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A NOTE INTRODUCING CAYLEY GRAPHS AND GROUP-COSET GRAPHS GENERATED BY GRAPH PACKINGS

Abstract. The aim of this paper is to construct a class of vertex-transitive graphs that includes the Kneser graphs as a special case. The class will be based on the notion of packing of graphs. Certain families of graphs within this class will be examined more closely, and some of their properties, such as hamiltonicity, will be investigated.

Keywords: Cayley graphs, hamiltonicity, packing of graphs.

Mathematics Subject Classification: 05C25, 05C45, 05C70.

1. INTRODUCTION

Computer networks and, more specifically, problems of passing information in networks, form an important application of graph theory. Many of the fundamental ideas and definitions used in graph theory were motivated by problems relating to computer networks.

From any computer, a corresponding graph can be defined in a simple way: the vertices of the graph represent processors, and the edges of the graph represent the physical connections between processors. Properties or problems relating to networks are referred to as "topological" if they can be defined as properties of the graph determined by the network (notice that different physical configurations of networks can result in isomorphic graphs). Such problems have become the subject of particularly intensive research.

When choosing the topology of a network we should take into account its efficiency and effectiveness on the one hand and costs and technical potential on the other.

Among many essential parameters one which is particularly important is regularity which means that every router is connected with the same number of other routers. A natural extension of this would be to be able to use the same procedures for every vertex of a network. In other words we want all the vertices to be 'the same'. This is the concept of a vertex-transitive graph. Graphs whose definitions are based on some algebraic structures (namely Cayley graphs and other similar constructions) play a particularly important role in planning networks.

1.1. CAYLEY GRAPHS AND GROUP-COSET GRAPHS

The following definitions and notations come from [12]. We assume that the reader is familiar with basic definitions of graph theory. In this paper we consider only simple finite graphs. Let $G = (V, E)$ be such a graph and let $\mathrm{Aut}\,(G)$ be its automorphism group. A graph G is *vertex-transitive* if and only if for every two vertices $x, y \in V$ there exists an automorphism $\theta \in \mathrm{Aut}\,(G)$ such that $\theta(x) = y$. Such graphs will be shortly denoted as VT.

One of the most important classes of VT graphs is the class of Cayley graphs. Let Γ be a finite group and S a subset of Γ with the following properties:

a) $1 \notin S$, where 1 is the identity element in Γ;

b) if $x \in S$ then $x^{-1} \in S$.

A Cayley graph is a graph $G = \mathrm{Cay}\,(\Gamma, S)$, the vertices of which are the elements of Γ, where the edge set is defined as follows:

$$E(G) = \{\{x, y\} \colon x^{-1}y \in S\}.$$

Remark. *As the vertices x, y are elements of a group we shall denote the edge connecting them by $\{x, y\}$ and not as usual by xy. The symbol xy will instead indicate the product in group Γ.*

Remark. *We do not insist that the set S be a generating set for Γ. However, in applications this assumption is usually fulfilled because the following (easy to show) statement is true:*

Theorem 1. *A Cayley graph $G = \mathrm{Cay}\,(\Gamma, S)$ is connected if and only if S generates Γ.*

The significance of Cayley graphs follows largely from the following theorem.

Theorem 2. *Every Cayley graph $G = \mathrm{Cay}\,(\Gamma, S)$ is VT.*

Not all VT graphs are Cayley graphs, however. The simplest example is the Petersen graph.

However, Sabidussi has shown that a slightly more general construction leads to the whole class of VT graphs.

Let H be a subgroup of a finite group Γ and let S be a subset of H such that:

a) $H \cap S = \emptyset$,

b) $S = S^{-1}$.

A group-coset graph is a graph $G = \mathrm{Cay}\,(\Gamma/H, S)$ the vertices of which are the elements of the cosets Γ/H (i.e. the layers xH, for every $x \in \Gamma$) and the edge set is defined as follows:

$$E(G) = \{\{xH, yH\} \colon x^{-1}y \in HSH\}.$$

Remark. *From point* a) *of the above definition it follows that* $1 \notin S$ *because every subgroup contains element* 1*.*

It can be shown that:

Theorem 3. *A group-coset graph* $G = \mathrm{Cay}\,(\Gamma/H, S)$ *is* VT*.*

The so called *Sabidussi's representation theorem* states that (see [7]):

Theorem 4. *Every* VT *graph is a group-coset graph.*

1.2. PACKING OF GRAPHS AND CAYLEY GRAPHS

Two simple graphs G and H of order n are *packable* into a complete graph K_n if and only if there exist two edge-disjoint subgraphs of K_n, G' and H' such that G is isomorphic to G' and H to H'.

It is easy to see, that two graphs G and H are packable into K_n if and only if G is isomorphic to a subgraph of $\overline{H}$.

In the case where $G = H$ we call this a *2-packing* of G or simply *a packing* of G and G is called (2-)packable.

The proof of the following basic theorem can be found for instance in [2].

Theorem 5. *Let* G *be a graph of order* n*. If* $e(G) \leq n-2$ *then there exists a 2-packing of* G*.*

A 2-packing of a graph $G = (V, E)$ is often identified with a permutation σ such that if $xy \in E$ then $\sigma(x)\sigma(y) \notin E$.

Let $G = (V, E)$ be a graph of order n. From now on we will assume that G has at least one edge. Let us denote a group of all permutations of V by $\Pi(V)$ and the set of those permutations from $\Pi(V)$ which are 2-packings of G by $\mathrm{Pak}\,(G)$. We shall assume that G is packable.

Proposition 6. a) $\mathrm{id}_V \notin \mathrm{Pak}\,(G)$,

b) *if* $\theta \in \mathrm{Pak}\,(G)$ *then* $\theta^{-1} \in \mathrm{Pak}\,(G)$*.*

Proof. Statement a) follows from the fact that G has at least one edge.

Suppose the edge $e = xy$ of G belongs also to the set $\theta^{-1}(E)$. This means that there exists an edge $e' = x'y'$ in G where $x = \theta^{-1}(x')$ and $y = \theta^{-1}(y')$. Then $x' = \theta(x)$ and $y' = \theta(y)$, which means that $e = \theta(e')$, so e belongs to both E and $\theta(E)$. This contradicts the fact that θ is a packing of G. $\square$

We can now define a Cayley graph $\mathrm{Cay}\,(\Gamma, S)$ where $\Gamma = \Pi(V(G))$ and $S = \mathrm{Pak}\,(G)$. Such a graph will be called *a Cayley graph of type PP* (an abbreviation for "packing permutations"). We shall say that such a graph is generated by G.

It is easy to see that the following lemma holds.

Lemma 7. *Let $G = (V, E)$ be a graph and $\alpha, \beta \in \Pi(V)$. The graphs $\alpha(G)$ and $\beta(G)$ are edge-disjoint if and only if the permutation $\beta^{-1}\alpha$ is a packing of G.*

In other words, the vertices of a Cayley graph of type PP generated by G are all possible images of G through $n!$ permutations. Any two images are connected if they are edge-disjoint.

Group-coset graphs are also associated in a natural way with the issues concerning packing of graphs. Let us notice, that

$$\mathrm{Pak}\,(G) \cap \mathrm{Aut}\,(G) = \emptyset$$

unless G has no edges.

We can then define a group-coset graph $\mathrm{Cay}\,(\Gamma/H, S)$ where $\Gamma = \Pi(V(G))$, $S = \mathrm{Pak}\,(G)$ and $H = \mathrm{Aut}\,(G)$. Such a graph will be called *a group-coset graph of type PP*. We shall say that such a graph is generated by G.

The geometric interpretation of these graphs is similar to the interpretation of Cayley graphs of type PP, however the images having the same edge set are now identified.

Section 2 shows that Kneser graphs are a special case of group-coset graphs of type PP. Section 3.1 examines the structure of a different special case of Cayley and group-coset graphs of type PP, showing that the graphs in this case are Hamiltonian. and that in fact every edge is in a hamiltonian cycle. Section 3.2 considers a third special case, and shows that this results in group-coset graphs that are isomorphic to disjoint unions of some number of copies of a Kneser graph. Section 4 poses problems for further study.

2. RELATIONSHIPS WITH OTHER CLASSES

We shall identify the vertices of G of order n with the elements of $\{1, \ldots, n\}$ and write $\Pi(n)$ instead of $\Pi(V(G))$.

Let $n = 2k+1$. Let us examine closer the group-coset graphs of type PP generated by a clique K_{k+1} (more precisely by a graph $G = K_{k+1} \cup kK_1$). We consider a graph $H = \mathrm{Cay}\,(\Pi(n)/\mathrm{Aut}\,(G), \mathrm{Pak}\,(G))$.

As one can easily see,

$$|\mathrm{Aut}\,(G)| = k! \cdot (k+1)!,$$

so the order of the graph H is

$$\frac{n!}{k! \cdot (n-k)!} = \binom{n}{k}.$$

On the other hand, the degree of every vertex in H is $k+1$, because every two disjoint cliques of size $k+1$ have exactly one common vertex. One can check that for $k=2$, H is the Petersen graph.

Let us now remind the definitions of Kneser graphs. The vertices of a *Kneser graph* $K(n,k)$ are k-element subsets of an n-element set (n and k are any numbers, $k \leq n$). Two vertices are connected with an edge if the corresponding sets are disjoint. One can check, that $K(5,2)$ is Petersen graph.

Let us suppose again, that $n = 2k+1$. Treating the set $\{1,2,3,\dots,n\}$ as the vertex set of a graph we can find a bijection between the cliques of size $k+1$ and k-element subsets. Simply map a clique of size $k+1$ to the k-element subset consisting of the label of the k copies of K_1 (that is, the labels **not** used on the vertices of that clique). Verifying that the definitions of adjacency of vertices in a Kneser graph $K(2k+1,k)$ and in graph $H = \mathrm{Cay}\,(\Pi(n)/\mathrm{Aut}\,(G), \mathrm{Pak}\,(G)$, where $G = K_{k+1} \cup kK_1$, are coherent is left to the reader.

We now have

Proposition 8. *Kneser graphs $K(2k+1,k)$ are some special cases of group-coset graphs of type PP.*

3. THE RESULTS

Throughout this section we let $a_1 a_2 \dots a_n$ denote an n-element permutation which is equivalent to: $\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$.

We shall need the following result about hamiltonicity of regular graphs.

Theorem 9. [4] *Every 2-connected regular simple graph with vertex degrees at least $|G|/3$ is Hamiltonian.*

3.1. GRAPHS $G = K_{1,m} \cup nK_1$

Example 1. *An introductory example.*
Let us consider a graph $G = K_{1,1} \cup K_1$ with vertex-set V=\{1,2,3\} and edge-set E=\{\{1,2\}\}. Figure 1 presents this example.

- *Automorphisms of G:*
  $\mathrm{Aut}\,(G) = \{123, 213\}$, $|\mathrm{Aut}\,(G)| = 2$
- *The following permutations constitute the packings of this graph:*
  $\mathrm{Pak}\,(G) = \{132, 231, 312, 321\}$, $|\mathrm{Pak}\,(G)| = 4$

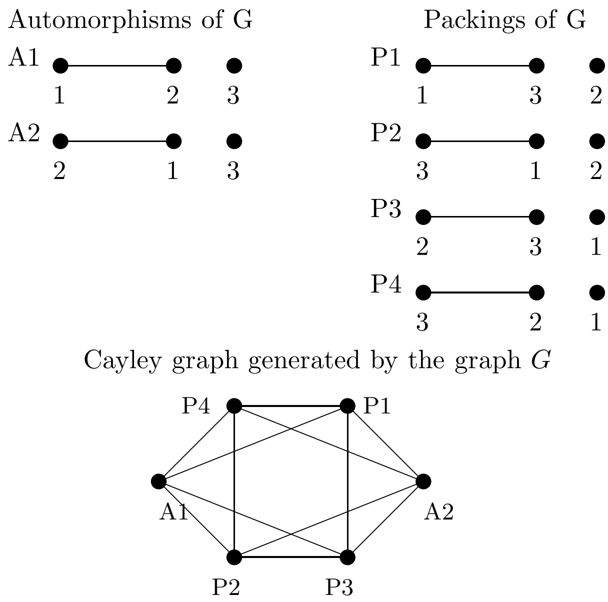

Fig. 1. The simplest example of a Cayley graph of type $\mathrm{Cay}\,(\Pi(V(G)), \mathrm{Pak}\,(G))$ for $G = K_{1,1} \cup K_1$

Example 2. *An example of a Cayley graph generated by $G = K_{1,2} \cup K_1$.*
Let us consider a graph $G = K_{1,2} \cup K_1$ with vertex-set $V = \{1, 2, 3, 4\}$ and edge-set $E = \{\{1, 2\}, \{1, 3\}\}$ — this graph is presented in the Figure 2.

- *Automorphisms of G:*
  $\mathrm{Aut}\,(G = \{1234, 1324\}$, $|\mathrm{Aut}\,(G)| = 2$
- *The following permutations constitute the packings of this graph:*
  $\mathrm{Pak}\,(G) = \{2341, 2431, 3241, 3421, 4123, 4213, 4132, 4312, 4231, 4321\}$,
  $|\mathrm{Pak}\,(G)| = 10$
- *The remaining permutations:*
  $C1 = 1243$, $C2 = 1423$, $C3 = 1342$, $C4 = 1432$, $C5 = 2134$, $C6 = 2314$, $C7 = 2143$, $C8 = 2413$, $C9 = 3124$, $C10 = 3214$, $C11 = 3142$, $C12 = 3412$.

This example is presented in the Figure 3. The Cayley graph generated by $G = K_{1,2} \cup K_1$ is hamiltonian from Theorem 9.

We shall consider the graph $G = K_{1,m} \cup nK_1$ for any integers $m, n > 0$. First, we consider $G = K_{1,m} \cup K_1$, which will serve as a base case for induction. It is easy to see that the following lemma holds.

Lemma 10. *Two graphs G_1, G_2 isomorphic to $K_{1,m} \cup K_1$ are edge-disjoint if and only if the centre of the star in G_1 is an isolated vertex in G_2 or the isolated vertex in G_1 is the centre of the star in G_2.*

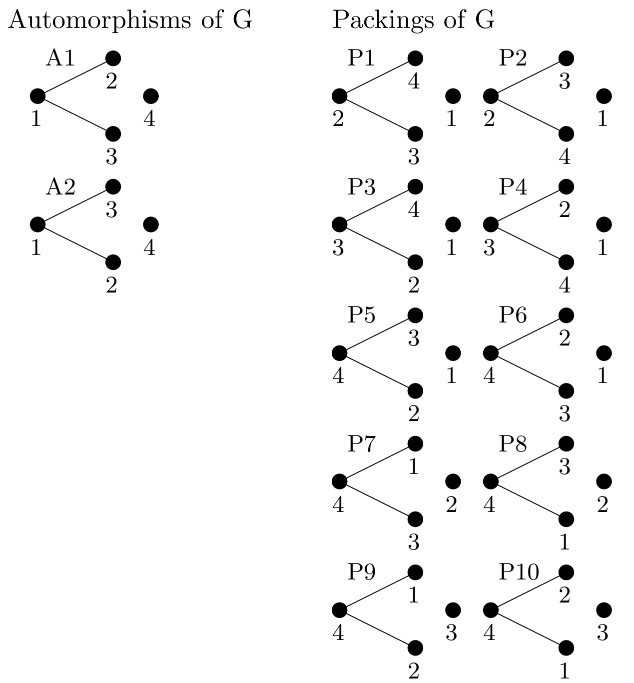

Fig. 2. The permutations of graph G, $G = K_{1,2} \cup K_1$

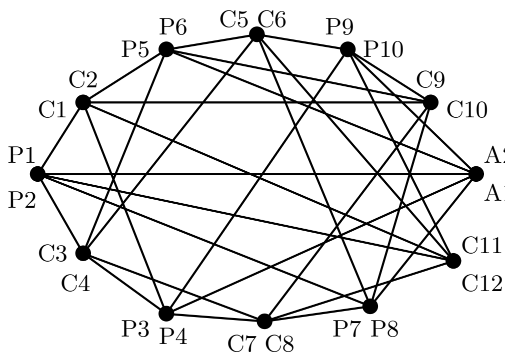

Fig. 3. The Cayley graph generated by $G = K_{1,2} \cup K_1$ (each double vertex represents a $\overline{K_2}$, with adjacent pairs of double vertices representing a $K_{2,2}$)

A permutation of type $\Pi_{j,i}(G)$ is the family of all permutations of the vertices of a graph G that map j onto i. Formally, for every permutation $p \in \Pi_{j,i}(G)$, $p(j) = i$, where $i, j \geq 1$.

Let us assume now, that we have families of permutations of a graph G: $\Pi_{j,a_j}(G)$ for fixed j, where $1 \le j \le m+2$ and $a_j = 1, \ldots, m+2$. Then the following lemma holds:

Lemma 11. *A graph $\alpha(G)$, where α is a permutation of G and α belongs to $\Pi_{j,a_j}(G)$, is edge-disjoint with a certain graph $\beta(G)$, where β belongs to $\Pi_{j,b_j}(G)$, for any $1 \le a_j, b_j, j \le m+2$.*

Proof. We identify the vertices of G with the elements of $\{1, \ldots, m+2\}$ in such a way that vertex 1 is is the centre of the star in G and vertex $m+2$ is the isolated vertex. Let $\alpha(G) = a_1 \ldots a_j \ldots a_{m+2}$.

We define a permutation $\beta(G)$, so that the obtained graph is edge-disjoint with the graph $\alpha(G)$. Let us take any b_j so that $1 \le b_j \le m+2$ and b_j is in position j in the permutation. Then:

$$\beta(G) = \begin{cases} a_{m+2} \ldots b_j \ldots, & \text{if } b_j \ne a_{m+2}, \\ \ldots b_j \ldots a_1, & \text{if } b_j = a_{m+2}. \end{cases}$$

(The "…" in the definition of β can be filled by any values that have not been used.) On the basis of Lemma 10 we know that the graphs for permutations $\alpha(G), \beta(G)$ are edge-disjoint. □

Theorem 12. *Cayley graphs of type PP (as defined in Section 1.2) generated by graphs $G = K_{1,m} \cup nK_1$ are hamiltonian and each edge in such a graph belongs to some hamiltonian cycle.*

Proof. The proof will proceed by a double induction, with case 1 completing the induction on m, which will provide the base case for the induction on n.

Case 1. Firstly we prove the theorem for $n = 1$. So, let $G = K_{1,m} \cup K_1$. We use the induction with respect to m:

Step 1. Example 1 shows that for $m = 1$ the Cayley graph is hamiltonian. Moreover, the reader can easily verify that each edge in this graph belongs to some hamiltonian cycle.

Step 2. Assume that the theorem is true for $m - 1 \ge 1$. We shall prove it for m.

Let us consider a graph $\mathrm{Cay} = \mathrm{Cay}(\Pi(V(K_{1,m} \cup K_1)), \mathrm{Pak}(K_{1,m} \cup K_1))$.

Let us choose an edge $e = \{\alpha, \beta\} \in E(\mathrm{Cay})$. The first position of the permutation denotes the centre of the star and the last position $(m+2)$ denotes the isolated vertex. In positions $1 < k < m+2$ there are star arms. Let us fix any k. We create the families of permutations of type $\Pi_{k,i}(G)$ where $i = 1, \ldots, m+2$. So we get $m+2$ families of permutations of G, each containing $(m+1)!$ permutations. The chosen edge may join either two permutations from one family or two permutations from different families.

The idea of the proof is to remove a vertex on position k from every graph which is an image of G after permutation in every family.

We obtain a Cayley graph $\mathrm{Cay}(\Pi(K_{1,m-1} \cup K_1), \mathrm{Pak}(K_{1,m-1} \cup K_1))$. On the basis of the induction hypothesis we know that in every family there is a hamiltonian cycle passing through any chosen edge. By adding the removed vertex as the arm of the graph we do not disconnect the cycle because graphs remain edge-disjoint. Now we have to combine $m+2$ cycles into one cycle.

Let us distinguish two cases.

1. Edge e joins permutations from one family.
Let us assume that $\alpha, \beta \in \Pi_{k,a_k}(G)$. By induction hypothesis we get a cycle passing through this edge. Next we choose two further permutations α', β' which are adjacent on this cycle (by the assumption on m we have at least six permutations on the cycle). Let $\alpha' = a_1 \dots a_k \dots a_{m+2}$. By Lemma 10 we get $\beta' = a_{m+2} \dots a_k \dots$ or $\beta' = \dots a_k \dots a_1$. Now we shall point out two adjacent permutations on each of the remaining cycles (by induction hypothesis we get the required cycles), which will then be used in creating a hamiltonian cycle.

On the cycle in $\Pi_{k,a_1}(G)$ let us choose: $\alpha'' = \dots a_1, \dots a_{m+2}$, $\beta'' = a_{m+2} \dots a_1 \dots$.

On the cycle in $\Pi_{k,a_{m+2}}$ let us choose: $\alpha''' = a_1 \dots a_{m+2} \dots$, $\beta''' = \dots a_{m+2} \dots a_1$.

For the remaining families $\Pi_{k,j}(G)$ and the cycles contained in them we put $\alpha'''' = a_1 \dots j \dots a_{m+2}$, $\beta'''' = a_{m+2} \dots j \dots a_1$.

Let us notice that, by Lemma 10, every chosen pair of permutations gives edge-disjoint graphs, so in our Cayley graph these permutations are joined with an edge. On the basis of Lemma 11 all remaining cycles (except for the chosen) may be arranged in any order.

Now we combine all the cycles by the chosen permutations in one of two ways, depending on β'. The way of combining is presented in Figure 4. Next we remove edges joining the pairs chosen on each of the cycles to get the required hamiltonian cycle.

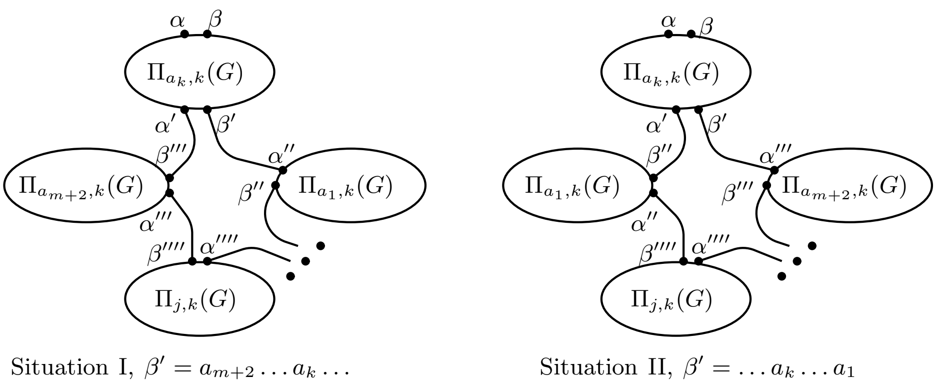

Fig. 4. Combining cycles into a hamiltonian cycle for edges from the same family dependent on β'

2. Edge e joins permutations from distinct families.
Let us assume that $\alpha \in \Pi_{k,a_k}(G)$ and $\beta \in \Pi_{k,a_l}(G)$. Let $\alpha = a_1 \dots a_k \dots a_{m+2}$. In such a situation we may have:

1° $\beta = \dots a_{m+2} \dots a_1$ $(l = m+2)$

2° $\beta = a_{m+2} \dots a_1 \dots$ $(l = 1)$

3° $\beta = \dots a_i \dots$ for $i \neq a_{m+2}$, $i \neq a_1$

The first two situations have been already discussed in the previous case. Now we only need to remark, that if $\alpha = a_1 \dots a_k \dots a_{m+2}$ then, by Lemma 10, $\beta = \dots a_{m+2} \dots a_1 \dots$ or $\beta = a_{m+2} \dots a_1 \dots$. Remark, that in the first situation $\beta = \beta'''$, in the second situation $\beta = \beta''$ and $\alpha = \alpha'$. By reasoning similar to that of previous case we obtain a Hamilton cycle.

We shall again choose permutations which are adjacent on the cycles and combine them into one hamiltonian cycle.

On the cycle for Π_{k,a_k} in addition to α we choose a permutation adjacent to it ie. $\beta' = a_{m+2} \dots a_k \dots a_1$.

On the cycle for $\Pi_{k,a_i}(G)$ in addition to β we choose $\alpha' = a_1 \dots a_i \dots a_{m+2}$. We can do this, because Lemma 10 implies that β is either $a_{m+2} \dots a_i \dots$ or $\dots a_i \dots a_1$.

On the cycle for $\Pi_{k,a_1}(G)$ we put $\alpha'' = a_n \dots a_1 \dots a_{m+2}$, $\beta'' = a'_n \dots a_1 a_{m+2} \dots a_n$, for any a_n, a'_n.

On the cycle for $\Pi_{k,a_{m+2}}(G)$ we put $\alpha''' = a_1 \dots a_{m+2} \dots a'_n$, $\beta''' = \dots a_{m+2} \dots a_1$.

On remaining cycles for families $\Pi_{k,j}(G)$ we put $\alpha'''' = a_1 \dots j \dots a_{m+2}$, $\beta'''' = a_{m+2} \dots j \dots a_1$.

All these pairs are adjacent in Cayley graph on the basis of Lemma 10, and on the basis of induction hypothesis they are adjacent also on the cycles. Now we combine the cycles as in Figure 5 and we remove edges joining the chosen pairs of permutations to obtain the required cycle.

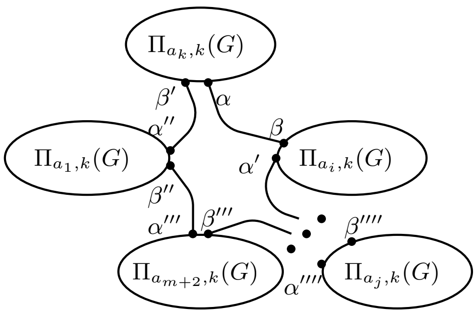

Fig. 5. Combining cycles into a hamiltonian cycle for edges from two distinct families

Case 2. Let us assume now that G has n isolated vertices.

It is easy to see that for $K_{1,1} \cup 2K_1$ the thesis of the theorem holds. We mention this graph because in some cases in the proof we shall construct a cycle built of at least 4 cycles obtained from the induction hypothesis.

We proceed by induction on n. Let us assume that for $n-1 \geq 1$ and for any m the thesis is true. We shall show that it is true for n. The idea of the proof is analogous to the one presented in Case 1. Let us assume that the permutations of G will be denoted as previously but the last n positions are taken by the isolated vertices. Again we construct families of permutations of the graph but now they are of type $\Pi_{n+m+1,a_i}(G)$, for any $1 \leq a_i \leq n+m+1$.

From each of the graphs we remove the isolated vertex standing on the last position in the permutation. Every family of graphs with the vertex removed is isomorphic to a Cayley graph $\mathrm{Cay}\,(\Pi(V(K_{1,m} \cup (n-1)K_1)), \mathrm{Pak}\,(K_{1,m} \cup (n-1)K_1)$. By our induction hypothesis we obtain hamiltonian cycles in all families that use any given edge. Now we add the removed vertex (graphs remain edge-disjoint because it is an isolated vertex) and we shall combine cycles into a hamiltonian cycle.

Let us choose an edge $e = \{\alpha, \beta\}$ in the Cayley graph and again let us distinguish two cases.

1°. Edge e joins permutations from the same family.
Assume that $\alpha, \beta \in \Pi_{m+n+1,a_{m+n+1}}(G)$. By our induction hypothesis we know that there exists a cycle passing through this edge. Next we choose two further permutations α', β' which are adjacent on the cycle (there are at least 6 permutations on the cycle by assumption on n). Let $\alpha' = a_1 \ldots a_{m+n+1}$, $\beta' = u_1 \ldots a_{m+n+1}$. We can assume that $a_1 \neq u_1$, since every permutation in this family appears somewhere on the hamiltonian cycle, and it is not possible that all of them (other than α and β) start with a_1. Thus, we can choose α and β from a place on the cycle where the first element of the permutation changes from a_1 (in α) to something else (in β). So, let $a_1 \neq u_1$.

We choose a pair of permutations on every cycle:

— for the family $\Pi_{m+n+1,a_1}(G)$: $\beta'' = u_1 \ldots a_1$, $\alpha'' = \ldots u_1 a_1$; a_1 and u_1 are isolated in $\alpha''(G)$ (there are at least two such vertices);

— for the family $\Pi_{m+n+1,u_1}(G)$: $\beta''' = \ldots a_1 u_1$, $\alpha''' = a_1 \ldots u_1$;

— for the remaining families $\Pi_{m+n+1,j}(G)$: $\beta'''' = u_1 \ldots a_1 j$, $\alpha'''' = a_1 \ldots u_1 j$.

Now we combine all the cycles as in the Figure 6 and we remove edges joining the chosen pairs to get a hamiltonian cycle.

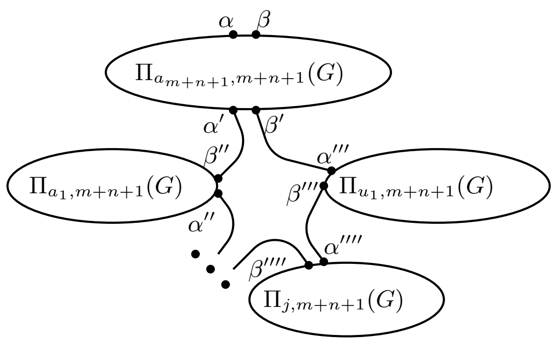

Fig. 6. Combining cycles into a hamiltonian cycle for edges from the same family when $u_1 \neq a_1$

2°. Edge e joins permutations from different families.
Let $\alpha \in \Pi_{m+n+1,a_{m+n+1}}(G)$ and $\beta \in \Pi_{m+n+1,u_{m+n+1}}(G)$ and let
$\alpha = a_1 \dots a_{m+n}a_{m+n+1}$, $\beta = u_1 \dots u_{m+n}u_{m+n+1}$.

For α we choose $\beta' = a_{m+n} \dots a_1 a_{m+n+1}$ and for β we choose $\alpha' = u_{m+n} \dots u_1 u_{m+n+1}$.

From the remaining cycles we choose pairs as follows:

— for the family Π_{m+n+1,a_1}: $\beta'' = a_{m+n} \dots a_{m+n+1}a_1$, $\alpha'' = a_{m+n+1} \dots a_{m+n}a_1$;

— for the family $\Pi_{m+n+1,u_{m+n}}$: $\beta''' = a_{m+n+1} \dots a_1 u_{m+n}$, $\alpha''' = a_1 \dots a_{m+n+1}u_{m+n}$;

— for the remaining families $\Pi_{m+n+1,j}$: $\beta'''' = a_{m+n+1} \dots a_1 j$, $\alpha'' = a_1 \dots a_{m+n+1} j$.

If $u_{m+n} = a_1$ then we neglect the doubled family. All cycles can be arranged in any order and joined by chosen permutations (because the centre of the first graph is mapped on the isolated vertex of the second).

Figures 7, 8 show how to combine these cycles into a hamiltonian cycle. After removing edges joining the chosen pairs we get the required hamiltonian cycle. $\square$

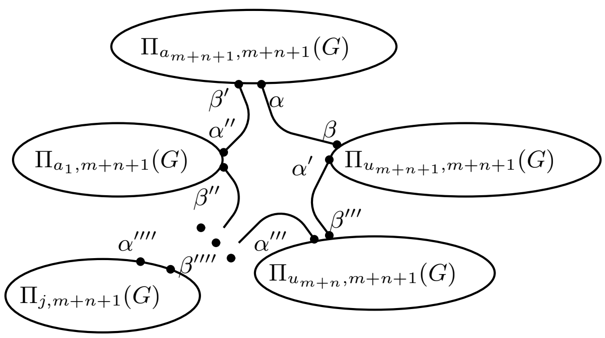

Fig. 7. Combining cycles into a hamiltonian cycle for edges from different families and $u_{m+n} \neq a_1$

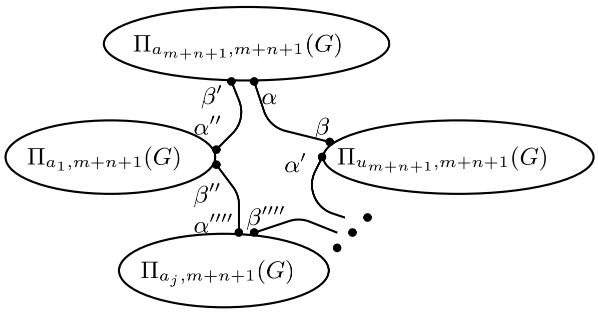

Fig. 8. Combining cycles into a hamiltonian cycle for edges from different families and $u_{m+n} = a_1$

3.2. GROUP-COSET GRAPHS. GRAPHS GENERATED BY $G = K_{n+1} \cup K_{1,n}$

Let us start with an example.

Remark. *Let $\{a_1 a_2 \dots a_n\}$ denote a permutation which is a representative of the coset.*

Example 3. *Let us calculate the cosets xH, $x \in \Pi(G)$, of the graph G from Example 1.*

$H = \mathrm{Aut}\,(G) = \{123, 213\}$,

$w_1 = 123H = \{123, 213\} = \{123\}$,

$w_2 = 132H = \{132, 312\} = \{132\}$,

$w_3 = 213H = \{213, 123\} = \{123\} = w_1$,

$w_4 = 231H = \{231, 321\} = \{231\}$,

$w_5 = 312H = \{312, 132\} = \{132\} = w_2$,

$w_6 = 321H = \{321, 231\} = \{231\} = w_4$.

Figure 9 presents this example.

Let $G = K_{n+1} \cup K_{1,n}$. There are $(2n+2)!$ permutations of G. Automorphisms of a star and of a clique constitute automorphisms of G. There are $(n+1)!n!$ automorphisms of G. Let us calculate the number of vertices of the group-coset graph:

$$\frac{|\Pi(G)|}{|\mathrm{Aut}\,(G)|} = \frac{(2n+2)!}{(n+1)!n!} = \binom{2n+1}{n}(2n+2).$$

One can notice that if $n > 1$ then the number of packings of G is the same as in the case of the graphs $K_{n+1} \cup nK_1$, discussed earlier. If $n = 1$ then it is easy to check that the group-coset graph is a triangle which is isomorphic with the group-coset graph dealt with in Example 3.

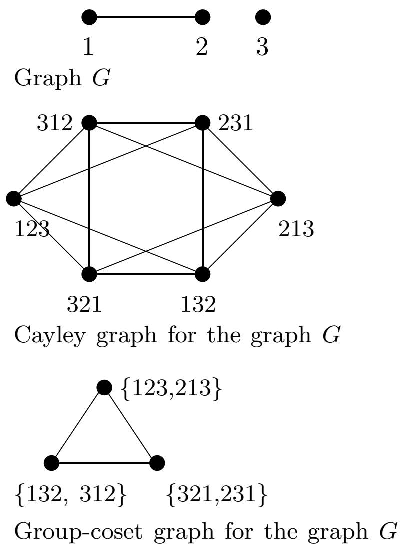

Fig. 9. The example of group-coset graph generated by $K_2 \cup K_1$

Let us now consider graphs G for $n \geq (>)1$. The number of different (not automorphic) packings is $n+1$ so the degree of every vertex in the group-coset graph generated by G equals $n+1$.

Lemma 13. *If two graphs G, H of type $K_{n+1} \cup K_{1,n}$ are edge-disjoint then the same vertex is the centre of the star in both graphs.*

Proof. In order to avoid common edges, the vertices of the clique of G must move to the vertices of the arms of the star of H, together with one vertex from the clique of H. The remaining $n-1$ vertices from the clique of H, together with the centre of the star of H, must therefore have formed the star of G. To avoid common edges, the vertices from the clique of H must be the vertices of the arms of the star of G, leaving the centre of the star of H to have come from the centre of the star of G. $\square$

Let us notice that all automorphisms of graphs of type $K_{n+1} \cup K_{1,n}$ also have the same vertex in the center of the star.

Let us introduce some helpful notation. Let G be a graph of type $K_{n+1} \cup K_{1,n}$. $\Pi(G)$ will denote the set of all permutations of vertices of this graph and H will denote the set of automorphisms of G. Π_H will denote the set of all left-cosets of the group of those permutations, ie. $\Pi_H = \{xH \colon x \in \Pi(G)\}$. Now we shall group the cosets using some rules. The family of cosets $\Pi_H(i,j)$ will consist of those cosets which are generated by permutations $x \in \Pi(G)$, for which $x(i) = j$, ie.

$$\Pi_H(i,j) = \{xH \in \Pi_H \colon x(i) = j\}.$$

We shall denote a permutation of G as a sequence $a_1 a_2 \ldots a_n a_{n+1} \ldots a_{2n+2}$, where $a_1, \ldots, a_{n+1}$ denote the images of vertices from the clique of G, a_{2n+2} is an image of the centre of the star of G and the remaining elements are the images of the arms of the star in G. We shall still denote the coset by its representative, ie. $\{a_1 \dots a_{2n+2}\}$. The remaining permutations are obtained by permuting elements $a_1, \dots, a_{n+1}$ and elements $a_{n+2}, \dots, a_{2n+1}$.

Remark. *It is quite easy to notice, that all the graphs in any fixed coset have the same vertex in the centre of the star. This is so because the automorphisms of G always have the same element $2n+2$ on the last $(2n+2)$ position.*

Composing any permutation from the given coset with any permutation x we shall obtain a permutation y such that elements in the last positions in x and in y are equal.

Lemma 13 implies that in the group-coset graph there are no edges between any two cosets from $\Pi_H(2n+2,k)$, $\Pi_H(2n+2,l)$, where $k \neq l$.

Lemma 14. *Let $\Pi_H(2n+2,k)$, $\Pi_H(2n+2,l)$ be the families of cosets defined above. If $k \neq l$ then the families have no common elements, ie.*

$$\Pi_H(2n+2,k) \cap \Pi_H(2n+2,l) = \emptyset$$

Proof. Let us assume that there exists a coset $xH \in \Pi_H(2n+2,k)$ and a coset $yH \in \Pi_H(2n+2,l)$ such that $xH = yH$ for some permutations x, y of G. Every graph given by a permutation in any coset has the same edges as any other graph given by a permutation from the same coset.

As we have noticed, automorphisms in H have the same vertex in the centre and this vertex is a fixed point of the permutation: $h(2n+2) = 2n+2$ for each $h \in H$.

In the coset xH we obtain permutations which map the centre of the star in G $(2n+2)$ onto k in $xh(G)$ for each h in H. In case of the coset yH $2n+2$ is mapped on l.

We know that the automorphisms of graphs of type $K_{n+1} \cup K_{1,n}$ have the same vertex in the center of the star, so we get a contradiction. □

Theorem 15. *A group-coset graph generated by $G = K_{n+1} \cup K_{1,n}$ consists of $2n+2$ disjoint copies of a group-coset graph generated by $G' = K_{n+1} \cup nK_1$ for $n \geq 2$.*

Proof. Lemma 14 and above Remark imply that coset families $\Pi_H(2n+2,k)$ constitute a partition of the set of all cosets for each $1 \leq k \leq 2n+2$. There are $2n+2$ cosets and each of them has

$$\frac{\binom{2n+1}{n}(2n+2)}{2n+2} = \binom{2n+1}{n}$$

elements.

The sets are of the same size as in the case of the graphs $K_{n+1} \cup nK_1$. So, we can define an isomorphism between $\Pi_H(2n+2,k)$ and a group-coset graph generated by $G' = K_{n+1} \cup nK_1$ for $n \geq 2$:

$$f \colon V(G') \to \Pi_H(2n+2,k).$$

Let $yH' \in V(G')$. Then $yH' = \{p_1, \dots, p_m \in \Pi(G')\}$ and $f(yH') = \{f(p_1), \dots, f(p_m)\}$, where $m = |\mathrm{Aut}\,(G')| = n!(n+1)!$

$$f(p_j) = \begin{cases} p_j(i),\ i \neq k, 1 \leq k \leq 2n+1, \\ 2n+2,\ i = k \\ p_j(k),\ i = 2n+2. \end{cases} \qquad 1 \leq j \leq 2n+1$$

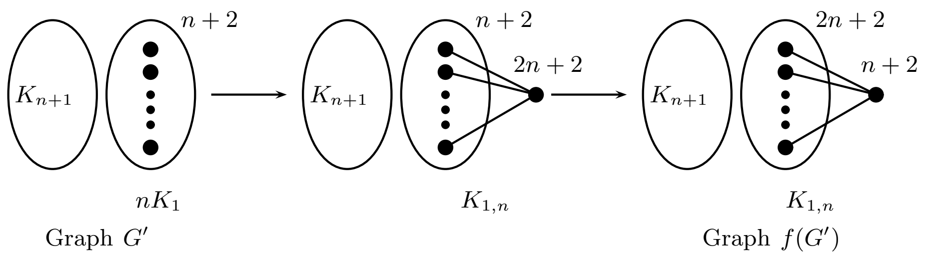

Fig. 10. An example of transformation of a graph belonging to the coset $\{1 \dots (n+1)(n+2) \dots (2n+1)\}$ into a graph in the coset $\Pi_H(2n+2, n+2)$

We only have to show that f is an isomorphism.

1. f is an extension of a $2n{+}1$-element permutation to a $2n{+}2$-element permutation by adding one position with value $2n+2$ and transposing it with position k. So f is well-defined.
2. Point 1 implies that f is a bijection.
3. Let $e = \{xH', yH'\}$ be an edge in a group-coset graph generated by G'. In graphs corresponding to the permutations which do not belong to the same coset the edge sets are disjoint. This means that all isolated vertices are mapped to vertices of the clique. After adding a vertex and joining it with isolated vertices we obtain a star. After permutations the added vertex remains the centre of the star and the graphs remain edge-disjoint. So the cosets which are images of xH' and yH' through f include edge-disjoint graphs (graphs belonging to cosets $f(xH')$, $f(yH')$). This means that in the group-coset graph generated by G cosets $f(xH')$ and $f(yH')$ are joined by an edge.

We obtain a copy of a group-coset graph generated by G' (Fig. 10).

Now we define functions f for the families $\Pi_H(2n+2, k), 1 \leq k \leq 2n+2$ and we obtain $2n+2$ disjoint copies of the group-coset graph generated by G' in the group-coset graph generated by G. ☐

Example 4. *The group-coset graph generated by* $G = K_3 \cup K_{1,2}$.

$|\Pi(G)| = 6! = 720,$

$|\mathrm{Aut}\,(G)| = 3!2! = 6 * 2 = 12,$

$\frac{|\Pi(G)|}{|\mathrm{Aut}\,(G)|} = \frac{720}{12} = 60.$

The degree of each vertex is 3. On the basis of Theorem 15 we obtain six copies of the a group-coset graph generated by $G' = K_3 \cup 2K_1$ (see Fig. 11).

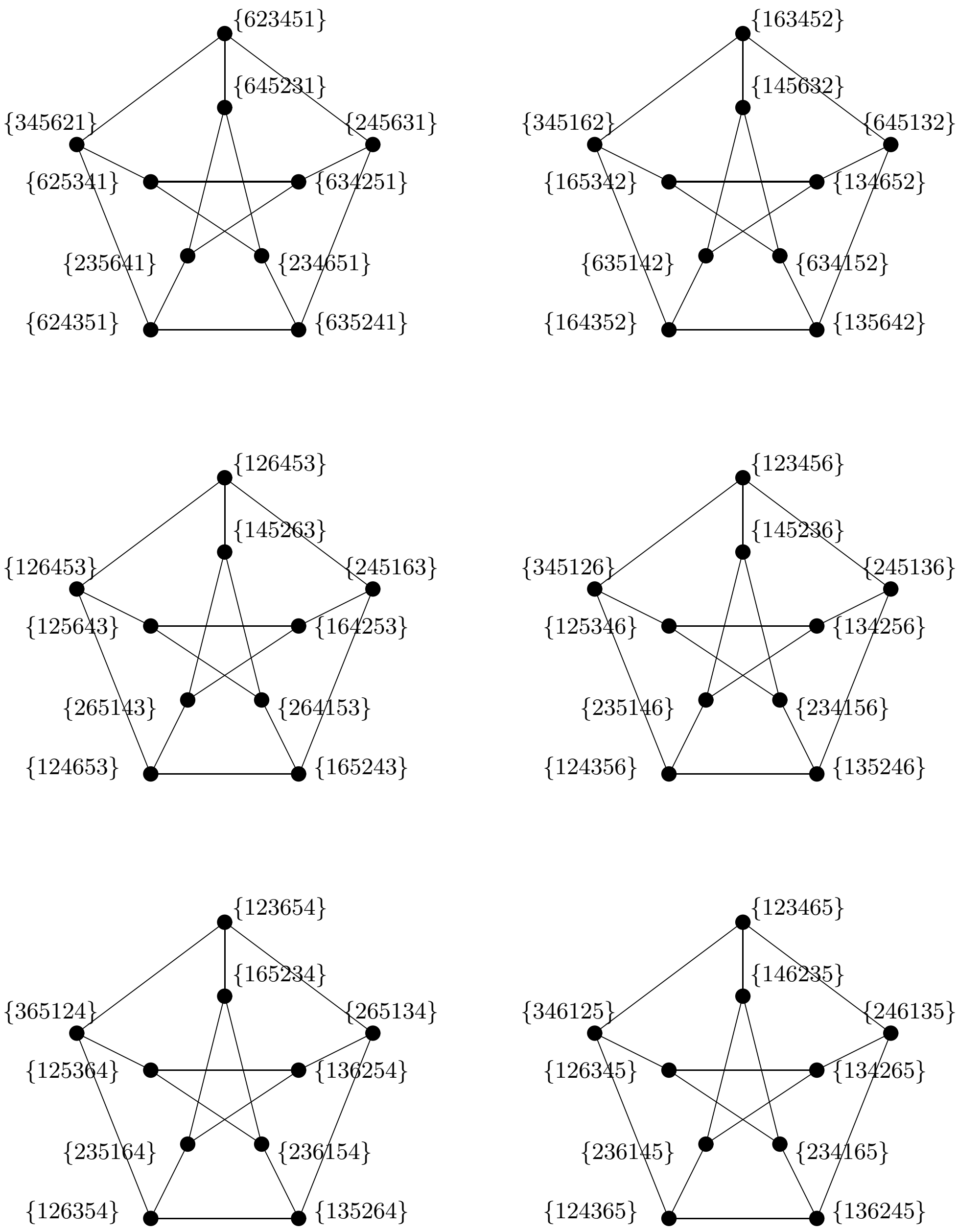

Fig. 11. The group-coset graph generated by $G = K_3 \cup K_{1,2}$

4. PROBLEMS AND SOME FURTHER GENERALISATIONS

There arise three natural problems which can be considered in connection with the special classes of Cayley graphs introduced in this note.

Firstly, we can ask which vertex-transitive graphs can be formed as Cayley or group-coset graphs of type PP?

The second problem would consist of applying some classical questions concerning Cayley graphs (or more broadly, VT graphs), such as the problem of the existence of the hamiltonian cycle, to these special families of VT graphs.

Finally, we can ask how the properties of a graph G influence the properties of a graph of type PP generated by G.

In conclusion, let us mention some possibilities of further generalisations. Let us assume that we have a graph G. Instead of a symmetric group $\Pi(V)$ we can consider one of its subgroups Γ'. We can also manipulate set S. Instead of taking $S = \mathrm{Pak}\,(G)$ we can use for example a set S' containing those permutations α for which the value of

$$|E(G) \cap E(\alpha(G)|$$

is constant, or more general, the permutations such, that

$$|E(G) \cap E(\alpha(G)| \in I,$$

where I is a subset of $\{0, 1, \ldots, |E|\}$. Of course, the relationships between groups Γ', H' and the set S' have to be maintained.

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REFERENCES

[1] Bollobás B.: *Extremal Graph Theory.* London, Academic Press 1978.

[2] Burns D., Schuster S.: *Every $(p, p-2)$ graph is contained in its complement.* J. Graph Theory **1** (1977), 277–279.

[3] Hillis D.: *The connection machine.* New York, MIT Press 1985.

[4] Jackson B.: *Hamilton cycles in regular 2-connected graphs.* J. Comb. Theory (B) **29** (1980), 27–46.

[5] Leighton F. T.: *Introduction to parallel algorithms and architecture.* Morgan Kaufmann, 1992.

[6] Rumeur de J.: *Communications dans les réseaux de processeurs.* Paris, Masson 1994.

[7] Sabidussi G.: *Vertex-transitive graphs.* Monatsh. Math. **68** (1964), 426–438.

[8] Woźniak M.: *Packing of Graphs.* Dissertationes Mathematicae **362** (1997), 1–78.

[9] Woźniak M.: *Wprowadzenie do problemów komunikacji w grafach.* Kraków, Wydawnictwa AGH 1999.

[10] Woźniak M.: *Packing of graphs and permutations — a survey.* Discrete Math. **276** (2004), 379–391.

[11] Yap H.P.: *Some Topics In Graph Theory.* London Mathematical Society, Lectures Notes Series 108, Cambridge, Cambridge University Press 1986.

[12] Yap H.P.: *Packing of graphs — a survey.* Discrete Math. **72** (1988), 395–404.

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