(G) - D$ is dominated by some $u \in D$. If every vertex $v \in V(G)$ is dominated by some $u \in D$, then D is a *total dominating set* of G . Further, D is a *strong dominating set* if every vertex $v \in V(G) - D$ is strongly dominated by some $u \in D$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set. Similarly, we define the *total domination number* $\gamma_t(G)$ and the *strong domination number* $\gamma_{st}(G)$ of G . For convenience, a subset which realizes $\gamma(G)$, $\gamma_t(G)$, $\gamma_{st}(G)$ will be called a $\gamma(G)$ -set, a $\gamma_t(G)$ -set, and a $\gamma_{st}(G)$ -set, respectively.

The domination number is a well-studied parameter (see [2]).

Our purpose is to study the total domination number and the strong domination number with respect to $G \oplus u_x v_x$.

Let $uv \in E(G)$ be a subdivided edge of G . Observe that for any $y \notin \{u, x, v\}$ we have $N_G(y) = N_{G \oplus u_x v_x}(y)$. In other words, every vertex different from u, x and v is adjacent in $G \oplus u_x v_x$ to the same vertices as in G . Thus, in the proofs of the following results we consider only vertices u, x and v with respect to the property of a dominating set.

Additionally, observe that G possesses a total dominating set if and only if $\delta(G) \geq 1$.

Theorem 1. *Let G be a graph such that $\delta(G) \geq 1$. Then*

$$\gamma_t(G) \leq \gamma_t(G \oplus u_x v_x) \leq \gamma_t(G) + 1,$$

for any edge $uv \in E(G)$.

Proof. Let $uv \in E(G)$ be a subdivided edge of G .

First, we show that $\gamma_t(G \oplus u_x v_x) \leq \gamma_t(G) + 1$.

Let D be a $\gamma_t(G)$ -set. If $u \in D$, then it is easy to see that $D \cup \{x\}$ is a total dominating set of $G \oplus u_x v_x$ and $\gamma_t(G \oplus u_x v_x) \leq |D \cup \{x\}| = \gamma_t(G) + 1$. Assume that $u, v \notin D$. Since D is a total dominating set of G then u is adjacent to say $u_1 \in D$, similarly v is adjacent to say $v_1 \in D$ and $v_1 \neq u$. Then $D \cup \{u\}$ is a dominating set of $G \oplus u_x v_x$ (observe that x is adjacent to $u \in D \cup \{u\}$ and v is adjacent to $v_1 \in D \cup \{u\}$ in $G \oplus u_x v_x$). Moreover, u is adjacent to $u_1 \in D \cup \{u\}$ in $G \oplus u_x v_x$. Thus $D \cup \{u\}$ is a total dominating set of $G \oplus u_x v_x$. Consequently, $\gamma_t(G \oplus u_x v_x) \leq |D \cup \{u\}| = \gamma_t(G) + 1$, as required.

It suffice to show that $\gamma_t(G) \leq \gamma_t(G \oplus u_x v_x)$.

Let D_1 be a $\gamma_t(G \oplus u_x v_x)$ -set. Observe that u or v belongs to D_1 (otherwise D_1 would not be a total dominating set). Without loss of generality we can assume that $u \in D_1$.

Let $x \in V(G) - D_1$. Then D_1 also is a total dominating set of G . So, $\gamma_t(G) \leq |D_1| = \gamma_t(G \oplus u_x v_x)$.

Let $x \in D_1$. If additionally $v \in D_1$, then D_1 also is a total dominating set of G and $\gamma_t(G) \leq |D_1| = \gamma_t(G \oplus u_x v_x)$. If $v \in V(G) - D_1$, then $(D_1 - \{x\}) \cup \{v\}$ is a total

dominating set of G with $\gamma_t(G) \leq |D_1| = |(D_1 - \{x\}) \cup \{v\}| = \gamma_t(G \oplus u_x v_x)$, which completes the proof. \square

We now obtain similar result with respect to the strong domination number.

Theorem 2. *Let G be a connected graph. Then*

$$\gamma_{st}(G) \leq \gamma_{st}(G \oplus u_x v_x) \leq \gamma_{st}(G) + 1,$$

for any edge $uv \in E(G)$.

Proof. If $G \cong P_2$, then $G \oplus u_x v_x \cong P_3$ and $\gamma_{st}(G) = \gamma_{st}(G \oplus u_x v_x) = 2$. Thus the theorem is true. Assume that $G \not\cong P_2$ and let $uv \in E(G)$. Observe that $\deg_G(u) \geq 2$ or $\deg_G(v) \geq 2$.

First, we prove that $\gamma_{st}(G) \leq \gamma_{st}(G \oplus u_x v_x)$. Let D_1 be a $\gamma_{st}(G \oplus u_x v_x)$ -set. Consider the following cases:

- 1) Let $u, v \in D_1$. Then $x \in V(G \oplus u_x v_x) - D_1$, (otherwise $D_1 - \{x\}$ would be a strong dominating set of $G \oplus u_x v_x$, which contradicts D_1 being the smallest strong dominating set of $G \oplus u_x v_x$). Observe that D_1 is also a strong dominating set of G . It means that $\gamma_{st}(G) \leq |D_1| = \gamma_{st}(G \oplus u_x v_x)$.
- 2) Let $u, v \in V(G \oplus u_x v_x) - D_1$. It must be that $x \in D_1$. Additionally, assume that $\deg_G(u) \geq \deg_G(v)$. Then u strongly dominates v in G and $(D_1 - \{x\}) \cup \{u\}$ is a strong dominating set of G . Consequently, $\gamma_{st}(G) \leq |(D_1 - \{x\}) \cup \{u\}| = |D_1| = \gamma_{st}(G \oplus u_x v_x)$, as desired.
- 3) Let $u \in D_1$ and $v \in V(G \oplus u_x v_x) - D_1$.
 - a) Suppose that $x \in D_1$. Notice that $\deg_{G \oplus u_x v_x}(u) \geq 2$, otherwise u would be a hanging vertex and $D_1 - \{u\}$ would be a strong dominating set of $G \oplus u_x v_x$ with the cardinality strictly less than $\gamma_{st}(G \oplus u_x v_x)$.
 If $\deg_{G \oplus u_x v_x}(v) \leq 2$, then also is $\deg_G(v) \leq 2$ and u strongly dominates v in G . If $\deg_{G \oplus u_x v_x}(v) \geq 3$, then it must exist a vertex $y \neq x$ which strongly dominates v in $G \oplus u_x v_x$ and also in G . It means that $D - \{x\}$ is a strong dominating set of G and $\gamma_{st}(G) \leq |D_1 - \{x\}| = |D_1| - 1 < \gamma_{st}(G \oplus u_x v_x)$.
 - b) Suppose that $x \in V(G \oplus u_x v_x) - D_1$. Recall that $u \in D_1$ and $v \in V(G \oplus u_x v_x) - D_1$. Then, of course, $D - \{x\}$ is a strong dominating set of G and $\gamma_{st}(G) \leq |D_1 - \{x\}| = |D_1| - 1 < \gamma_{st}(G \oplus u_x v_x)$, as desired.

Now, we prove the second part of this theorem, i.e. $\gamma_{st}(G \oplus u_x v_x) \leq \gamma_{st}(G) + 1$.

Recall that $G \not\cong P_2$, so $\deg_G(u) \geq 2$ or $\deg_G(v) \geq 2$. Let D be a $\gamma_{st}(G)$ -set. If $u \in D$, then observe that x is strongly dominated by u or v in $G \oplus u_x v_x$. Moreover, $D \cup \{v\}$ is a strong dominating set of $G \oplus u_x v_x$. Thus $\gamma_{st}(G \oplus u_x v_x) \leq |D \cup \{v\}| \leq \gamma_{st}(G) + 1$. Similarly, if $v \in D$, then $D \cup \{u\}$ is a strong dominating set of $G \oplus u_x v_x$. Now, assume that $u, v \in V(G) - D$. Further, since D is a strong dominating set of G so there exists a vertex $u_1 \in D$ and a vertex $v_1 \in D$ which strongly dominates u and v , respectively. Then $D \cup \{u\}$ is a strong dominating set of $G \oplus u_x v_x$, if $\deg_G(u) \geq 2$ or $D \cup \{v\}$ is a strong dominating set of $G \oplus u_x v_x$, if $\deg_G(v) \geq 2$. Consequently, $\gamma_{st}(G \oplus u_x v_x) \leq \gamma_{st}(G) + 1$, as required. \square

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