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A NOTE ON A LIST COLOURING OF HYPERGRAPHS

Abstract. In the note we present two results. The first of them gives a sufficient condition for a colouring of a hypergraph from an assigned list. It generalises the analogous fact for graphs. The second result states that for every $k \geq 3$ and every $l \geq 2$, a distance between the list chromatic number and the chromatic number can be arbitrarily large in the class of k -uniform hypergraphs with the chromatic number bounded below by l . A similar result for k -uniform, 2-colorable hypergraphs is known but the proof techniques are different.

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1. INTRODUCTION

The subhypergraph $\mathcal{H}[X]$ of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ induced by $X \subseteq V$ is defined as follows: $V(\mathcal{H}[X]) = X$ and $\mathcal{E}(\mathcal{H}[X]) = \{e \in \mathcal{E}(\mathcal{H}) : e \subseteq X\}$.

The *vertex degree* $d_{\mathcal{H}}(v)$ of a vertex v in \mathcal{H} is the number of edges in \mathcal{H} containing v . The *edge degree* $d_{\mathcal{H}}(f)$ of the edge f in \mathcal{H} is the number of edges other than f , each having a non-empty intersection with f .

By $\Delta(\mathcal{H})$ we denote the vertex maximum degree in the hypergraph \mathcal{H} .

The concept of the colouring of graphs from the list was introduced independently by Vizing [6] and Erdős, Rubin and Taylor [3]. This idea was extended by Kostochka, Stiebitz and Wirth [4] to hypergraphs.

Now, let us consider a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and the family of sets $L(v)$, $v \in V$, of positive integers assigned to the set V . Such an L we call a *list* for \mathcal{H} . An *L -colouring* of \mathcal{H} is a mapping g of $V(\mathcal{H})$ into the set of colours $\bigcup_{v \in V(\mathcal{H})} L(v)$ so that $g(v) \in L(v)$ for all $v \in V(\mathcal{H})$. Adding the condition $|\{g(v) : v \in f\}| \geq 2$ for each $f \in \mathcal{E}$, we obtain the definition of a *proper L -colouring* of \mathcal{H} . If \mathcal{H} admits a proper L -colouring, then \mathcal{H} is said to be *L -colourable*. The hypergraph \mathcal{H} is *k -choosable* if it is L -colourable for every list L for \mathcal{H} satisfying $|L(v)| = k$ for every

$v \in V(\mathcal{H})$. A *choice number* $ch(\mathcal{H})$ is the smallest positive integer k such that \mathcal{H} is k -choosable. A *chromatic number* $\chi(\mathcal{H})$ is the smallest positive integer k such that \mathcal{H} is L -colourable for the list $L(v) = \{1, \dots, k\}$, $v \in V$.

Note that for any hypergraph \mathcal{H} we can construct the subhypergraph \mathcal{H}_S , called a *Sperner's reduction of \mathcal{H}* , obtained by deleting all edges, containing the other edges as subsets. It is very simple to check that for any hypergraph \mathcal{H} and any list L , g is a proper L -colouring of \mathcal{H} if and only if g is a proper L -colouring of \mathcal{H}_S .

2. RESULTS

For $i \in N$ and a list L for \mathcal{H} , the set of all vertices v of a hypergraph \mathcal{H} so that $i \in L(v)$ is denoted by $P_i(\mathcal{H}, L)$.

In 1997 Reed [5] showed that if $|L(v)| \geq 2 \cdot e \cdot \Delta(G[P_i(G, L)])$ for any vertex v of G and $i \in N$, then there exists a proper L -colouring of G , where e denotes the Euler constant. It is our purpose to study a similar problem for hypergraphs. For an arbitrary hypergraph $\mathcal{H} = (V, \mathcal{E})$ and a list L for \mathcal{H} we introduce the following notations:

$$\begin{aligned} \mathcal{E}_i(\mathcal{H}, L) &= \{f \in \mathcal{E}(\mathcal{H}_S) : \forall v \in f i \in L(v)\} = \{f \in \mathcal{E}(\mathcal{H}_S) : f \subseteq P_i(\mathcal{H}, L)\}, \\ \mathcal{E}_i^h(\mathcal{H}, L) &= \{h\} \cup \mathcal{E}_i(\mathcal{H}, L), i \in N, h \in \mathcal{E}(\mathcal{H}_S). \end{aligned}$$

The edge degree of the edge f in the subhypergraph $(V, \mathcal{E}_i^f(\mathcal{H}, L))$ of \mathcal{H} , will be denoted by $\Delta_i^f(\mathcal{H}, L)$.

$$\begin{aligned} \Delta_f(\mathcal{H}, L) &= \sum_{i \in \bigcup_{v \in f} L(v)} \Delta_i^f(\mathcal{H}, L), \\ \Delta_L(\mathcal{H}) &= \max_{f \in \mathcal{E}(\mathcal{H}_S)} \Delta_f(\mathcal{H}, L) = \Delta_L(\mathcal{H}_S). \end{aligned}$$

For example if $\mathcal{H} = (V, \mathcal{E})$, $V = \{1, \dots, 6\}$, $\mathcal{E} = \{f_1, \dots, f_5\}$, $f_1 = \{1, 2, 3\}$, $f_2 = \{2, 3, 4, 5\}$, $f_3 = \{4, 5, 6\}$, $f_4 = \{1, 3\}$, $f_5 = \{3, 4, 5\}$ and $L(1) = L(5) = \{2, 6, 7\}$, $L(2) = L(3) = L(4) = \{1, 4, 6, 8\}$, $L(6) = \{1, 2, 7\}$ then $\mathcal{E}_i(\mathcal{H}, L) = \emptyset$ for $i \in N - \{6\}$, $\mathcal{E}_6(\mathcal{H}, L) = \{f_4, f_5\}$, $\Delta_i^f(\mathcal{H}, L) = 0$ for $i \in N - \{6\}$ and $f \in \mathcal{E}(\mathcal{H}_f)$, $\Delta_6^{f_4}(\mathcal{H}, L) = \Delta_6^{f_5}(\mathcal{H}, L) = \Delta_6^{f_3}(\mathcal{H}, L) = 1$, $\Delta_{f_3}(\mathcal{H}, L) = \Delta_{f_4}(\mathcal{H}, L) = \Delta_{f_5}(\mathcal{H}, L) = 1$, $\Delta_L(\mathcal{H}) = 1$.

To prove of the first theorem, we must use the well-known *Local Lovász Lemma (LLL)*.

Let A_1, \dots, A_n be events in an arbitrary probability space. A graph $G = (V, E)$ on the set of vertices $V = \{1, \dots, n\}$ is a dependency graph for the events A_1, \dots, A_n if for each i , $1 \leq i \leq n$, the event A_i is mutually independent of a set of all the other events A_j , so that $\{i, j\}$ are not elements of E . By \bar{A}_i we denote the opposite event to A_i .

Lemma 1. (LLL, [2]) *Let G be the dependency graph for the events A_1, \dots, A_n in an arbitrary finite probability space (Ω, P) . If $\Delta(G) \leq d$, $P(A_i) \leq p$ for each i , $1 \leq i \leq n$, and the following inequality holds $p(d+1)e \leq 1$, then $P(\bigcap_{i=1}^n \bar{A}_i) > 0$.*

Theorem 1. *If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph such that for every $f \in \mathcal{E}$, $|f| \geq k \geq 2$ and if L is a list for \mathcal{H} satisfying $\min_{v \in V} |L(v)| \geq \sqrt[k]{e \cdot (\Delta_L(\mathcal{H}) + 1)}$, then there exists a proper L -colouring of \mathcal{H} .*

Proof. We truncate each $L(v)$, so that it has exactly $\min_{v \in V} |L(v)|$ colours and construct the probability space (Ω, P) with Ω being the set of all, not necessarily proper, L -colourings of \mathcal{H} . The event that every vertex v of an edge f is coloured with i will be denoted by $A_{f,i}$ unless $i \in L(v)$. Let \mathcal{M} denote the family of such events. It is easy to observe that $A_{g,i}$ is mutually independent of all other events of the family \mathcal{M} but at most the events of the family $A_{g,i}^* = \{A_{f,j} : \exists x \in f \cap g, j \in L(x)\}$. Application of Lemma (LLL) to the dependency graph of the family \mathcal{M} with suitable constants $p = \left(\frac{1}{\min_{v \in V} |L(v)|}\right)^k$ and $d = \Delta_L(\mathcal{H})$ gives the proposition.

Erdős, Rubin and Taylor [3] found for every assumed positive integer b a fine construction of a graph whose chromatic number is equal to two and its choice number is greater than b . Moreover, there is a stronger result essentially implied by Vizing's [6] and Erdős, Rubin and Taylor's [3] proofs: *For every k and s there exists a k -uniform 2-colorable hypergraph with list chromatic number at least s .* We use the below tool for generalisation of that result giving a really different proof. \square

Pigeonhole Principle. *Suppose that q_1, \dots, q_n are positive integers. If $X = X_1 \cup \dots \cup X_t$ is a partition of the set X and $|X| \geq \sum_{i=1}^t q_i - t + 1$ then $|X_i| \geq q_i$ for some $i \in \{1, \dots, t\}$.*

Theorem 2. *For any non-negative integers $s \geq 0, k \geq 3, l \geq 2$ there exists a k -uniform hypergraph $\mathcal{H}_{k,s,l}$ so that $ch(\mathcal{H}_{k,s,l}) - \chi(\mathcal{H}_{k,s,l}) > s$ and $\chi(\mathcal{H}_{k,s,l}) \geq l$.*

Proof. Let a function f be defined by

$$f(x) = \left\lceil \frac{x(x-1) - 2 + (k-2)(k+1)}{2(k-2)(k+1)} \right\rceil - 1 - x.$$

The function $f(x)$ tends to infinity with $x \rightarrow \infty$. Hence, for given s, k and l there exists an integer $x_0 \geq l(k-1)$ satisfying $f(x_0) \geq s$. For this integer x_0 , let

$$b = x_0 + f(x_0) = \left\lceil \frac{x_0(x_0-1) - 2 + (k-2)(k+1)}{2(k-2)(k+1)} \right\rceil - 1.$$

Note that $b \geq s + x_0 \geq 4$.

Let the set V of vertices of the hypergraph $\mathcal{H}_{k,s,l}$ be given by the sum of disjoint sets V_1, \dots, V_{x_0} , all of the cardinality $\binom{2b-1}{b}$. We define the set of edges of $\mathcal{H}_{k,s,l}$ as the family of all k -element subsets of $\bigcup_{i \in \{1, \dots, x_0\}} V_i$, having at most one element in intersection with V_i , for every $i \in \{1, \dots, x_0\}$. It is easy to observe that $\chi(\mathcal{H}_{k,s,l}) = \lceil \frac{x_0}{k-1} \rceil$. Note that it is sufficient to prove that $\mathcal{H}_{k,s,l}$ is not b -choosable, i.e., $ch(\mathcal{H}_{k,s,l}) > b$. The last fact and the definition of the number b will give $ch(\mathcal{H}_{k,s,l}) - \chi(\mathcal{H}_{k,s,l}) > b - x_0 \geq s$.

To prove this, let $C = \{1, \dots, 2b - 1\}$ be a set of colours and let the family of b -element subsets of C be described as follows: $P_b(C) = \{A_j : j = 1, \dots, \binom{2b-1}{b}\}$. Moreover, let $V_i = \{v_{i,j} : j = 1, \dots, \binom{2b-1}{b}\}$, $i = 1, \dots, x_0$. We define a list L for the hypergraph $\mathcal{H}_{k,s,l}$ by $L(v_{i,j}) = A_j$ for all i and j . Let g be a proper L -colouring of the hypergraph $\mathcal{H}_{k,s,l}$. Note that:

- g uses at least b different colours for the vertices of set V_i , $i = 1, \dots, x_0$;
- for $i_1, i_2 \in \{1, \dots, x_0\}$, $i_1 \neq i_2$, there exist $j_1, j_2 \in \{1, \dots, 2b - 1\}$ and colour $c \in C$ such that $g(v_{i_1, j_1}) = g(v_{i_2, j_2}) = c$. It follows by previous statement and the fact that $|C| = 2b - 1$. Select one of such colours c and call it $\{i_1, i_2\}$ -colour. Let $X = \{\{i_1, i_2\} : i_1, i_2 \in \{1, \dots, x_0\}\}$ and $X_c = \{\{i_1, i_2\} : c \text{ is } \{i_1, i_2\}\text{-colour, } c \in C\}$.

Note that $X = X_1 \cup \dots \cup X_{2b-1}$ and for the sequence of integers $q_1 = \dots = q_{2b-1} = \binom{k}{2}$ we have

$$|X| = \binom{x_0}{2} \geq \sum_{i=1}^{2b-1} q_i - (2b - 1) + 1.$$

Hence, by Pigeonhole Principle it follows that there exists $c_0 \in C$, such that $|X_{c_0}| \geq q_{c_0} = \binom{k}{2}$. It implies that there are at least k pairwise different integers $i_1, \dots, i_k \in \{1, \dots, x_0\}$ and for any i_t there exists an integer j , $1 \leq j \leq 2b - 1$, satisfying $g(v_{i_t, j}) = c_0$. By the definition of $\mathcal{H}_{k,s,l}$ we know that there exists an edge of $\mathcal{H}_{k,s,l}$ included in the set of vertices coloured c_0 . Hence, $ch(\mathcal{H}_{k,s,l}) > b$, which completes the proof.

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