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WEAKLY CONVEX AND CONVEX DOMINATION NUMBERS

Abstract. Two new domination parameters for a connected graph G: the weakly convex domination number of G and the convex domination number of G are introduced. Relations between these parameters and the other domination parameters are derived. In particular, we study for which cubic graphs the convex domination number equals the connected domination number.

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1. INTRODUCTION

Let $G = (V, E)$ be a connected undirected graph. The *neighbourhood* of a vertex $v \in V$ in G is the set $N_G(v)$ of all vertices adjacent to v in G. For a set $X \subseteq V$, *the open neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and *the closed neighbourhood* $N_G[X] = N_G(X) \cup X$. Let X be a set of vertices and let $u \in X$. We say that a vertex v is a *private neighbour* of u, with respect to X, if $N_G[v] \cap X = \{u\}$.

The *degree* $\deg_G(v)$ of a vertex v is the number of edges incident to v, $\deg_G(v) = |N_G(v)|$. If $\deg_G(v) = r$ for every vertex $v \in V$ in G, then G is said to be *regular* of degree r, or simply r-regular. A 3-regular graph is also called a *cubic graph.* A 2-regular graph of order n is *a cycle* and is denoted by C_n. If $\deg_G(v) = 1$, then v is called *an end-vertex* or *a leaf* of G. A vertex which is a neighbour of an end-vertex let us call a *support.* A vertex $x \in V$ is called *a universal vertex* (or *a dominating vertex*) if $deg(x) = |V(G)| - 1$. A set $D \subset V$ is a *dominating set* of G if $N_G[D] = V$. A dominating set D is a *perfect dominating set* if $|N_G(v) \cap D| = 1$ for each $v \in V - D$. Further, D is a *connected dominating set* if D is dominating and the subgraph $\langle D \rangle$ induced by D, is connected. A set $D \subset V$ is *independent* in G if no two vertices of D are adjacent. A set is *independent dominating* if it is independent and dominating. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G, while the *connected domination number* $\gamma_c(G)$ is the minimum cardinality of a connected dominating set in G. The *distance* $d_G(u,v)$ between two vertices u and v in a connected graph G is the length of the shortest $(u-v)$ path in G. A $(u-v)$ path of length $d_G(u,v)$ is called $(u-v)$-*geodesic*. The *diameter* diam G of a connected graph G is $\max_{u,v\in V(G)} d_G(u,v)$. A set $X \subseteq V$ is called *weakly convex* (or *isometric* [4]) in G if for every two vertices $a, b \in X$ exists $(a-b)$-geodesic whose vertices belong to X. A set X is *convex* in G if vertices from all $(a-b)$-geodesic belong to X for every two vertices $a, b \in X$. A set $X \subseteq V$ is a *weakly convex dominating set* if X is weakly convex and dominating. Further, X is a *convex dominating set* if it is convex and dominating. The *weakly convex domination number* $\gamma_{\text{wcon}}(G)$ of a graph G is the minimum cardinality of a weakly convex dominating set, while the *convex domination number* $\gamma_{\text{con}}(G)$ of a graph G is the minimum cardinality of a convex dominating set. Convex and weakly convex domination numbers were first introduced by Jerzy Topp, Gdansk University of Technology, 2002. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$ and the *join* $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv\colon\ u \in V(G_1), v \in V(G_2)\}$. The *corona* $G = H \circ K_1$ is the graph G constructed from a copy of H, where for each vertex $v \in V(H)$, a new vertex $v^{'}$ and a pendant edge $vv^{'}$ are added.

2. RELATIONS BETWEEN $\gamma_{\text{wcon}}, \gamma_{\text{con}}$ AND THE OTHER DOMINATION PARAMETRES

Since every convex dominating set is weakly convex dominating set and every weakly convex dominating set is connected dominating set, we have following inequality chain.

Lemma 1. *For any connected graph G is*

$$\gamma(G) \leq \gamma_c(G) \leq \gamma_{\text{wcon}}(G) \leq \gamma_{\text{con}}(G).$$

We show that the differences $\gamma_{\text{con}} - \gamma_c$ and $\gamma_{\text{con}} - \gamma_{\text{wcon}}$ can be arbitrarily large.

Theorem 1. *For any $k, r \in N$ where $r \geq 3$, there exists a graph G such that $\gamma_c(G) = \gamma_{\text{wcon}}(G) = r$ and $\gamma_{\text{con}}(G) - \gamma_c(G) = \gamma_{\text{con}}(G) - \gamma_{\text{wcon}}(G) = k$.*

Proof. Assume first that $k \in N$, $r = 3$.

Let H be a complete bipartite graph isomorphic to $K_{k+1,2}$ and let x be a vertex of minimum degree in H. Let y and z be two neighbours of x. Let G be a graph which results if we identify y with one support of P_6 (a path with six vertices) and z with another support of P_6. The set $\{x, y, z\}$ is a minimum connected dominating set and a weakly convex set in G, so we have $\gamma_c(G) = \gamma_{\text{wcon}}(G) = 3$. It is easy to observe that $V(H)$ is a minimum convex dominating set of G and therefore $\gamma_{\text{con}}(G) = k+3$. Thus $\gamma_{\text{con}}(G) - \gamma_c(G) = \gamma_{\text{con}}(G) - \gamma_{\text{wcon}}(G) = k$.

If $r \geq 4$, then the graph G is constructed as follows. Let x and y be two non-end-vertices at distance two in the corona $C_r \circ K_1$. Let u and v be two vertices of a complete bipartite graph $K_{k,2}$ which form its partite set. Let G be a graph which results if we identify x with u and y with v (see Fig. 1 for $r = 8, k = 2$).

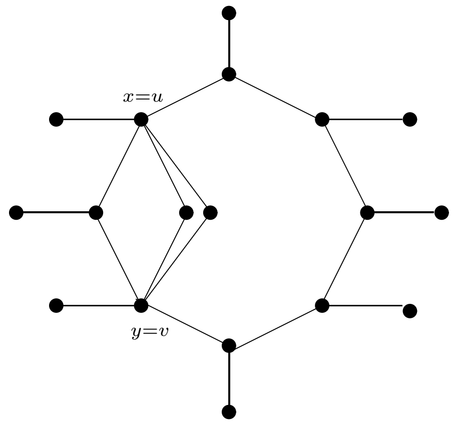

Fig. 1

In such a graph G, the minimum connected dominating set consists of all supports. This set is also weakly convex, so $\gamma_c(G) = \gamma_{\text{wcon}}(G) = r$. It is obvious that the minimum convex dominating set consists of all non-end-vertices, so $\gamma_{\text{con}}(G) = n - r = k + r$. Thus $\gamma_{\text{con}}(G) - \gamma_c(G) = \gamma_{\text{con}}(G) - \gamma_{\text{wcon}}(G) = k$. □

Theorem 2. *For any positive integers k and $l, k \geq 3$, there is a graph G, for which $\gamma_c(G) = k$ and $\gamma_{\text{con}}(G) = k + l$.*

Proof. We begin with the corona $C_k \circ K_1$ and then, to receive a graph G, we replace any non-end edge uv by l inner disjoint $(u - v)$ paths of length two (see Fig. 2 for $k = 7$, $l = 3$).

It is easy to observe that in such a graph G, the minimum connected dominating set consists of all supports of G and certainly $\gamma_c(G) = k$. It is also easy to observe that the minimum convex dominating set consists of all non-end vertices of G and $\gamma_{\text{con}}(G) = k + l$. □

Duchet and Meyniel [1] have shown that for any connected graph G is $\gamma_c(G) \leq \leq 2\beta_0(G) - 1$ and $\gamma_c(G) \leq 2\Gamma(G) - 1$, where $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of G and β_0 is the maximum cardinality of an independent dominating set of G.

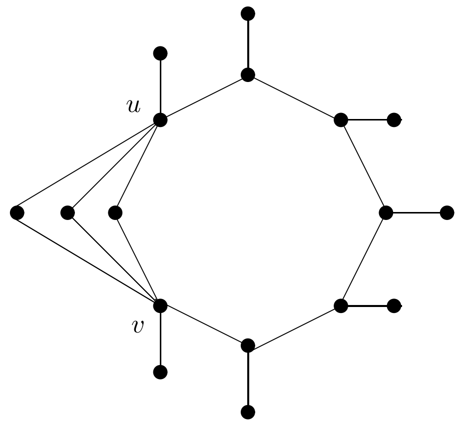

Fig. 2

The next theorem shows that there is no case for the convex domination number of a graph.

Theorem 3. *Every of differences $\gamma_{\mathrm{con}} - \beta_0$ and $\gamma_{\mathrm{con}} - \Gamma$ can be arbitrarily large.*

Proof. The graph G is constructed as follows. We begin with a cycle C_{2k+2} with $2k+2$ vertices and label consecutive vertices $v_1, \ldots, v_k, z, w_k, w_{k-1}, \ldots, w_1, u$. Then we add the edges $v_i w_i, v_i w_{i+1}$ and $w_i v_{i+1}$, for $i = 1, \ldots, k-1$ and to each of vertices u and z we add an end edge (see Fig. 3 for $k = 3$). This completes the construction of G. If we want to find a minimum convex dominating set of G, we must put there supports u and z. Vertices v_i and w_i, where $i = 1, \ldots, k$ belong to the shortest paths between u and z, so we must also put them to a minimum convex dominating set.

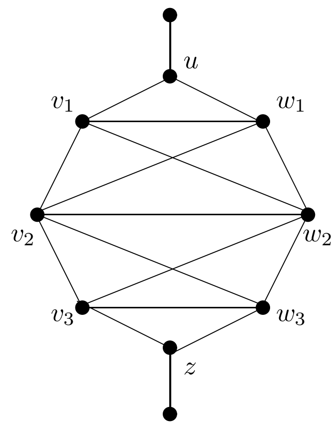

Fig. 3

Thus we have $\gamma_{\rm con}(G) = 2k+2$. If we want to find a maximum independent set of G, we must put there the end-vertices. Then the neighbours of these vertices, formed by vertices of degree 3 from the cycle C_{2k+2}, do not belong to this set. The rest of vertices from the cycle, i.e., vertices of degree 4 and 5, we can put into $\lceil \frac{k}{2} \rceil$ disjoint sets, where the induced subgraphs by these sets are complete graphs (if k is even, then we have only graphs K_4 and if k is odd, then except for the graphs K_4 we have one graph K_2). Only one vertex from each of these subgraphs can belong to a maximum independent set, so $\beta_0(G) = \lceil \frac{k}{2} \rceil + 2$ and $\gamma_{\rm con}(G) - \beta_0(G) = 2k - \lceil \frac{k}{2} \rceil = \lfloor \frac{3k}{2} \rfloor$. Observe, that the maximum independent set in this graph is also the maximum minimal dominating set of G. Thus $\Gamma(G) = \beta_0(G) = \lceil \frac{k}{2} \rceil + 2$ and $\gamma_{\rm con}(G) - \Gamma(G) = \lfloor \frac{3k}{2} \rfloor$. $\square$

3. EQUALITY $\gamma_c = \gamma_{\rm con}$ FOR CUBIC GRAPHS

Now we study cubic graphs, for which γ_c is equal to $\gamma_{\rm con}$.

Theorem 4. *If G is a cubic graph and there exists a minimum connected dominating set D in G such that $\langle D \rangle$ is a star, then $\gamma_c(G) = \gamma_{\rm con}(G)$.*

Proof. Let G be a cubic graph and let D be a minimum connected dominating set of G such that $\langle D \rangle$ is a star. If $\langle D \rangle = K_1$ or $\langle D \rangle = K_{1,1}$, then D consists of one or two vertices and D is convex. So we have required equality. If $\langle D \rangle = K_{1,2}$, then a minimum connected dominating set consists of three vertices, so in such a graph G we have $\gamma_c(G) = 3$.

Graph G is cubic, and since $\sum_{v \in V} \deg_G(v) = 2\,|E|$, it is of even order. For $n = 4$ we have $G = K_4$, so $\gamma_c(G) = \gamma_{\rm con}(G) = 1$. It is easy to check, that for every cubic graph G of order $n = 6$ we have $\gamma_c(G) = \gamma_{\rm con}(G) = 2$, (except for a graph G^* in Fig. 4, for which $\gamma_c(G^*) = 3$, but in this graph does not exist a minimum connected dominating set D such that $\langle D \rangle$ is a star). Thus, since $\gamma_c(G) = 3$ and $G \neq G^*$, we have $n \geq 8$.

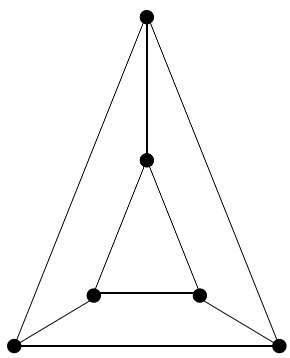

Fig. 4. Graf G^*

Let us denote vertices from a set D successively v_1, v_2, v_3, and vertices from $V - D$ successively $v_4, v_5, \ldots, v_n$. Suppose, that $\gamma_c(G) < \gamma_{\text{con}}(G)$. Then D is not a convex set and there exists a vertex belonging to $V - D$ (without loss of generality, a vertex v_4) such that it is a neighbour of v_1 and v_3. Each of vertices $v_5, \ldots, v_n$ must have a neighbour in D. So, since G is cubic, we have $n \leq 7$, which gives a contradiction. Thus D is a convex set in G and $\gamma_c(G) = \gamma_{\text{con}}(G)$.

Now suppose that $\langle D \rangle = K_{1,3}$. We know that $n \geq 8$ and we consider two cases: $n = 8$ and $n > 8$. Let us denote by v_1 a vertex of degree 3 in $\langle D \rangle$ and the rest of vertices from $\langle D \rangle$ successively by v_2, v_3, v_4.

1) $n = 8$.

   Vertices from $V - D$ we denote by $v_5, \ldots, v_8$. Suppose that $\gamma_c(G) < \gamma_{\text{con}}(G)$, that is a set D is not convex. Then there exists a vertex in $V - D$ (without loss of generality, a vertex v_5) such that it has two neighbours in D, say, vertices v_2, v_3.

   If any other vertex from $V - D$ has two neighbours in D, then $D - \{v_3\}$ (or $D - \{v_2\}$) is a connected dominating set, which gives a contradiction. Thus exactly one vertex from $V - D$ has two neighbours in D (this is a vertex v_5).

   If the vertex v_5 has three neighbours in D, then every of vertices v_6, v_7, v_8 has exactly one neighbour in D. Moreover, none of these vertices has a common neighbour in D and the subgraph $\langle v_6, v_7, v_8 \rangle$ induced by v_6, v_7 and v_8 is K_3. In such a graph G, vertices v_4, v_6, v_7, v_8 create a convex dominating set of cardinality $|D|$, which gives a contradiction. Thus, the vertex v_5 has exactly two neighbours in D, each of vertices v_6, v_7, v_8 has exactly one neighbour in D and none of these vertices has a common neighbour in D. Then some of vertices belonging to $V - D$ must be neighbours of v_4, which gives a contradiction. Thus D is a convex set.

2) $n > 8$.

   If $n > 8$, then $n \geq 10$, because n is even. Let us denote by $v_5, \ldots, v_n$ the vertices from $V - D$. Suppose that D is not convex. Then there exists a vertex in $V - D$ having two neighbours in D. Thus, since G is cubic, we have $n \leq 9$, which gives a contradiction. □

Theorem 5. *If G is a cubic graph of order n and there exists a minimum connected dominating set D in G such that $\langle D \rangle$ is a cycle C_p, then:*

a) $p \geq 4$, $n \geq 8$*; if* $p = 3$*, then* $n = 6$ *and* $G = G^*$ *or*

b) *D is a perfect dominating set and every vertex from D has exactly one private neighbour or*

c) $n = 2p$ *or*

d) *if* $p < 6$*, then* $\gamma_c(G) = \gamma_{\text{con}}(G)$.

Proof. Let G be a cubic graph of order n and let D be a minimum connected dominating set of G such that $\langle D\rangle$ is a cycle C_p. Suppose that a) does not hold. Then $p = 3$, $n \geq 8$. Thus there exist at least 5 vertices in $V - D$, and since G is cubic and $p = 3$, then $|V - D| \leq 3$, which gives a contradiction.

Suppose that b) does not hold. If D is not a perfect dominating set, then there exists a vertex $v \in V - D$ such that it has at least two neighbours in D, let us denote it u_1, u_2. Then $D - \{u_1\}$ (or $D - \{u_2\}$) is a connected dominating set, which gives a contradiction. If there exists a vertex $u \in D$ having two private neighbours, then $d(u) > 3$, which gives contradiction.

If a) and b) hold, then $n = |D| + |V - D| = |D| + |D| = 2\,|D| = 2p$.

If $p < 6$, then $p = 3$ and $G = G^*$ or $p = 4$ or $p = 5$. For $p = 3$ and $G = G^*$ we have $\gamma_c(G^*) = \gamma_{\mathrm{con}}(G^*) = 3$. If $p = 4$ or $p = 5$, then suppose that $\gamma_c(G) < \gamma_{\mathrm{con}}(G)$, that is D is not convex. Then there exists a vertex $v \in V - D$ such that v is a neighbour of two non-adjacent vertices from D, which contradicts b). □

Theorem 6. *If G is a cubic graph and there exists the minimum connected dominating set D in G such that* $\mathrm{diam}(\langle D\rangle) \leq 2$, *then* $\gamma_c(G) = \gamma_{\mathrm{con}}(G)$.

Proof. Let G be a cubic graph and let D be a minimum connected dominating set of G such that $\mathrm{diam}(\langle D\rangle) \leq 2$. If $\mathrm{diam}(\langle D\rangle) = 1$, then there exists a dominating vertex in G and a minimum connected and simultaneously convex dominating set consists of this vertex and we have required equality. Suppose that $\langle D\rangle$ has a diameter equal to 2. It is easy to observe, that $|V(\langle D\rangle)| \geq 3$. If $|V(\langle D\rangle)| = 3$, then $\langle D\rangle = C_3$ or $\langle D\rangle = K_{1,2}$. So from Theorem 5 and Theorem 4 we have $\gamma_c(G) = \gamma_{\mathrm{con}}(G)$. If $|V(\langle D\rangle)| = 4$, then:

a) $\langle D\rangle = K_{1,3}$. Then from Theorem 4 we have $\gamma_c(G) = \gamma_{\mathrm{con}}(G)$.

b) $\langle D\rangle = C_4$. Then from Theorem 5 we have $\gamma_c(G) = \gamma_{\mathrm{con}}(G)$.

c) $\langle D\rangle = K_1 + (K_1 \cup K_2)$ Then $|V - D| = 4$, in the other case we would find in G a connected dominating set D_1 such that $|D_1| < |D|$. Every vertex from $V - D$ is a neighbour of exactly one vertex from D and it is easy to observe that D is convex. So we have required equality.

d) $\langle D\rangle = C_4 \cup \{e\}$, where e is any chord of the cycle C_4. Then $|V - D| \leq 2$, which gives $n \leq 6$, contradiction, because in cubic graphs we have $\gamma_c \leq 3$ for $n \leq 6$.

If $|V(\langle D\rangle)| = 5$, then:

a) $\langle D\rangle = C_5$. Then, from Theorem 5 we have $\gamma_c(G) = \gamma_{\mathrm{con}}(G)$.

b) $\langle D\rangle = C_5 \cup \{e\}$, where e is any chord of the cycle C_5. Then $|V - D| \leq 3$ and we find in G a connected dominating set D_1 such that $|D_1| < |D|$, contradiction.

c) $\langle D\rangle = C_5 \cup \{e_1, e_2\}$, where e_1, e_2 are chords of a cycle C_5 having no common vertex. Then $|V - D| = 1$ and a vertex $v \in V - D$ is of degree 1, which is impossible in cubic graphs.

If $|V(\langle D \rangle)| = 6$, then every vertex in $\langle D \rangle$ is of degree 3. Thus $D - \{v\}$, where v is any vertex from D, is a connected dominating set, which gives a contradiction. If $\langle D \rangle$ has more than six vertices, then to have a diameter equal to two, a vertex of degree greater than three must exists in G, which is impossible in cubic graphs. $\square$

Theorem 7. *If G is cubic and $\gamma(G) = \gamma_c(G)$, then $\gamma_c(G) = \gamma_{\mathrm{con}}(G)$.*

Proof. In [2] Arumugam and Paulraj showed that if G is cubic and $\gamma(G) = \gamma_c(G)$, then $\gamma_c \leq 3$. Thus from Theorem 6 follows immediately that $\gamma_c(G) = \gamma_{\mathrm{con}}(G)$. $\square$

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