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SOME THEOREMS OF RABINOWITZ TYPE FOR NONLINEARIZABLE EIGENVALUE PROBLEMS

Abstract. We discuss the structure of the solution set for nonlinearizable eigenvalue problems in a Hilbert space.

Keywords: nonlinear eigenvalue problem, bifurcation point, bifurcation interval.

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1. INTRODUCTION

Suppose $G\colon \mathbf{R}\times E \mapsto E$, where E is a Banach space, G is continuous and $G(\lambda,0)=0$ for every $\lambda\in\mathbf{R}$. Equations of the form $(*)$ $u=G(\lambda,u)$ are called nonlinear eigenvalue problems. It is often the case of applications that $G(\lambda,u)=\lambda Lu+H(u)$, where L is a linear and compact operator, H is compact. The theory of global bifurcation by Rabinowitz may be applicable to $(*)$ if G is Fréchet differentiable at $(\lambda,0)$ which means $H(u)=o\|u\|$. With this assumption $(*)$ is called a linearizable eigenvalue problem. Such problems have been investigated very well.

The purpose of this paper, which has been inspired by the works of Rabinowitz, is to prove some global bifurcation results for nonlinearizable eigenvalue problems. Since our work is a continuation of [2], we begin with the remainder of some basic facts.

Let $(E,\|\ \|_E)$ be a real Banach space imbedded in a Hilbert space $(H,||\ ||_H)$ and L be a linear symmetric operator in H, such that $Range\ L\subset E$, $L|_E\colon E\mapsto E$ is compact. If there exists $\mu\in\mathbf{R}$ and $0\neq v\in E$ such that $v=\mu Lv$, μ is said to be a characteristic value of L. The set of characteristic values of L will be denoted by $r(L)$. The multiplicity of $\mu\in r(L)$ is defined by $k_\mu=dim\bigcup_{i=1}^{\infty}\ Ker(I-\mu L)^i$.

We consider the equation

$$u = L(\lambda u + F(u)), \tag{1}$$

where $F\colon E \mapsto H$ is a nonlinear operator satisfying the following conditions

$$L \circ F\colon E \mapsto E \text{ is compact and } \exists_{M>0}\ \forall_{u\in E}\ \|F(u)\|_H \le M\|u\|_H. \tag{2}$$

By a solution of (1) is meant a pair $(\lambda, u) \in \mathbf{R} \times E$ satisfying (1). In particular, (1) has the line of trivial solutions. It was shown in [2] that (1) possesses no nontrivial solutions when $\lambda \notin [\mu - M, \mu + M]$ for $\mu \in r(L)$.

Theorem 1. *If* (λ, u) *is a nontrivial solution of* (1)*, then*

$$dist(\lambda, r(L)) \le M. \tag{3}$$

Moreover, the sufficient condition for $[\mu - M, \mu + M] \times \{0\}$ to be a bifurcation interval for (1) was also formulated in [2], namely

Theorem 2. *If* $\mu \in r(L)$ *is of odd multiplicity and* $dist(\mu, r(L) - \{\mu\}) > 2M$*, then* $[\mu - M, \mu + M] \times \{0\}$ *is a bifurcation interval for* (1)*.*

Our aim is to give the global version of Theorem 2. We show the existence of at least one unbounded continuum of nontrivial solutions of (1) bifurcating from $[\mu - M, \mu + M] \times \{0\}$. In particular the case of bifurcation from simple characteristic values is discussed. We also present some information about the set of bifurcation points in the bifurcation interval.

2. THE MAIN RESULTS

Denoting by B the set of bifurcation points of (1) we have $\mathcal{B} := B \cap \big([\mu - M, \mu + M] \times \{0\}\big) \neq \varnothing$. Let $\mathcal{S}$ denote the closure of the set of nontrivial solutions of (1). A component of $\mathcal{S}$ is a closed connected subset of $\mathcal{S}$ maximal with respect to inclusion.

We formulate the global bifurcation result being the generalization of Theorem 2.

Theorem 3. *If* $\mu \in r(L)$ *is of odd multiplicity and* $dist(\mu, r(L) - \{\mu\}) > 2M$*, then* $\mathcal{C} = \bigcup_{(\lambda,0)\in\mathcal{B}} \mathcal{C}_\lambda$*, where* $\mathcal{C}_\lambda$ *is a component of* $\mathcal{S}$ *containing* $(\lambda, 0)$*, is unbounded.*

Proof. An argument similar to that of Theorem 2 is applicable here. By assumption, there exists $\varepsilon_0 > 0$ such that $dist(\mu, r(L) - \{\mu\}) = 2M + \varepsilon_0$. We argue by contradiction, so suppose $\mathcal{C}$ is bounded. The compactness of $L \circ (\lambda I + F)$ implies $\bar{\mathcal{C}}$ is compact and consequently $\mathcal{C} \cup ([\mu - M, \mu + M] \times \{0\})$ is compact and connected. Thus, there exists a bounded open set $\mathcal{O}$ such that $\mathcal{C} \cup ([\mu - M, \mu + M] \times \{0\}) \subset \mathcal{O}$ and $\mathcal{S} \cap \partial\mathcal{O} = \varnothing$. Let $\mathcal{O}_\lambda = \{u \in E\colon (\lambda, u) \in \mathcal{O}\}$ and $\underline{\lambda} = \mu - M - \varepsilon/2$, $\bar{\lambda} = \mu + M + \varepsilon/2$ for $0 < \varepsilon < \varepsilon_0$ sufficiently small. It follows from Theorem 1 that $\mathcal{O}_\lambda$ makes sense for $\lambda \in [\underline{\lambda}, \bar{\lambda}]$. Note that

$$dist\big(\underline{\lambda}, r(L)\big) = dist\big(\bar{\lambda}, r(L)\big) = M + \varepsilon/2.$$

Denote $\Phi(\lambda) = I - L \circ (\lambda I + F)$. It is clear that the Leray–Schauder degree $d(\Phi(\lambda), \mathcal{O}_\lambda, 0)$ is well defined for all $\lambda \in [\underline{\lambda}, \bar{\lambda}]$. By the homotopy invariance property of degree

$$d(\Phi(\lambda), \mathcal{O}_\lambda, 0) = \text{ const for } \lambda \in \left[\underline{\lambda}, \bar{\lambda}\right].$$

Since (1) has no nontrivial solutions for $\lambda \notin [\mu - M, \mu + M]$, we have

$$\begin{aligned} d(\Phi(\underline{\lambda}), B(0,\delta), 0) = d(\Phi(\underline{\lambda}), \mathcal{O}_{\underline{\lambda}}, 0) = d(\Phi(\bar{\lambda}), \mathcal{O}_{\bar{\lambda}}, 0) = \\ = d(\Phi(\bar{\lambda}), B(0,\delta), 0) \text{ for some } \delta > 0. \end{aligned} \tag{4}$$

Consider now the first term in (4). Notice that $u \neq L(\underline{\lambda} u + tF(u))$ for $u \in \partial B(0,\delta)$ and $t \in [0,1]$. If not,

$$u = L\big(\underline{\lambda} u + tF(u)\big) \text{ for some } u \in \partial B(0,\delta),\ t \in [0,1]. \tag{5}$$

Since $\underline{\lambda} \notin r(L)$, $I - \underline{\lambda} L$ is invertible, so that (5) is equivalent to

$$u = t(I - \underline{\lambda} L)^{-1} LF(u).$$

Hence

$$\|u\|_H \leq |t| \cdot \|(I - \underline{\lambda} L)^{-1} L\| \cdot \|F(u)\|_H \leq \|(I - \underline{\lambda} L)^{-1} L\| \cdot M \cdot \|u\|_H,$$

so

$$1 \leq \|(I - \underline{\lambda})^{-1} L\| \cdot M.$$

From the spectral mapping theorem for symmetric operators, we have a contradiction $\|(I - \underline{\lambda} L)^{-1} L\|^{-1} = dist(\underline{\lambda}, r(L)) \leq M$.

So, using the homotopy invariance again, we obtain that

$$d\left(\Phi(\underline{\lambda}), B(0,\delta), 0\right) = d\left(I - \underline{\lambda} L, B(0,\delta), 0\right) = i(\underline{\lambda}).$$

The same argument can be used for $\bar{\lambda}$. Hence, we obtain $i(\underline{\lambda}) = i(\bar{\lambda})$. Since μ is the only characteristic value of L in $[\underline{\lambda}, \bar{\lambda}]$ and k_μ is odd, then $i(\underline{\lambda}) \cdot (-1)^{k_\mu} = i(\bar{\lambda}) \neq 0$, which is impossible. The proof is complete. □

Remark 1. *Theorem 3 shows in particular the existence of at least one unbounded continuum of $\mathcal{S}$, bifurcating from $[\mu - M, \mu + M] \times \{0\}$.*

Assume additionally that

$$\forall_{E - \{0\} \ni u_n \to 0} \exists_{(u_{k_n}) \subset (u_n)} \exists_{m \in [-M,M]} \lim_{n \to \infty} \frac{L\big(F(u_{k_n})\big) - mLu_{k_n}}{\|u_{k_n}\|_E} = 0. \tag{6}$$

Hence $A := \{m \in [-M, M] \colon \exists_{u_n \to 0} \lim_{n\to\infty} \frac{L\left(F(u_n)\right) - mLu_n}{\|u_n\|_E} = 0\} \neq \varnothing$. This condition is satisfied in particular when $L \circ F$ is Fréchet differentiable at 0 and $(L \circ F)'(0) = = m^* L$. Then, it is clear that $A = \{m^*\}$.

Theorem 4. *Under the assumptions of Theorem 3, if moreover* (6) *holds, then* $\mathcal{C} = \bigcup_{(\mu-m,0)\in\mathcal{B}} \mathcal{C}_{\mu-m}$*, where* $m \in A$.

Proof. Let $(\lambda, 0) \in [\mu - M, \mu + M] \times \{0\}$ be a bifurcation point for (1). It means that there exists a sequence $(\lambda_n, u_n) \to^{\mathbf{R}\times E} (\lambda, 0)$ satisfying (1). Dividing (1) by $\|u_n\|_E$ and setting $u_n / \|u_n\|_E = w_n$ yields the equation

$$w_n = L\left(\lambda_n w_n + \frac{F(u_n)}{\|u_n\|_E}\right)$$

and consequently

$$w_{k_n} = (\lambda_{k_n} + m)Lw_{k_n} + \frac{L(F(u_{k_n})) - mLu_{k_n}}{\|u_{k_n}\|_E}, \tag{7}$$

where u_{k_n}, m are chosen in such a way that the second term on the right $\to 0$ as $n \to \infty$. Since L is compact, a subsequence of Lw_{k_n} converges. Hence the left-hand side of (7) has a convergent subsequence $\tilde{w}_{k_n} \to w$ with $\|w\|_E = 1$ and satisfying $w = (\lambda + m)Lw$. Consequently $\lambda + m \in r(L)$. Since $|\lambda - \mu| \leq M$ and $|m| \leq M$ we obtain $|\lambda + m - \mu| \leq 2M$, which is only possible when $\lambda = \mu - m$. Hence $\mathcal{C}_\lambda = \mathcal{C}_{\mu-m}$, which completes the proof. □

The next results are immediate consequences of Theorem 4.

Corollary 1. *If* A *is finite, then* $\mathcal{B}$ *is finite.*

Corollary 2. *If* A *consists of isolated points, then the bifurcation interval* $[\mu - M, \mu + M] \times \{0\}$ *contains only isolated bifurcation points.*

Moreover, replacing (6) by

$$\exists_{m^*\in[-M,M]}\ (L \circ F)'(0) = m^* L \tag{8}$$

we have the particular case of Theorem 4, namely

Theorem 5. *Under the assumptions of Theorem 3 and* (8)*, there exists the unbounded continuum of* $\mathcal{S}$, $\mathcal{C}_{\mu-m^*}$*, bifurcating from* $(\mu - m^*, 0)$.

Proof. In this case $A = \{m^*\}$, $\mathcal{C} = \mathcal{C}_{\mu-m^*}$, $\mathcal{C}$ is unbounded. □

In the case of bifurcation from a whole interval, which means that each point of the interval is the bifurcation point, we formulate the next necessary condition.

Corollary 3. *If* $\mathcal{B} = [\alpha, \beta] \times \{0\}$*, then* $[\mu - \beta, \mu + \alpha] \subset A$*. In particular,* $\mathcal{B} = [\mu - M, \mu + M] \times \{0\}$ *implies* $A = [-M, M]$.

3. BIFURCATION FROM SIMPLE CHARACTERISTIC VALUES

We can say much more about solutions of (1) bifurcating from $[\mu - M, \mu + M] \times \{0\}$ when μ is simple. To describe it precisely, let v denote the normalized characteristic function of L corresponding to $\mu \in r(L)$, $k_\mu = 1$.

Let E' denote the dual space of E, $\langle , \rangle$ the duality between E' and E, and $v^* \in E'$ a normalized characteristic function of L^* corresponding to μ. If $\hat{E} := \{y \in E : \langle v^*, y\rangle = 0\}$, then $E = span\{v\} \oplus \hat{E}$ and each $u \in E$ can be written as $u = \alpha v + y$, where $\alpha = \langle v^*, u\rangle$ and $y \in \hat{E}$.

For each $\eta \in (0, 1)$ define

$$Q_\eta := \{(\lambda, u) \in \mathbf{R} \times E\colon\ |\lambda - \mu| \le M,\ |\langle v^*, u\rangle| > \eta \, \|u\|_E\} \,.$$

It is clear that Q_η consists of two disjoint subsets Q_η^+, Q_η^-, where

$$\begin{aligned} Q_\eta^+ &= \{(\lambda, u) \in \mathbf{R} \times E\colon\ |\lambda - \mu| \le M,\ \langle v^*, u\rangle > \eta \, \|u\|_E\} \,, \\ Q_\eta^- &= \{(\lambda, u) \in \mathbf{R} \times E\colon\ |\lambda - \mu| \le M,\ \langle v^*, u\rangle < -\eta \, \|u\|_E\} \,. \end{aligned}$$

The following result localizes the possible solutions of (1) bifurcating from $[\mu - M, \mu + M] \times \{0\}$.

Theorem 6. *If $\mu \in r(L)$ is simple, $dist(\mu, r(L) - \{\mu\}) > 2M$ and F satisfies (6), then there exists $\delta_0 > 0$ such that*

$$\forall_{0<\delta<\delta_0} (\mathcal{S} - \mathcal{B}) \cap ([\mu - M, \mu + M] \times B(0, \delta)) \subset Q_\eta.$$

Moreover,

$$(\lambda, u) \in (\mathcal{S} - \mathcal{B}) \cap ([\mu - M, \mu + M] \times B(0, \delta))$$

implies

$$u = \alpha v + y, \ \text{where}\ |\alpha| > \eta \, \|u\|_E \ \text{and}\ y = o(\alpha) \ \text{at}\ \alpha = 0.$$

Proof. To prove the first part of Theorem 6, we argue by contradiction. If there is no δ_0 as in the statement, then there exist two sequences: $\delta_n \mapsto 0$ and

$$\begin{gathered} (\lambda_n, u_n) \in (\mathcal{S} - \mathcal{B}) \cap \big([\mu - M, \mu + M] \times B(0, \delta_n)\big), \\ (\lambda_n, u_n) \notin Q_\eta. \end{gathered}$$

Hence

$$|\langle v^*, u_n\rangle| \le \eta \, \|u_n\|_E \,. \tag{9}$$

Since λ_n is bounded, there exists a subsequence $\lambda_{k_n} \mapsto \lambda_0 \in [\mu - M, \mu + M]$ and $m \in A$ chosen to u_{k_n}. Substituting (λ_{k_n}, u_{k_n}) to (1), we have

$$\frac{u_{k_n}}{\|u_{k_n}\|_E} = \lambda_{k_n} L \left(\frac{u_{k_n}}{\|u_{k_n}\|_E} \right) + \frac{L\big(F(u_{k_n})\big)}{\|u_{k_n}\|_E}.$$

The same compactness argument as in Theorem 4 shows that along some subsequence, again labeled k_n

$$\lim_{n\to\infty} \frac{u_{k_n}}{||u_{k_n}||_E} = w, \|w\|_E = 1, w = (\lambda_0 + m)Lw,\ \lambda_0 = \mu - m.$$

Hence $w = \mp v$. Dividing (9) by $\|u_{k_n}\|_E$ and passing to the limit as $n \to \infty$, we have

$$1 = |\langle v^*, v\rangle| \le \eta < 1$$

which is impossible.

To prove the second part of Theorem 6, we observe that $(\lambda, u) = (\lambda, \alpha v + y) \in Q_\eta$ implies

$$|\alpha| = |\langle v^*, u\rangle| > \eta \, \|u\|_E \, .$$

Hence $u \to 0$ and $y \to 0$ as $\alpha \to 0$. It remains to show that $\lim_{\alpha \to 0} \frac{y}{|\alpha|} = 0$. Consider any sequence

$$(\lambda_n, u_n) \in (\mathcal{S} - \mathcal{B}) \cap \big([\mu - M, \mu + M] \times B(0, \delta)\big)$$

such that $\alpha_n = \langle v^*, u_n\rangle \to 0$. Repeating the compactness argument again, we have

$$\exists_{m \in A} \lim_{n \to \infty} (\lambda_{k_n}, u_{k_n}) = (\mu - m, 0) \quad \text{and} \quad \lim_{n \to \infty} \frac{u_{k_n}}{\|u_{k_n}\|_E} = w \in \mathit{span}\{v\}.$$

Hence, setting $u_{k_n} = \alpha_{k_n} v + y_{k_n}$, we have

$$\lim_{n \to \infty} \frac{|\alpha_{k_n}|}{\|u_{k_n}\|_E} = \lim_{n \to \infty} \left| \left\langle v^*, \frac{u_{k_n}}{\|u_{k_n}\|_E} \right\rangle \right| = 1.$$

Therefore

$$\lim_{n \to \infty} \frac{y_{k_n}}{|\alpha_{k_n}|} = \lim_{n \to \infty} \left(\frac{y_{k_n}}{\|u_{k_n}\|_E} \cdot \frac{\|u_{k_n}\|_E}{|\alpha_{k_n}|} \right) = \lim_{n \to \infty} \frac{u_{k_n} - \alpha_{k_n} v}{\|u_{k_n}\|_E} = 0.$$

This achieves the proof, since the computation above is true for every sequence (λ_n, u_n).

Corollary 4. *There exists a continuum $\breve{\mathcal{C}} \subset \mathcal{C}$ meeting $[\mu - M, \mu + M] \times \{0\}$ and $[\mu - M, \mu + M] \times \partial B(0, \delta)$ such that*

$$\breve{\mathcal{C}} \subset Q_\eta^+ \cup \big([\mu - M, \mu + M] \times \{0\}\big) \quad \mathit{or} \quad \breve{\mathcal{C}} \subset Q_\eta^- \cup \big([\mu - M, \mu + M] \times \{0\}\big).$$

Corollary 5. *Under the assumption* (8)*, $\mathcal{C} = \mathcal{C}_{\mu - m^*}$ can be decomposed into two subcontinua $\mathcal{C}^+$, $\mathcal{C}^-$ such that*

$$\mathcal{C}^+ \subset Q_\eta^+ \cup \{(\mu - m^*, 0)\} \quad \mathit{and} \quad \mathcal{C}^- \subset Q_\eta^- \cup \{(\mu - m^*, 0)\}.$$

Moreover,

$$(\lambda, u) \in \mathcal{C}^+(\mathcal{C}^-) \cap ([\mu - M, \mu + M] \times B(0, \delta)) \;\; \mathit{and} \;\; (\lambda, u) \ne (\mu - m^*, 0)$$

implies

$$u = \alpha v + y, \quad \mathit{where} \quad \alpha > 0 (\alpha < 0) \quad \mathit{and} \quad y = o(\alpha) \quad \mathit{at} \quad \alpha = 0.$$

The results obtained may be applied to nonlinear Sturm–Liouville boundary value problems with nondifferentiable nonlinearities [3].

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