Opuscula Mathematica • Vol. 24/2 • 2004

Ewa Drgas-Burchardt

A NOTE ON A LIST COLOURING OF HYPERGRAPHS

Abstract. In the note we present two results. The first of them gives a sufficient condition for a colouring of a hypergraph from an assigned list. It generalises the analogous fact for graphs. The second result states that for every $k \geq 3$ and every $l \geq 2$, a distance between the list chromatic number and the chromatic number can be arbitrarily large in the class of $k-$uniform hypergraphs with the chromatic number bounded below by l. A similar result for k-uniform, 2-colorable hypergraphs is known but the proof techniques are different.

Keywords: hypergraph, list colouring.

Mathematics Subject Classification: 05C65, 05C15.

1. INTRODUCTION

The subhypergraph $\mathcal{H}[X]$ of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ induced by $X \subseteq V$ is defined as follows: $V(\mathcal{H}[X]) = X$ and $\mathcal{E}(\mathcal{H}[X]) = \{e \in \mathcal{E}(\mathcal{H})\colon\ e \subseteq X\}$.

The *vertex degree* $d_{\mathcal{H}}(v)$ *of a vertex* v *in* $\mathcal{H}$ is the number of edges in $\mathcal{H}$ containing v. The *edge degree* $d_{\mathcal{H}}(f)$ *of the edge* f *in* $\mathcal{H}$ is the number of edges other than f, each having a non-empty intersection with f.

By $\Delta(\mathcal{H})$ we denote the vertex maximum degree in the hypergraph $\mathcal{H}$.

The concept of the colouring of graphs from the list was introduced independently by Vizing [6] and Erdös, Rubin and Taylor [3]. This idea was extended by Kostochka, Stiebitz and Wirth [4] to hypergraphs.

Now, let us consider a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and the family of sets $L(v)$, $v \in V$, of positive integers assigned to the set V. Such an L we call a *list* for $\mathcal{H}$. An *L-colouring of* $\mathcal{H}$ is a mapping g of $V(\mathcal{H})$ into the set of colours $\bigcup_{v \in V(\mathcal{H})} L(v)$ so that $g(v) \in L(v)$ for all $v \in V(\mathcal{H})$. Adding the condition $|\{g(v)\colon v \in f\}| \geq 2$ for each $f \in \mathcal{E}$, we obtain the definition of a *proper L-colouring of* $\mathcal{H}$. If $\mathcal{H}$ admits a proper L-colouring, then $\mathcal{H}$ is said to be *L-colourable*. The hypergraph $\mathcal{H}$ is *k-choosable* if it is L-colourable for every list L for $\mathcal{H}$ satisfying $|L(v)| = k$ for every

$v \in V(\mathcal{H})$. A *choice number* $ch(\mathcal{H})$ is the smallest positive integer k such that $\mathcal{H}$ is k-choosable. A *chromatic number* $\chi(\mathcal{H})$ is the smallest positive integer k such that $\mathcal{H}$ is L-colourable for the list $L(v) = \{1, \ldots, k\}$, $v \in V$.

Note that for any hypergraph $\mathcal{H}$ we can construct the subhypergraph $\mathcal{H}_S$, called *a Sperner's reduction of* $\mathcal{H}$, obtained by deleting all edges, containing the other edges as subsets. It is very simple to check that for any hypergraph $\mathcal{H}$ and any list L, g is a proper L-colouring of $\mathcal{H}$ if and only if g is a proper L-colouring of $\mathcal{H}_S$.

2. RESULTS

For $i \in N$ and a list L for $\mathcal{H}$, the set of all vertices v of a hypergraph $\mathcal{H}$ so that $i \in L(v)$ is denoted by $P_i(\mathcal{H}, L)$.

In 1997 Reed [5] showed that if $|L(v)| \geq 2 \cdot e \cdot \Delta(G[P_i(G, L)])$ for any vertex v of G and $i \in N$, then there exists a proper L-colouring of G, where e denotes the Euler constant. It is our purpose to study a similar problem for hypergraphs. For an arbitrary hypergraph $\mathcal{H} = (V, \mathcal{E})$ and a list L for $\mathcal{H}$ we introduce the following notations:

$$\mathcal{E}_i(\mathcal{H}, L) = \{f \in \mathcal{E}(\mathcal{H}_S) \colon \forall_{v \in f} i \in L(v)\} = \{f \in \mathcal{E}(\mathcal{H}_S) \colon f \subseteq P_i(\mathcal{H}, L)\},$$

$$\mathcal{E}_i^h(\mathcal{H}, L) = \{h\} \cup \mathcal{E}_i(\mathcal{H}, L), i \in N, h \in \mathcal{E}(\mathcal{H}_S).$$

The edge degree of the edge f in the subhypergraph $(V, \mathcal{E}_i^f(\mathcal{H}, L))$ of $\mathcal{H}$, will be denoted by $\Delta_i^f(\mathcal{H}, L)$.

$$\Delta_f(\mathcal{H}, L) = \textstyle\sum_{i \in \bigcup_{v \in f} L(v)} \Delta_i^f(\mathcal{H}, L),$$

$$\Delta_L(\mathcal{H}) = \max_{f \in \mathcal{E}(\mathcal{H}_S)} \Delta_f(\mathcal{H}, L) = \Delta_L(\mathcal{H}_S).$$

For example if $\mathcal{H} = (V, \mathcal{E})$, $V = \{1, \ldots 6\}$, $\mathcal{E} = \{f_1, \ldots, f_5\}$, $f_1 = \{1, 2, 3\}$, $f_2 = \{2, 3, 4, 5\}$, $f_3 = \{4, 5, 6\}$, $f_4 = \{1, 3\}$, $f_5 = \{3, 4, 5\}$ and $L(1) = L(5) = = \{2, 6, 7\}$, $L(2) = L(3) = L(4) = \{1, 4, 6, 8\}$, $L(6) = \{1, 2, 7\}$ then $\mathcal{E}_i(\mathcal{H}, L) = \emptyset$ for $i \in N - \{6\}, \mathcal{E}_6(\mathcal{H}, L) = \{f_4, f_5\}$, $\Delta_i^f(\mathcal{H}, L) = 0$ for $i \in N - \{6\}$ and $f \in \mathcal{E}(\mathcal{H}_f)$, $\Delta_6^{f_4}(\mathcal{H}, L) = \Delta_6^{f_5}(\mathcal{H}, L) = \Delta_6^{f_3}(\mathcal{H}, L) = 1$, $\Delta_{f_3}(\mathcal{H}, L) = \Delta_{f_4}(\mathcal{H}, L) = \Delta_{f_5}(\mathcal{H}, L) = 1$, $\Delta_L(\mathcal{H}) = 1$.

To prove of the first theorem, we must use the well-known *Local Lovász Lemma (LLL)*.

Let $A_1, \ldots, A_n$ be events in an arbitrary probability space. A graph $G = (V, E)$ on the set of vertices $V = \{1, \ldots, n\}$ is a dependency graph for the events $A_1, \ldots, A_n$ if for each i, $1 \leq i \leq n$, the event A_i is mutually independent of a set of all the other events A_j, so that $\{i, j\}$ are not elements of E. By $\bar{A}_i$ we denote the opposite event to A_i.

Lemma 1. (LLL, [2]) *Let G be the dependency graph for the events $A_1, \ldots, A_n$ in an arbitrary finite probability space (Ω, P). If $\Delta(G) \leq d$, $P(A_i) \leq p$ for each i, $1 \leq i \leq n$, and the following inequality holds $p(d+1)e \leq 1$, then $P(\bigcap_{i=1}^n \bar{A}_i) > 0$.*

Theorem 1. *If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph such that for every $f \in \mathcal{E}$, $|f| \geq k \geq 2$ and if L is a list for $\mathcal{H}$ satisfying $\min_{v \in V} |L(v)| \geq \sqrt[k]{e \cdot (\Delta_L(\mathcal{H}) + 1)}$, then there exists a proper L-colouring of $\mathcal{H}$.*

Proof. We truncate each $L(v)$, so that it has exactly $\min_{v \in V} |L(v)|$ colours and construct the probability space (Ω, P) with Ω being the set of all, not necessarily proper, L-colourings of $\mathcal{H}$. The event that every vertex v of an edge f is coloured with i will be denoted by $A_{f,i}$ unless $i \in L(v)$. Let $\mathcal{M}$ denote the family of such events. It is easy to observe that $A_{g,i}$ is mutually independent of all other events of the family $\mathcal{M}$ but at most the events of the family $A^*_{g,i} = \{A_{f,j} \colon \exists_{x \in f \cap g} j \in L(x)\}$. Application of Lemma (LLL) to the dependency graph of the family $\mathcal{M}$ with suitable constants $p = \left(\frac{1}{\min_{v \in V} |L(v)|}\right)^k$ and $d = \Delta_L(\mathcal{H})$ gives the proposition.

Erdös, Rubin and Taylor [3] found for every assumed positive integer b a fine construction of a graph whose chromatic number is equal to two and its choice number is greater then b. Moreover, there is a stronger result essentially implied by Vizing's [6] and Erdös, Rubin and Taylor's [3] proofs: *For every k and s there exists a k-uniform 2-colorable hypergraph with list chromatic number at least s.* We use the below tool for generalisation of that result giving a really different proof. □

Pigeonhole Principle. *Suppose that $q_1, \ldots, q_n$ are positive integers. If $X = X_1 \cup \ldots \cup X_t$ is a partition of the set X and $|X| \geq \sum_{i=1}^{t} q_i - t + 1$ then $|X_i| \geq q_i$ for some $i \in \{1, \ldots, t\}$.*

Theorem 2. *For any non-negative integers $s \geq 0, k \geq 3$, $l \geq 2$ there exists a k-uniform hypergraph $\mathcal{H}_{k,s,l}$ so that $ch(\mathcal{H}_{k,s,l}) - \chi(\mathcal{H}_{k,s,l}) > s$ and $\chi(\mathcal{H}_{k,s,l}) \geq l$.*

Proof. Let a function f be defined by

$$f(x) = \left\lceil \frac{x(x-1) - 2 + (k-2)(k+1)}{2(k-2)(k+1)} \right\rceil - 1 - x.$$

The function $f(x)$ tends to infinity with $x \to \infty$. Hence, for given s, k and l there exists an integer $x_0 \geq l(k-1)$ satisfying $f(x_0) \geq s$. For this integer x_0, let

$$b = x_0 + f(x_0) = \left\lceil \frac{x_0(x_0-1) - 2 + (k-2)(k+1)}{2(k-2)(k+1)} \right\rceil - 1.$$

Note that $b \geq s + x_0 \geq 4$.

Let the set V of vertices of the hypergraph $\mathcal{H}_{k,s,l}$ be given by the sum of disjoint sets $V_1, \ldots, V_{x_0}$, all of the cardinality $\binom{2b-1}{b}$. We define the set of edges of $\mathcal{H}_{k,s,l}$ as the family of all k-element subsets of $\bigcup_{i \in \{1,\ldots,x_0\}} V_i$, having at most one element in intersection with V_i, for every $i \in \{1, \ldots, x_0\}$. It is easy to observe that $\chi(\mathcal{H}_{k,s,l}) = \lceil \frac{x_0}{k-1} \rceil$. Note that it is sufficient to prove that $\mathcal{H}_{k,s,l}$ is not b-choosable, i.e., $ch(\mathcal{H}_{k,s,l}) > b$. The last fact and the definition of the number b will give $ch(\mathcal{H}_{k,s,l}) - \chi(\mathcal{H}_{k,s,l}) > b - x_0 \geq s$.

To prove this, let $C = \{1, \ldots, 2b-1\}$ be a set of colours and let the family of b-element subsets of C be described as follows: $P_b(C) = \left\{A_j : j = 1, \ldots, \binom{2b-1}{b}\right\}$. Moreover, let $V_i = \{v_{i,j} : j = 1, \ldots, \binom{2b-1}{b}\}, i = 1, \ldots, x_0$. We define a list L for the hypergraph $\mathcal{H}_{k,s,l}$ by $L(v_{i,j}) = A_j$ for all i and j. Let g be a proper L-colouring of the hypergraph $\mathcal{H}_{k,s,l}$. Note that:

— g uses at least b different colours for the vertices of set V_i, $i = 1, \ldots, x_0$;

— for $i_1, i_2 \in \{1, \ldots, x_0\}$, $i_1 \neq i_2$, there exist $j_1, j_2 \in \{1, \ldots, 2b-1\}$ and colour $c \in C$ such that $g(v_{i_1,j_1}) = g(v_{i_2,j_2}) = c$. It follows by previous statement and the fact that $|C| = 2b-1$. Select one of such colours c and call it $\{i_1, i_2\}$-*colour*. Let $X = \{\{i_1, i_2\} : i_1, i_2 \in \{1, \ldots, x_0\}\}$ and $X_c = \{\{i_1, i_2\} :$ c is $\{i_1, i_2\}\}$-*colour*, $c \in C$.

Note that $X = X_1 \cup \ldots \cup X_{2b-1}$ and for the sequence of integers $q_1 = \ldots = q_{2b-1} = \binom{k}{2}$ we have

$$|X| = \binom{x_0}{2} \geq \sum_{i=1}^{2b-1} q_i - (2b-1) + 1.$$

Hence, by Pigeonhole Principle it follows that there exists $c_0 \in C$, such that $|X_{c_0}| \geq q_{c_0} = \binom{k}{2}$. It implies that there are at least k pairwise different integers $i_1, \ldots, i_k \in \{1, \ldots, x_0\}$ and for any i_t there exists an integer j, $1 \leq j \leq 2b-1$, satisfying $g(v_{i_t j}) = c_0$. By the definition of $\mathcal{H}_{k,s,l}$ we know that there exists an edge of $\mathcal{H}_{k,s,l}$ included in the set of vertices coloured c_0. Hence, $ch(\mathcal{H}_{k,s,l}) > b$, which completes the proof.

REFERENCES

[1] Borowiecki M., Drgas-Burchardt E., Mihók P.: *Generalized list colourings of graphs.* Discussiones Mathematicae Graph Theory **15** (2) (1995), 185–193.

[2] Erdös P., Lovász L.: *Problems and results on 3-chromatic hypergraphs and some related questions.* Colloquia Mathematica Societatis Jänos Bolai 10, Infinite and Finite Sets, vol. II (1973) 609–627.

[3] Erdös P., Rubin A. L., Taylor H.: *Choosability in graphs*, in: *Proceedings West Coast Conference on Combinatorics.* Graph Theory and Computing, Arcata CA, Sept. 5–7, 1979, Congr. Numer. 26.

[4] Kostochka A. V., Stiebitz M., Wirth B.: *The colour theorems of Brooks and Gallai extended.* Discrete Mathematics **162** (1996), 299–303.

[5] Reed B.: *The list colouring constants.* J. Graph Theory **31** (1999), 149–153.

[6] Vizing V. G., *Colouring the vertices of graph in prescribed colours.* Diskret. Analiz. **29**, Metody Diskret. Anal. v Teorii Kodov i Shem **101** (1976), 3–10 (Russian).

Ewa Drgas-Burchardt
E.Drgas-Burchardt@wmie.uz.zgora.pl

University of Zielona Góra
Faculty of Mathematics Computer Science and Econometrics
ul. Podgórna 50, 65-246 Zielona Góra, Poland

Received: October 29, 2003.