Opuscula Mathematica • Vol. 24/2 • 2004

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NEARLY PERFECT SETS IN PRODUCTS OF GRAPHS

Abstract. The study of nearly perfect sets in graphs was initiated in [2]. Let $S \subseteq V(G)$. We say that S is a nearly perfect set (or is nearly perfect) in G if every vertex in $V(G) - S$ is adjacent to at most one vertex in S. A nearly perfect set S in G is called maximal if for every vertex $u \in V(G) - S$, $S \cup \{u\}$ is not nearly perfect in G. The minimum cardinality of a maximal nearly perfect set is denoted by $n_p(G)$. It is our purpose in this paper to determine maximal nearly perfect sets in two well-known products of two graphs, i.e. in the Cartesian product and in the strong product. Lastly, we give upper bounds of $n_p(G_1 \times G_2)$ and $n_p(G_1 \otimes G_2)$, for some special graphs G_1, G_2.

Keywords: dominating sets, product of graphs.

Mathematics Subject Classification: 05C69, 05C70.

1. INTRODUCTION

Let G be a simple graph and $u \in V(G)$. By $N_G(u)$ we denote the open neighbourhood of u; i.e., $N(u) = \{v \colon uv \in E(G)\}$. We say that the subset $A \subseteq V(G)$ is called *stable* in G if a subgraph of G induced by A is without edges. The Cartesian product of two graphs G_1 and G_2 is a graph $G_1 \times G_2$ such that $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(x_1, y_1)(x_2, y_2) \in E(G_1 \times G_2)$ if $x_1x_2 \in E(G_1)$ and $y_1 = y_2$ or $y_1y_2 \in E(G_2)$ and $x_1 = x_2$. The strong product of two graphs G_1 and G_2 is a graph $G_1 \otimes G_2$ such that $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = E(G_1 \times G_2) \cup \{(x_1, y_1)(x_2, y_2) \colon x_1x_2 \in \in E(G_1)$ and $y_1y_2 \in E(G_2)\}$.

For concepts not defined here, see [1].

2. MAIN RESULTS

Let S_i is a nearly perfect set in G_i, for $i = 1, 2$. Consider an ordering of the vertices of G_i in such a way that the first vertices of the ordering are those of S_i. With this convention any vertex $(x, y) \in V(G_1) \times V(G_2) - S_1 \times S_2$ has at least two neighbours in $S_1 \times S_2$ if and only if x (as a vertex of G_1) has at least two neighbours in S_1 or y (as a vertex of G_2) has at least two neighbours in S_2. Moreover, no vertex $(x, y) \in (V(G_1) - S_1) \times (V(G_2) - S_2)$ has a neighbour in $S_1 \times S_2$. As a consequence, we have the following result.

Proposition 1. *If S_i is a nearly perfect set in G_i for $i = 1, 2$, then $S_1 \times S_2$ is a nearly perfect set in $G_1 \times G_2$.*

Theorem 1. $n_p(G_1 \times G_2) \leq \min\{|V(G_1)| \cdot n_p(G_2), |V(G_2)| \cdot n_p(G_1)\}$.

Proof. Without loss of generality we will show that $n_p(G_1 \times G_2) \leq |V(G_1)| \cdot n_p(G_2)$. Suppose that S_2 is a maximal nearly perfect set in G_2, such that $|S_2| = n_p(G_2)$. We know that $V(G_1)$ is a maximal nearly perfect set in G_1. Thus, by Proposition 1, it follows that $V(G_1) \times S_2$ is a nearly perfect set in $G_1 \times G_2$. Now, we will prove that $V(G_1) \times S_2$ is maximal. Let (x, y) be an arbitrary vertex outside $V(G_1) \times S_2$. From the maximality of S_2 (as a subset of G_2), it follows that $S_2 \cup \{y\}$ is not nearly perfect in G_2. Hence, we can find a vertex $v \in V(G_2) - (S_2 \cup \{y\})$ having at least two neighbours in $S_2 \cup \{y\}$. In $G_1 \times G_2$, (x, v) has at least two neighbours into $\{(x, y)\} \cup \{(x, s_2)\colon \quad s_2 \in S_2\}$ which is obviously a subset of $V(G_1) \times S_2 \cup \{(x, y)\}$. Consequently, $V(G_1) \times S_2 \cup \{(x, y)\}$ is not nearly perfect in $G_1 \times G_2$. Thus, $V(G_1) \times S_2$ is a maximal nearly perfect set in $G_1 \times G_2$ and $n_p(G_1 \times G_2) \leq |V(G_1) \times S_2| = |V(G_1)| \cdot n_p(G_2)$, as required. Hence, the theorem is proved. $\square$

Proposition 2. *If S_i is a nearly perfect stable set in G_i for $i = 1, 2$, then $S_1 \times S_2$ is a nearly perfect set in $G_1 \otimes G_2$.*

Proof. Let $(x, y) \in V(G_1 \otimes G_2) - S_1 \times S_2$. Our purpose is to show that

$$|N_{G_1 \otimes G_2}(x, y) \cap S_1 \times S_2| \leq 1. \tag{1}$$

Without loss of generality, we have two cases to consider.

1. Let $x \in S_1$ and $y \in V(G_2) - S_2$. Since S_2 is nearly perfect in G_2, then $|N_{G_2}(y) \cap S_2| \leq 1$. This implies that $|N_{G_1 \otimes G_2}(x, y) \cap (\{x\} \times S_2)| \leq 1$, by the definition $G_1 \otimes G_2$. Moreover, $|N_{G_1 \otimes G_2}(x, y) \cap S_1 \times S_2| \leq 1$ by the assumption that S_1 is stable in G_1.
2. Let $x \in V(G_1) - S_1$ and $y \in V(G_2) - S_2$. Suppose, by the contrary that $|N_{G_1 \otimes G_2}(x, y) \cap S_1 \times S_2| \geq 2$. More precisely, there exist $(u_1, v_1), (u_2, v_2) \in S_1 \times S_2$ such that $(x, y)(u_1, v_1) \in E(G_1 \otimes G_2)$ and $(x, y)(u_2, v_2) \in E(G_1 \otimes G_2)$. Observe that $u_1 \neq u_2$ or $v_1 \neq v_2$. Since (x, y) is adjacent to both (u_1, v_1) and (u_2, v_2), that means that either x is adjacent to u_1 and u_2 or y is adjacent to v_1 and v_2. One of these two situation, at least, leads to a contradiction with the maximality of S_1 and S_2.

All these two contradictions prove the inequality (1) which says that $S_1 \times S_2$ is a nearly perfect set in $G_1 \otimes G_2$, as required. $\square$

Theorem 2. *If S_i is a maximal nearly perfect stable set in G_i for $i = 1, 2$, then $S_1 \times S_2$ is a maximal nearly perfect set in $G_1 \otimes G_2$.*

Proof. By Proposition 2, $S_1 \times S_2$ is nearly perfect in $G_1 \otimes G_2$. It remains to show that $S_1 \times S_2$ is maximal. Let $(x, y) \in V(G_1 \otimes G_2) - S_1 \times S_2$. We prove that $S_1 \times S_2 \cup \{(x, y)\}$ is not nearly perfect in $G_1 \otimes G_2$. Without loss of generality, consider the two following cases.

1. Let $x \in S_1$ and $y \in V(G_2) - S_2$. Since $S_2 \cup \{y\}$ is not a nearly perfect stable set in G_2, there is a vertex $v \notin S_2 \cup \{y\}$ which is adjacent to two vertices of this set, say $s_2 \in S_2$ and $p_2 = y$ (it is possible since S_2 is maximal). Hence (x, v) is adjacent to (x, y) and to (x, s_2) which implies that $S_1 \times S_2 \cup \{(x, y)\}$ can not be a nearly perfect set in $G_1 \otimes G_2$.
2. Let $x \in V(G_1) - S_1$ and $y \in V(G_2) - S_2$. Since $S_1 \cup \{x\}$ and $S_2 \cup \{y\}$ are not nearly perfect sets in G_1 and G_2, respectively, there is a vertex $u \in V(G_1) - (S_1 \cup \{x\})$ adjacent to two vertices in $S_1 \cup \{x\}$ (say $s_1 \in S_1$ and x) and a vertex v as well, adjacent to $s_2 \in S_2$ and to y. From the definition of $G_1 \otimes G_2$, (u, v) is adjacent to (x, y) and to (s_1, s_2). Hence $S_1 \times S_2 \cup \{(x, y)\}$ is not a nearly perfect set in $G_1 \otimes G_2$.

Finally, $S_1 \times S_2$ is a maximal nearly perfect set in $G_1 \otimes G_2$, as required. □

By Theorem 2, it follows that if S_i is a maximal nearly perfect stable set in G_i for $i = 1, 2$, then $n_p(G_1 \otimes G_2) \leq |S_1 \times S_2|$.

Corollary 1. *If $n_p(G_i) = 1$ for $i = 1, 2$, then $n_p(G_1 \otimes G_2) = 1$.*

It has been proved in [2] that the maximal nearly perfect sets in graphs P_{3k+1}, C_{3k}, for $k \geq 1$, with cardinalities $n_p(P_{3k+1})$ and $n_p(C_{3k})$ are stable.

Corollary 2. *If $G, H \in \{P_{3k+1}, C_{3k}, G_0\}$, where $n_p(G_0) = 1$ for $k \geq 1$, then $n_p(G \otimes H) \leq n_p(G) \cdot n_p(H)$.*

In particular, $G_0 = K_n$, or $G_0 = W_n$ (W_n is a wheel, i.e. $W_n = K_1 + C_n$) for $n \geq 3$. A full characterization of graphs G_0, with $n_p(G_0) = 1$ is given in [2].

Acknowledgements

The authors thank the referee for many helpful suggestions.

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Received: November 16, 2003.