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CORONA THEOREM AND ISOMETRIES

Abstract. The aim of this note is to discuss a new operator theory approach to Corona Problem. An equivalent operator problem invariant under unitary equivalence is stated. The related condition involves certain joint spectra of commuting subnormal operators. A special case leading to isometries is studied. As a result one obtains a relatively short proof of Corona Theorem for a wide class of domains in the plane, where Marshall's Theorem on the approximation by inner functions holds.

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1. INTRODUCTION

In this note we consider the algebra $H^\infty(\Omega)$ of all bounded holomorphic functions on a domain $\Omega \subset \mathbb{C}^m$. It is a Banach algebra normed by $\|f\|_\Omega := \sup\{|f(z)| : z \in \Omega\}$, whose unit $\mathbf{1}$ is the constant function equal 1.

Let $\mathbb{F} = (\phi_1, \dots, \phi_k)$ be a tuple of length k (a finite collection) of elements of $H^\infty(\Omega)$. One says that \mathbb{F} is *uniformly separated from zero* (or that \mathbb{F} forms a *corona data on Ω*), if

$$\inf_{z \in \Omega} (|\phi_1(z)| + \dots + |\phi_k(z)|) > 0. \quad (1)$$

We say that Corona Theorem holds on Ω if the ideal $\mathbb{I}(\mathbb{F})$ generated by any tuple \mathbb{F} uniformly separated from $\mathbf{0}$ is not proper i.e. $\mathbb{I}(\mathbb{F}) = H^\infty(\Omega)$. Equivalently, assuming (1), there should exist $g_1, \dots, g_k \in H^\infty(\Omega)$ satisfying the Corona Equation:

$$\phi_1(z)g_1(z) + \dots + \phi_k(z)g_k(z) = 1, \quad z \in \Omega. \quad (2)$$

Another meaning for (2) is that $\mathbf{0} := (0, \dots, 0) \in \mathbb{C}^k$ is not in the joint spectrum $\sigma_{H^\infty(\Omega)}(\phi_1, \dots, \phi_k)$ of the tuple \mathbb{F} in the Banach algebra $H^\infty(\Omega)$.

Let $\mathfrak{M}(\Omega) := \mathfrak{M}(H^\infty(\Omega))$ be the Gel'fand spectrum of $H^\infty(\Omega)$, the set of all nonzero multiplicative and linear functionals (characters)

$$\chi: H^\infty(\Omega) \ni f \rightarrow \chi(f) =: \widehat{f}(\chi) \in \mathbb{C}. \quad (3)$$

We have Gel'fand's Formula

$$\widehat{\mathbb{F}}(\mathfrak{M}(H^\infty(\Omega))) = \sigma_{(H^\infty(\Omega))}(\mathbb{F}) \quad \text{for all } \mathbb{F} \in (H^\infty(\Omega))^k, \quad (4)$$

where for $\mathbb{F} = (\phi_1, \dots, \phi_k)$ with $\phi_j \in H^\infty(\Omega)$ its Gel'fand transform sends a character $\chi \in \mathfrak{M}(\Omega)$ into $\widehat{\mathbb{F}}(\chi) = (\widehat{\phi}_1(\chi), \dots, \widehat{\phi}_k(\chi)) \in \mathbb{C}^k$. Let $\widetilde{\Omega}$ be the w^* -closure in $\mathfrak{M}(\Omega)$ of the canonical image of Ω (via the mapping $\Omega \ni z \rightarrow \chi_z \in \mathfrak{M}(\Omega)$, with $\widehat{f}(\chi_z) = f(z)$). The w^* -continuity of this transform and compactness of \mathfrak{M} yield $\widehat{\mathbb{F}}(\widetilde{\Omega}) = [\mathbb{F}(\Omega)]^-$, where the bar stands for the Euclidean topology closure (in \mathbb{C}^k).

Hence (1) means $\mathbf{0} \notin \widehat{\mathbb{F}}(\widetilde{\Omega})$, while (2) reads $\mathbf{0} \notin \widehat{\mathbb{F}}(\mathfrak{M}(\Omega))$. Because Gel'fand transforms separate the points in $\mathfrak{M}(\Omega)$, the transforms of all finite tuples separate points from compact subsets of this space. Hence Corona Theorem for Ω is equivalent to the density of Ω in $\mathfrak{M}(\Omega)$ in the sense that

$$\widetilde{\Omega} = \mathfrak{M}(\Omega). \quad (5)$$

If Ω is the unit disc \mathbb{D} in \mathbb{C} , an analogy to the Sun eclipse-related phenomena has led Kakutani to naming the set $\mathfrak{M}(\mathbb{D}) \setminus \widetilde{\mathbb{D}}$ the *corona of $H^\infty(\mathbb{D})$* . Therefore Corona Theorem for Ω asserts exactly that *the corona for $H^\infty(\Omega)$ is an empty set*. For $\Omega = \mathbb{D}$ it holds true, as shown in 1962 by L. Carleson [5].

The paper is organized as follows. In Section 2 equivalent formulations of corona problem are discussed. The most important one is related to analytic Toeplitz operators of multiplication by bounded analytic functions on Ω . Assuming the strict pseudoconvexity we show that the equality of two joint spectra is equivalent to Corona Theorem for this domain.

In Section 3 we restrict our attention to the case where $\mathbb{F} = (\phi_1, \dots, \phi_k)$ is a tuple of inner functions on Ω . The related Toeplitz operators are isometries and we link our operator version of (2) in this special case with a uniqueness theorem for joint spectra of isometries [15].

In the one-dimensional case it turns out, that for many domains like the unit disc, or Denjoy domains, the linear density of the inner functions in $H^\infty(\Omega)$ proved by Marshall *et al.* ([4, 11]) allows one a limitation to the isometric case. Indeed, our operator version “extends to uniform limits and linear combinations”.

2. EQUIVALENT VERSIONS OF CORONA THEOREM

Another equivalent to (1) \Rightarrow (2) condition is (SI) – meaning the validity (for any $k \in \mathbb{N}$) of the following “*Spectral Inclusion theorem for k -tuples*”

$$\sigma_{H^\infty(\Omega)}(\mathbb{F}) \subseteq [\mathbb{F}(\Omega)]^- \quad (\forall \mathbb{F} \in (H^\infty(\Omega))^k), \quad (\text{SI}_{(k)})$$

where $[\mathbb{F}(\Omega)]^-$ is the Euclidean closure of the image of Ω under \mathbb{F} . The latter is treated as a \mathbb{C}^k -valued mapping. Since the opposite inclusion is elementary by (4), spectral Inclusion $(SI_{(k)})$ asserts in fact the equality. As we shall see, this is an equality comparing two different types of joint spectra.

Also, Corona Theorem is equivalent to the local version of (SI). For boundary points $z_0 \in \partial\Omega$ one defines the *cluster set of \mathbb{F} at the point z_0* , $cl(\mathbb{F}; z_0)$ consisting of all the adherent (limit) points in \mathbb{C}^k of sequences $\{\mathbb{F}(z_j)\}_{j=1}^\infty$ with $\{z_j\}_{j=1}^\infty \subset \Omega$ and $\lim_{j \rightarrow \infty} |z_j - z_0| = 0$. (Here $|\cdot|$ denotes the Euclidean norm in \mathbb{C}^m .) The same definition applies when $z_0 \in \Omega$, and then $cl(\mathbb{F}; z_0) = \{\mathbb{F}(z_0)\}$ is a one-point set. Finally, put $cl(\mathbb{F}; z_0) = \emptyset$ in the case when $z_0 \notin \bar{\Omega} (= \Omega \cup \partial\Omega)$. Here it is convenient (but not necessary -cf. [6]) to assume that our domain $\Omega \subseteq \mathbb{C}^m$ is bounded. The coordinate functions tuple $\mathbf{z} := (z_1, \dots, z_m)$ belongs then to $(H^\infty(\Omega))^m$, yielding the canonical projection $\pi := \hat{\mathbf{z}}: \mathfrak{M}(\Omega) \rightarrow \mathbb{C}^m$. For $z \in \mathbb{C}^m$ let $\mathfrak{M}_z(\Omega) := \pi^{-1}\{z\}$ be the *fibre of the spectrum of $H^\infty(\Omega)$ over the point z* .

Lemma 1. *For $\Omega \subset \mathbb{C}^m$ either pseudoconvex, or a polydomain (i.e., Cartesian product of m bounded plane domains), one must have $\pi(\mathfrak{M}(\Omega)) = \bar{\Omega}$ (see [6, 13]).*

Similar considerations (cf. [16]) show that Corona Theorem is valid on Ω if and only if one has the following Local Spectral Inclusions

$$\hat{\mathbb{F}}(\mathfrak{M}_z(\Omega)) \subseteq cl(\mathbb{F}; z). \tag{LSI_{(k)}}$$

One more preliminary remark is easy to deduce from (4).

Lemma 2. *Assume Ω is as in Lemma 1. The equivalence of the three conditions: [(1) \Rightarrow (2)], (SI) and [(LSI) $\forall z \in \mathbb{C}^m$] holds true with any fixed length k of the tuples. For example, Corona Theorem for all pairs $(\phi_1, \phi_2) \in (H^\infty(\Omega))$ is equivalent to $(SI_{(2)})$ for such pairs.*

Note that for $k = 1$ the implication [(1) \Rightarrow (2)] and the inclusions $(SI_{(1)})$ are trivial, while the conclusion $(LSI_{(1)})$ is not easy (cf [13], where $(LSI_{(1)})$ is called “Cluster Values Theorem”) and it is here that some regularity of Ω is needed.

All previously listed conditions are in a sense classical. Now we shall formulate a new operator version of Corona Theorem whose merit lies in its invariance under unitary transformations. Its origins are in the following well-known construction.

2.1. TOEPLITZ OPERATORS ON HARDY SPACE $H^2(\Omega)$

Let $H^2(\Omega)$ be the Hilbert space of those analytic functions $f: \Omega \rightarrow \mathbb{C}$, whose squared modulus is majorized by some harmonic function $h: \Omega \rightarrow \mathbb{R}_+$. Let us fix one point, say $\omega_0 \in \Omega$. Denoting the least harmonic majorant of $|f|^2$ on Ω by h_f , following W. Rudin we may define $\|f\| := \sqrt{h_f(\omega_0)}$ and this yields a Hilbert space norm on $H^2(\Omega)$. For Ω being the unit disc: \mathbb{D} , this agrees with the classical Hardy space norm, if $\omega_0 = 0$.

Any $\varphi \in H^\infty(\Omega)$ defines the analytic Toeplitz operator

$$T_\varphi: H^2(\Omega) \ni g \rightarrow \varphi g \in H^2(\Omega).$$

This operator is bounded, with $\|T_\varphi\| = \|\varphi\|_\Omega$. It is pure subnormal for nonconstant φ (cf. e.g. [14]). It is an isometry if φ is inner i.e. if the boundary values of φ are of modulus one at almost all [with respect to harmonic measure on $\partial\Omega$] points of $\partial\Omega$ (cf. e.g. [14]).

For any tuple $\mathbb{F} = (\varphi_1, \dots, \varphi_k) \in (H^\infty(\Omega))^k$ we get $T_{\mathbb{F}} := (T_{\varphi_1}, \dots, T_{\varphi_k})$, the commuting tuple which is jointly subnormal.

Also, $\mathcal{B} := \{T_\varphi: \varphi \in H^\infty(\Omega)\}$ is a commutative Banach algebra, whose unit is $I = T_1$. \mathcal{B} is inverse-closed: if $z_0 \notin \sigma(T_\varphi)$ for some $\varphi \in H^\infty(\Omega)$, then $(T_\varphi - z_0 I)^{-1}$ also belongs to \mathcal{B} . Indeed, this resolvent is nothing else, but $T_\psi \in \mathcal{B}$ with $\psi(\omega) = (\varphi(\omega) - z_0)^{-1}$. This is so because $\psi \in H^\infty(\Omega)$, since obviously $\sigma(T_\varphi) = [\varphi(\Omega)]^-$.

On this Banach algebra we are going to consider two (*a priori* different) types of joint spectra. We begin with a look at the joint spectrum in the Banach algebra \mathcal{B} of $T_{\mathbb{F}}$, which is a subset of \mathbb{C}^k defined for $\mathbb{F} = (\phi_1, \dots, \phi_k) \in (H^\infty(\Omega))^k$ by

$$(\lambda_1, \dots, \lambda_n) \in \sigma_{\mathcal{B}}(T_{\mathbb{F}}) \Leftrightarrow \sum_{j=1}^k A_j (T_{\phi_j} - \lambda_j I) \neq I \quad (\forall A_j \in \mathcal{B}).$$

There is an obvious, but useful consequence of the inclusion between two commutative, norm-closed operator algebras \mathcal{B}_j containing $\tau := T_{\mathbb{F}}$ and I :

$$\mathcal{B}_1 \subset \mathcal{B}_2 \implies \sigma_{\mathcal{B}_2}(\tau) \subseteq \sigma_{\mathcal{B}_1}(\tau). \quad (6)$$

A different kind of joint spectrum is of “spatial character”: it takes into account just the way a given k -tuple $\tau = (T_1, \dots, T_k)$ of operators acts on the underlying Hilbert space. For $\lambda = (\lambda_1, \dots, \lambda_k) \in E \subset \mathbb{C}^k$ we define its complex conjugate

$$\lambda^* := (\lambda_1^*, \dots, \lambda_k^*)$$

and let $E^* := \{\lambda^*: \lambda \in E\}$, while E^- , or \bar{E} means the Euclidean closure throughout this paper. The symbol $*$ will be also used to denote the adjoint: T^* of a Hilbert space operator T , or the adjoint tuple $\tau^* = (T_1^*, \dots, T_k^*)$ -of a tuple $\tau = (T_1, \dots, T_k)$ of operators. This other joint spectrum of a tuple $\tau \in \mathcal{B}^k$ of such operators is

$$\sigma_*(\tau) := [(\sigma_p(T_1^*, \dots, T_k^*))^*]^-.$$

In other words, $\sigma_*(\tau)$ is the closure of the set $(\sigma_p(T_1^*, \dots, T_k^*))^*$ of all complex conjugates of joint eigenvalues of the adjoint tuple. (This set is easily seen to be contained in the Taylor joint spectrum of τ , denoted here $\sigma(\tau, H)$, where H stands for the Hilbert space $H^2(\Omega)$.)

Let us recall that Taylor’s nonsingularity of τ (denoted $0 \notin \sigma(\tau, H)$) means the exactness of the Koszul (chain) complex $(K_p(\tau, H), \delta_p)$. Here $K_p(\tau, H) = \bigwedge^p \mathbb{C}^k \otimes H$

is spanned by the vectors $x \otimes e_\alpha = x \otimes e_{\alpha(1)} \wedge \cdots \wedge e_{\alpha(p)}$ with $x \in H$ and α ranging over the multiindexes of length p . The chain derivations $\delta_p: K_p(\tau, H) \rightarrow K_{p-1}(\tau, H)$ map such a vector $x \otimes e_\alpha$ onto

$$\delta_p(x \otimes e_\alpha) := \sum_{j=1}^p (-1)^{j-1} T_{\alpha(j)} x \otimes e_{\alpha(1)} \wedge \cdots \wedge \widehat{e_{\alpha(j)}} \wedge \cdots \wedge e_{\alpha(p)},$$

where $\widehat{}$ is the omission sign. Now $\lambda \in \sigma(\tau, H)$ means non-exactness of the Koszul complex for $\tau - \lambda I := (T_1 - \lambda_1, \dots, T_k - \lambda_k)$. A simple characterization of Taylor’s nonsingularity occurs for $k = 2$: the pair should have $(0, 0) \notin \sigma_p(T_1, T_2)$, satisfy $H = T_1(H) + T_2(H)$ and have the system of equations $T_1 z = x, T_2 z = y$ solvable whenever the necessary condition: $T_2 x - T_1 y = 0$ holds.

One has $\sigma(\tau^*, H) = \sigma(\tau, H)^*$, since the adjoints of the δ_p are the derivations of the related cochain complex [19, 17].

The following condition (UICT) is our *Unitarily Invariant Corona Theorem*: For all positive integers k one has

$$\sigma_{\mathcal{B}}(\tau) = \sigma_*(\tau) \quad (\forall \tau \in \mathcal{B}^k). \tag{UICT}_{(k)}$$

Theorem 1. *Assume that either $\Omega \subset \mathbb{C}$, or Ω is strictly pseudoconvex with C^2 -defining function. The conditions $(SI_{(k)})$ and $(UICT_{(k)})$ are then equivalent. Therefore (UICT) (precisely, its nontrivial half: $\sigma_{\mathcal{B}}(\tau) \subseteq \sigma_*(\tau)$) is equivalent to Corona Theorem on Ω .*

Note that there are statements (on rigid inner functions) called “Operator Corona Theorems” that actually fail even in the unit disc setting ([2, 18]). Compared to (UICT), these are too strong conditions.

Proof. Since the mapping $H^\infty(\Omega) \ni \phi \rightarrow T_\phi \in \mathcal{B}$ is a Banach algebra isomorphism, for $\mathbb{F} = (\phi_1, \dots, \phi_k) \in (H^\infty(\Omega))^k$ we have $\sigma_{\mathcal{B}}(T_{\mathbb{F}}) = \sigma_{H^\infty(\Omega)}(\mathbb{F})$. Therefore it suffices to show the equality

$$\sigma_*(T_{\mathbb{F}}) = [\mathbb{F}(\Omega)]^-. \tag{7}$$

Indeed, the inclusions $(SI_{(k)})$ are in fact equalities, by elementary algebra: Since (2) cannot hold whenever all the $\phi_j(z_0)$ vanish for some $z_0 \in \Omega$, the assumption $0 \in \mathbb{F}(\Omega)$ implies $0 \in \sigma_{H^\infty(\Omega)}(\mathbb{F})$ and the inclusions result by subtracting $\phi_j(z_0)\mathbf{1}$ from ϕ_j .

The containment (\supseteq) in (7) follows easily from the Reproducing Kernel Property of $H^2(\Omega)$: for any $z \in \Omega$ continuity of the evaluation at z yields some $K_z \in H^2(\Omega)$ satisfying

$$f(z) = \langle f, K_z \rangle, \quad (\forall f \in H^2(\Omega)). \tag{8}$$

Here $\langle \cdot, \cdot \rangle$ is the inner product of this Hardy space. Applied to $f = \phi_j g$, as g runs through $H^2(\Omega)$, (8) gives

$$\langle g, T_{\phi_j}^* K_z \rangle = \langle \phi_j g, K_z \rangle = \phi_j(z)g(z) = \langle g, (\phi_j(z))^* K_z \rangle.$$

Hence $(\mathbb{F}(z))^* = (\phi_1(z)^*, \dots, \phi_k(z)^*) \in \mathbb{C}^k$ is a joint eigenvalue of $T_{\mathbb{F}}^*$ with eigenvector K_z , showing that $\mathbb{F}(\Omega) \subseteq \sigma_*(T_{\mathbb{F}})$. Since $\sigma_*(T_{\mathbb{F}})$ is closed, \supseteq of (7) follows.

The opposite inclusion is more delicate and it requires some regularity of Ω (as follows from an inspection of N. Sibony's counterexamples). $\sigma_*(T_{\mathbb{F}})$ is contained in Taylor's joint spectrum, $\sigma(T_{\mathbb{F}})$. This is an easy consequence of the definition of $\sigma(T_{\mathbb{F}}, H)$ in terms of non-exactness of the Koszul complex, while $0 \in [\sigma_p(T_{\mathbb{F}}^*)]^*$ means its non-exactness at stage 0.

Therefore the inclusion \subseteq in (7) follows from the description of Taylor's joint spectrum of tuples of k analytic Toeplitz operators:

$$\sigma(T_{\mathbb{F}}, H) = [\mathbb{F}(\Omega)]^-.$$

This result, related to the " L^2 -Corona Theorem" can be found in [19] and in a special case in [12]. Actually we do not need its full strength and the inclusion $\sigma_p(T_{\mathbb{F}}) \subseteq [\mathbb{F}(\Omega)]^-$ can be obtained directly from " L^2 -Corona Theorem" (asserting (2), but with $g_j \in H^2(\Omega)$, under the uniform separation assumptions (1) with $\psi_j \in H^\infty(\Omega)$ replaced by $\psi_j \in H^2(\Omega)$). This result based on L^2 -estimates for $\bar{\partial}$ can be found in a number of papers ([3, 10]). \square

Our final general observation is that for domains Ω regular as above the joint spectrum $\sigma_*(\tau)$ is actually equal to either of the joint spectra in the sense of R. Harte, or that of J. L. Taylor ([8, 17]). This follows *a posteriori* from the equality $\sigma(\tau, H) = \bar{\Omega}$ proved in [19]. Hence it suffices to show the following slightly weaker "Taylor variant" of the problem contained in the following theorem.

Theorem 2. *Corona Theorem on a domain Ω regular as in 1 is equivalent to the inclusions $\sigma_{\mathcal{B}}(\tau) \subseteq \sigma(\tau, H)$.*

3. CORONA THEOREM FOR INNER FUNCTIONS

Our limitation to the inner functions will be partially justified by invoking Marshall's result [11] on their density in $H^\infty(\mathbb{D})$, extended further in [4]. This technical limitation allows us to avoid the complications in the case of multiply connected domains covered by the Abrahamse–Douglas functional model for certain subnormal operators. Instead, we are dealing with a much simpler situation of the isometric Hilbert space operators.

In [14] pure isometry of Toeplitz operators with inner symbol was shown on a quite general bounded domain. For domains having smooth boundary this was known even much earlier. The concept of inner function requires fixing some measure μ on $\partial\Omega$, so that the functions in $H^2(\Omega)$ have boundary limits (along certain paths) at almost all $[d\mu]$ boundary points of Ω , yielding isometric embedding: $H^2(\Omega) \rightarrow L^2(\mu)$. Then "inner" means "having boundary values of modulus 1 almost everywhere $[d\mu]$ ". In most cases one takes as μ the harmonic measure for Ω evaluating at the norming point ω_0 fixed previously. By arguments in [14] or otherwise, it is easy to verify that nonconstant $\varphi \in H^\infty(\Omega)$ yields a pure subnormal operator. Its normal

extension acts on $L^2(\mu)$, again by the multiplication formula. (Pure operators are also called completely nonnormal. Pure subnormal isometries are precisely the isometries without nontrivial unitary part, known as completely nonunitary isometries.)

Now if (\mathbb{F}) is an n -tuple (eg. a pair) of inner functions, then $T_{\mathbb{F}}$ consists of pure isometries. Let $\mathcal{R}(\tau)$ denote the smallest inverse-closed algebra of operators on $H^2(\Omega)$ containing a given tuple τ . Since the Banach algebra \mathcal{B} of all analytic Toeplitz operators is inverse-closed, as noted earlier, for $\tau = T_{\mathbb{F}}$ we have

$$\mathcal{R}(\tau) \subseteq \mathcal{B}. \quad (9)$$

The following Uniqueness Theorem for joint spectra is established in [15] by a kind of induction on the “length of the tuple”. It is based on a vector-valued variant of Gamelin’s localization technique [6] combined with E. Amar’s separation of singularities [1] and reduction to the $k = 2$ -case. The basic idea in this case (for pairs) is to use the functional model for the first operator and just the description of model’s commutant-for the second one. Thus T_1 becomes the unilateral shift of some multiplicity, while T_2 is mapped by this unitary equivalence onto the multiplication by a rigged inner function. At each point from some neighbourhood of the “singularity of T_1 ” one estimates uniformly the operator norms of inverses of point-values of this rigged inner function. This allows separation of singularities from [1] to extend to our (operator-valued) setting.

For the sake of convenience we are still using the symbol H for the underlying Hilbert space even though it may be different from $H^2(\Omega)$.

Theorem 3. *If $\tau = (T_1, \dots, T_k)$ is a finite collection of commuting isometries on a Hilbert space H , with each T_j unitarily equivalent to an analytic Toeplitz operator on some $H^2(\Omega)$ space, then*

$$\sigma_{\mathcal{R}(\tau)}(\tau) = \sigma_*(\tau).$$

Note also that (9) and (6) imply $\sigma_{\mathcal{B}}(\tau) \subseteq \sigma_{\mathcal{R}(\tau)}(\tau)$. Therefore applying Theorem 3 we obtain the nontrivial inclusions in (UICT) of Theorem 1. Concluding, we may state our main result as follows.

Theorem 4. *If Ω is a domain regular as in Theorem 1, then Corona Theorem [(1) \Rightarrow (2)] holds true for the inner functions and for the Banach algebra generated in $H^\infty(\Omega)$ by such functions. In particular, it holds for $H^\infty(\Omega)$ if Ω is a planar domain unitarily equivalent either to the unit disc, or to a Denjoy domain.*

Proof. Having proved (UICT) for the inner tuples we may pass to tuples consisting of linear combinations of inner functions. Indeed, (UICT) asserts equality of two types of joint spectra of a given tuple. To both sides spectral mapping theorems (cf. eg. [13]) apply. Here the mapping is a polynomial (even linear). An this is how the equality holds for finite linear combinations of inner functions. Finally, using the result of [4] we pass to uniform limits of such combinations. The upper semicontinuity of joint spectra suffices to get the nontrivial part of (UICT) for these limits. \square

The class of plane domains where Corona Theorem is known to hold (see [7]) seems to be not wider than the class of those plane domains where Marshall's Theorem is established. The analysis behind the proofs is much easier in the latter case, which in a sense justifies the efforts presented in this note.

Unfortunately, the inner functions, although exist (and even span a dense subset in the topology of uniform convergence on compact subsets of Ω), fail to generate (in the uniform convergence topology) the Banach algebra $H^\infty(\Omega)$ in the multidimensional case. They are not even sufficient to separate the points of $\mathfrak{M}(\Omega)$, as noted by S.H. Kon [9] in the unit bidisc case.

Isn't this stumbling block an actual reason, why Corona Problem in many variables is still open?

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