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**A LOCAL EXISTENCE THEOREM OF THE SOLUTION
OF THE CAUCHY PROBLEM FOR BBGKY CHAIN
OF EQUATIONS REPRESENTED
IN CUMULANT EXPANSIONS IN THE SPACE E_ξ**

Abstract. It is proved convergence of solution in cumulant expansions of the initial value problem for BBGKY chain of equations of non-symmetrical one-dimensional system of particles which interact via a short-range potential in the space E_ξ of the sequences of continuous bounded functions.

Keywords: Non-symmetrical particle systems, space of the sequences of continuous bounded functions, BBGKY chain of equations, cumulant.

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1. INTRODUCTION

State of non-symmetrical one-dimensional infinite particle system can be described by the solution of the Cauchy problem for the BBGKY chain of equations represented in the form of cumulant expansions in the space E_ξ of the sequences of continuous bounded functions with the norm

$$\|f\| = \sup_{n \geq 0} \xi^{-n} \sup_{x_{-n_2}, \dots, x_{n_1}} |f_n(x_{-n_2}, \dots, x_{n_1})| \exp \left(\beta \sum_{i=-n_2}^{n_1} \frac{p_i^2}{2} \right), \quad \xi, \beta > 0.$$

The space E_ξ is different from the one L_α^1 in which the solution was constructed [1]. The cumulant structure of the solution solves the problem of divergence of each term of the series (the solution is represented through the functional series) for initial data belonging to the space E_ξ .

In this paper we prove convergence of the considered solution for $F(0) \in E_\xi$ on the finite time interval using method of interaction region. This method was formulated in [2], and was used to prove existence theorems for three-dimensional systems in [3, 4] and for non-symmetrical systems in [5].

2. FORMULATION OF THE PROBLEM

Consider a one-dimensional non-symmetrical system of particles of unit mass interacting with their nearest neighbors via the potential which satisfies the following conditions

$$\begin{cases} \Phi(q) \in C^2(0, R], \\ \Phi(q) = 0, \text{ if } q \in (R, +\infty), \end{cases} \quad (1)$$

from which it follows that the conditions of stability and regularity are fulfilled, and the following relations hold

$$\begin{aligned} \left| \sum_{i=-n_2}^{n_1-1} \Phi(q_i - q_{i+1}) \right| &\leq B(n_1 + n_2), B > 0, \\ \sup_i |F_i| &\equiv D < +\infty, \end{aligned} \quad (2)$$

where F_i is a force acting upon the i -th particle of its nearest neighbors side.

For the sake of simplicity we denote $n = n_1 + n_2, s = s_1 + s_2$.

The solution of the Cauchy problem for BBGKY chain of equations of non-symmetrical particle system in the space L_α^1 is given by the formula [1]

$$F_{|Y|}(t, Y) = \sum_{n=0}^{\infty} \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \geq 0}} \int_{(R^1 \times R^1)^{n_1+n_2}} d(X \setminus Y) U_{(n_2, n_1)}(t, X_Y) F_{|X|}(0, X), \quad (3)$$

where $Y = (x_{-s_2}, \dots, x_{s_1}), X = (x_{-(n_2+s_2)}, \dots, x_{s_1+n_1}), X_Y = (x_{-(n_2+s_2)}, \dots, x_{-(s_2+1)}, Y, x_{s_1+1}, \dots, x_{s_1+n_1})$. Cumulant $U_{(n_2, n_1)}(t, X_Y)$ has the form

$$U_{(n_2, n_1)}(t, X_Y) = \sum_{P: X_Y = \bigcup_i X_i} (-1)^{|P|-1} \prod_{X_i \subset P} S_{|X_i|}(-t, X_i),$$

where \sum_P — the sum over all ordered divisions of partly ordered set X_Y onto $|P|$ nonempty ordered subsets $X_i \subset X_Y$, which do not intersect each other, and set Y belongs entirely to one of subsets X_i .

Our aim is to give a mathematical meaning to the series (3) for initial data $F(0)$ from the space E_ξ . It is easy to see that each integrand in the formula (3) with an arbitrary fixed n differs from zero only in a bounded region of the configuration space, which will be called the interaction region and denoted by $\Omega_{n_1+n_2}(t)$. Thus, each term in (3) with a fixed n is meaningful and the corresponding integrals will be convergent. To prove the existence of $F_{|Y|}(t, Y)$ it is only required to show that the series (3) converges.

3. THE ESTIMATE OF THE SOLUTION REPRESENTED
IN CUMULANT EXPANSIONS IN NON-EXPLICIT FORM

Taking into account the invariance of the Gibbs distributions with respect to the acting of evolution operator $S(-t)$ and using the conditions (2) we have

$$S(-t, x_{k+1}, \dots, x_{k+n}) \exp\left(-\beta \sum_{i=k+1}^{k+n} \frac{p_i^2}{2}\right) \leq \exp(2\beta Bn) \exp\left(-\beta \sum_{i=k+1}^{k+n} \frac{p_i^2}{2}\right).$$

According to these inequalities and if initial data $F(0) \in E_\xi$ the following estimate holds

$$|F|_Y(t, Y) \leq \|F(0)\| \sum_{n=0}^{\infty} \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \geq 0}} \int_{R^n \times \Omega_n(t)} d(X \setminus Y) 2^n \times \exp(2\beta B(n+s)) \xi^{n+s} \exp\left(-\beta \sum_{i=-(n_2+s_2)}^{s_1+n_1} \frac{p_i^2}{2}\right). \tag{4}$$

Since the volume of the interaction region $V_n(t) \equiv V_n(t, p_{-(n_2+s_2)}, \dots, p_{-(s_2+1)}, p_{s_1+1}, \dots, p_{s_1+n_1})$ (4) is transformed in the following way

$$|F|_Y(t, Y) \leq \|F(0)\| \sum_{n=0}^{\infty} \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \geq 0}} 2^n \exp(2\beta B(n+s)) \xi^{n+s} \int_{R^n} dp_{-(n_2+s_2)} \dots \times dp_{-(s_2+1)} dp_{s_1+1} \dots dp_{s_1+n_1} V_n(t) \exp\left(-\beta \sum_{i=-(n_2+s_2)}^{s_1+n_1} \frac{p_i^2}{2}\right). \tag{5}$$

Denote by:

- $V_{n_1}(t)$ – the interaction region volume of the right-hand side particles $x_{s_1+1}, \dots, x_{s_1+n_1}$,
- $V_{n_2}(t)$ – the interaction region volume of the left-hand side particles $x_{-(n_2+s_2)}, \dots, x_{-(s_2+1)}$,

then $V_n(t) = V_{n_1}(t)V_{n_2}(t)$, besides denote $b = 2\xi \exp(2\beta B)$.

Therefore, the estimate (5) takes on the form

$$|F|_Y(t, Y) \leq \|F(0)\| \xi^s \exp(2\beta Bs) \exp\left(-\beta \sum_{i=-s_2}^{s_1} \frac{p_i^2}{2}\right) A_1 A_2, \tag{6}$$

where

$$A_1 = \sum_{n_1=0}^{\infty} b^{n_1} \int_{R^{n_1}} dp_{s_1+1} \dots dp_{s_1+n_1} V_{n_1}(t) \exp\left(-\beta \sum_{i=s_1+1}^{s_1+n_1} \frac{p_i^2}{2}\right),$$

$$A_2 = \sum_{n_2=0}^{\infty} b^{n_2} \int_{R^{n_2}} dp_{-(n_2+s_2)} \dots dp_{-(s_2+1)} V_{n_2}(t) \exp\left(-\beta \sum_{i=-(n_2+s_2)}^{-(s_2+1)} \frac{p_i^2}{2}\right).$$

For the sake of simplicity we shall prove convergence of A_1 (convergence of A_2 can be proved analogously).

4. ON THE ESTIMATE OF THE INTERACTION REGION VOLUME $V_{n_1}(t)$ IN EXPLICIT FORM

We fix the position q_{s_1} of the last particle from the group $Y = (x_{-s_2}, \dots, x_{s_1})$ and denote this position by a conditional origin of reference system along a straight line.

Denote by Υ_k — the maximum interval such that k particles located within it can interact, then

$$\begin{aligned} \Upsilon_k &= \Upsilon_{k-1} + \Delta s_k, \\ \Upsilon_{k+1} &= \Upsilon_k + \Delta s_{k+1}, \end{aligned}$$

where Δs_k is a maximum admissible distance by which the particle q_{s_1+k} increases the total run interval Υ_{k-1} of the particles $q_{s_1+1}, q_{s_1+2}, \dots, q_{s_1+k-1}$.

Estimate Δs_k — the maximum relative run of the particle q_{s_1+k} during the time $2t$. If some particle (say k -particle) has the initial momentum $|p_k|$, then we divide its motion into two parts during the time $2t$:

- other particles act with the force, limited by D , upon this particle during the time τ . Thus, its velocity increases to $|p_k| + D\tau$, but relative run of this particle to the other ones cannot be greater than R ;
- free run of this particle with constant velocity increases the total interval by $p_{\max}(2t - \tau) = (|p_k| + D\tau)(2t - \tau)$.

Find the maximum of the function

$$\begin{aligned} f(\tau) &= (p + D\tau)(2t - \tau) \rightarrow \max, \tau \in [0, 2t]. \\ \frac{df}{d\tau} &= D(2t - \tau) + (p + D\tau)(-1) = 0 \Rightarrow \tau = t - \frac{p}{2D} \\ &\begin{cases} \Delta s = R + 2pt, & \text{if } t \leq \frac{p}{2D}, \\ \Delta s = R + D\left(t + \frac{p}{2D}\right)^2, & \text{if } t \geq \frac{p}{2D}, \end{cases} \end{aligned}$$

but if $p \leq 2Dt$, then

$$\Delta s \leq R + Dt^2 + pt + \frac{p}{4D}2Dt \leq R + Dt^2 + 2pt.$$

Thus, the two expressions can be estimated in the following way

$$\Delta s_k \leq R + 2|p_k|t + Dt^2.$$

Now let us talk about the interaction region volume. For the sake of simplicity denote $V_{n_1} = V_{n_1}(t)$.

$$V_{n_1} = \int_0^{Z_1} dq_{s_1+1} \int_{q_{s_1+1}}^{Z_2} dq_{s_1+2} \dots \int_{q_{s_1+n_1-1}}^{Z_{n_1}} dq_{s_1+n_1}, \tag{7}$$

where

$$Z_1 = \Upsilon_1 = \Delta s_0 + \Delta s_1, \\ \Delta s_0 = 2|p_0|t + 2Dt^2, \quad \Delta s_1 = R + 2|p_1|t + Dt^2, \quad |p_0| = |p_{s_1}|, \quad |p_1| = |p_{s_1+1}|,$$

and

$$Z_k = \min(\Upsilon_k, q_{s_1+k-1} + \Delta s_{k-1} + \Delta s_k - R),$$

where $Z_k \leq \Upsilon_k$, because the distance between q_{s_1+k} and q_{s_1} cannot be greater than $\Upsilon_k = \Delta s_0 + \Delta s_1 + \dots + \Delta s_k$. From the other hand, if $q_{s_1+1}, q_{s_1+2}, \dots, q_{s_1+k-1}$ are very close to the origin of reference system, then q_{s_1+k} cannot be located near extreme point of Υ_k , otherwise q_{s_1+k-1} and q_{s_1+k} are not able to interact; therefore, the maximum admissible distance between them is $\Delta s_{k-1} + \Delta s_k - R$.

Using these notations, from (7) we obtain

$$V_{n_1} \leq \int_0^{Z_1} dq_{s_1+1} \dots \int_{q_{s_1+n_1-1}}^{q_{s_1+n_1-1} + \Delta s_{n_1-1} + \Delta s_{n_1} - R} dq_{s_1+n_1} = \\ = \int_0^{Z_1} dq_{s_1+1} \dots \int_{q_{s_1+n_1-2}}^{Z_{n_1-1}} (\Delta s_{n_1-1} + \Delta s_{n_1} - R) dq_{s_1+n_1-1}$$

or

$$V_{n_1} \leq V_{n_1-1}(\Delta s_{n_1-1} + \Delta s_{n_1} - R). \tag{8}$$

Finally, prove the convergence of the series

$$A_1 = \sum_{n_1=0}^{\infty} b^{n_1} g_{n_1}, \tag{9}$$

where

$$g_{n_1} = \int_{R^{n_1}} dp_{s_1+1} \dots dp_{s_1+n_1} V_{n_1}(t) \exp\left(-\beta \sum_{i=s_1+1}^{s_1+n_1} \frac{p_i^2}{2}\right).$$

According to the formula (8) we obtain

$$\begin{aligned}
 g_{n_1} &\leq \int_{R^{n_1}} dp_{s_1+1} \dots dp_{s_1+n_1} V_{n_1-1}(t) \exp\left(-\beta \sum_{i=s_1+1}^{s_1+n_1-1} \frac{p_i^2}{2}\right) \times \\
 &\quad \times (\Delta s_{n_1-1} + \Delta s_{n_1} - R) \exp\left(-\beta \frac{p_{s_1+n_1}^2}{2}\right) = \\
 &= \int_{R^{n_1-1}} dp_{s_1+1} \dots dp_{s_1+n_1-1} V_{n_1-1}(t) \times \\
 &\quad \times \exp\left(-\beta \sum_{i=s_1+1}^{s_1+n_1-1} \frac{p_i^2}{2}\right) \int_R dp_{s_1+n_1} (\Delta s_{n_1-1} + \Delta s_{n_1} - R) \exp\left(-\beta \frac{p_{s_1+n_1}^2}{2}\right). \tag{10}
 \end{aligned}$$

The last integral can be transformed in this way

$$\begin{aligned}
 \int_{-\infty}^{\infty} (2t|p_{s_1+n_1-1}| + 2t|p_{s_1+n_1}| + 2t^2D + R) \exp\left(-\beta \frac{p_{s_1+n_1}^2}{2}\right) dp_{s_1+n_1} = \\
 = (2t|p_{s_1+n_1-1}| + 2t^2D + R)I_0 + 2tI_1,
 \end{aligned}$$

where

$$\begin{aligned}
 I_0 &= \int_{-\infty}^{\infty} \exp\left(-\beta \frac{p^2}{2}\right) dp = \sqrt{\frac{2\pi}{\beta}}, \\
 I_1 &= \int_{-\infty}^{\infty} |p| \exp\left(-\beta \frac{p^2}{2}\right) dp = \frac{2}{\beta},
 \end{aligned}$$

then (10) takes on the form

$$\begin{aligned}
 \int_{R^{n_1-1}} dp_{s_1+1} \dots dp_{s_1+n_1-1} ((2t^2D + R)I_0 + 2tI_1) V_{n_1-1}(t) \exp\left(-\beta \sum_{i=s_1+1}^{s_1+n_1-1} \frac{p_i^2}{2}\right) + \\
 + I_0 \int_{R^{n_1-1}} dp_{s_1+1} \dots dp_{s_1+n_1-1} V_{n_1-1}(t) 2t|p_{s_1+n_1-1}| \exp\left(-\beta \sum_{i=s_1+1}^{s_1+n_1-1} \frac{p_i^2}{2}\right) = \\
 = ((2t^2D + R)I_0 + 2tI_1) g_{n_1-1} + \\
 + I_0 \int_{R^{n_1-1}} dp_{s_1+1} \dots dp_{s_1+n_1-1} V_{n_1-1}(t) 2t|p_{s_1+n_1-1}| \exp\left(-\beta \sum_{i=s_1+1}^{s_1+n_1-1} \frac{p_i^2}{2}\right). \tag{11}
 \end{aligned}$$

Note, the function $V_{n_1-1}(t)$ is no greater than linear one with respect to momentum $|p_{s_1+n_1-1}|$, therefore, $V_{n_1-1}(t) = \alpha + \gamma|p_{s_1+n_1-1}|$, where α, γ do not depend on $p_{s_1+n_1-1}$. Thus, the second term of the formula (11) has the form

$$2tI_0(MI_1 + NI_2), \quad \text{and} \quad g_{n_1-1} = MI_0 + NI_1,$$

where

$$I_2 = \int_{-\infty}^{\infty} p^2 \exp\left(-\beta \frac{p^2}{2}\right) dp = \frac{1}{\beta} \sqrt{\frac{2\pi}{\beta}},$$

$$M = \int_{R^{n_1-2}} dp_{s_1+1} \dots dp_{s_1+n_1-2} \alpha \exp\left(-\beta \sum_{i=s_1+1}^{s_1+n_1-2} \frac{p_i^2}{2}\right),$$

$$N = \int_{R^{n_1-2}} dp_{s_1+1} \dots dp_{s_1+n_1-2} \gamma \exp\left(-\beta \sum_{i=s_1+1}^{s_1+n_1-2} \frac{p_i^2}{2}\right).$$

It is obvious, that

$$MI_1 + NI_2 < \kappa(MI_0 + NI_1),$$

where

$$\kappa = \max\left(\frac{I_1}{I_0}, \frac{I_2}{I_1}\right) = \max\left(\sqrt{\frac{2}{\beta\pi}}, \sqrt{\frac{\pi}{2\beta}}\right) = \frac{1}{2} \sqrt{\frac{2\pi}{\beta}} = \frac{I_0}{2}.$$

Therefore,

$$2tI_0(MI_1 + NI_2) < 2tI_0\kappa(MI_0 + NI_1) = 2tI_0\kappa g_{n_1-1}.$$

Thus,

$$g_{n_1} < ((2t^2D + R)I_0 + 2tI_1 + 2t\kappa I_0) g_{n_1-1}.$$

The series (9) converges if

$$b((2t^2D + R)I_0 + 2tI_1 + 2t\kappa I_0) < 1$$

or

$$2\xi \exp(2\beta B) ((2t^2D + R)I_0 + 2tI_1 + tI_0^2) < 1,$$

hence

$$t < -\frac{2I_1 + I_0^2}{4DI_0} + \sqrt{\frac{1}{2DI_0} \left(\frac{\exp(-2\beta B)}{2\xi} - RI_0\right) + \left(\frac{2I_1 + I_0^2}{4DI_0}\right)^2}.$$

Thus, the series (6) converges, and the series (3) — converges uniformly if

$$0 \leq t < 2t_0 = -\frac{2I_1 + I_0^2}{4DI_0} + \sqrt{\frac{1}{2DI_0} \left(\frac{\exp(-2\beta B)}{2\xi} - RI_0\right) + \left(\frac{2I_1 + I_0^2}{4DI_0}\right)^2}. \quad (12)$$

Thus, from the results, obtained above, we can formulate the following theorem.

Theorem 1. *If the interaction potential satisfies the conditions (1) and the initial data $F(0) \in E_\xi$, then the series (3) converges uniformly in Y on any compact if*

$$0 \leq t < 2t_0 = -\frac{2I_1 + I_0^2}{4DI_0} + \sqrt{\frac{1}{2DI_0} \left(\frac{\exp(-2\beta B)}{2\xi} - RI_0\right) + \left(\frac{2I_1 + I_0^2}{4DI_0}\right)^2}.$$

From this theorem the local existence theorem follows:

Theorem 2. *If the interaction potential satisfies the conditions (1) and the initial data $F(0) \in E_\xi$, then the sequence $F(t)$ (3) exists and the series (3) converges uniformly on any compact on a finite time interval (12).*

5. CONCLUSION

Using physical aspects of interaction for non-symmetrical one-dimensional system, we proved convergence of the series (3) on the time interval (12) (local existence of the sequence $F(t)$).

REFERENCES

- [1] Gerasimenko V. I., Stashenko M. O.: *Non-equilibrium cluster expansions of non-symmetrical particle systems*. Sci. Bulletin of the Volyn State Univ. **4** (2002), 5–13 (Ukrainian).
- [2] Petrina D. Ya.: *Mathematical description of the evolution of infinite systems of classical statistical physics: locally perturbed one-dimensional systems*. Theor. Math. Fiz. **38** (1979), 230–250.
- [3] Petrina D. Ya., Gerasimenko V. I., Malyshev P. V.: *Thermodynamic limit for solutions of Bogolubov's equations*. Sov. Sci. Rev., Ser. C, Math. Phys. **7** (1988), 280–335.
- [4] Petrina D. Ya., Gerasimenko V. I., Malyshev P. V.: *Mathematical foundations of classical Statistical Mechanics. Continuous systems*. New York, Gordon and Breach 1989.
- [5] Gerasimenko V. I.: *Evolution of the infinite particle system with nearest-neighbors interaction*. Dokl. Akad. Nauk USSR, Ser. A **5** (1982), 10–13.

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