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EDGE DECOMPOSITIONS OF MULTIGRAPHS INTO MULTI-2-PATHS

Abstract. We establish the computational time complexity of the existence problem of a decomposition of an instance multigraph into isomorphic 3-vertex paths with multiple edges. If the two edge multiplicities are distinct, the problem is NPC; if mutually equal then polynomial.

Keywords: edge decomposition, multigraph, multipath, path, time complexity.

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1. INTRODUCTION

By a graph we mean a (loopless) multigraph. Let M and H be multigraphs. By an H-*decomposition* of M we mean a partition of the edge set of M into subsets which induce graphs isomorphic to H. In this paper we contribute to the complexity of the following problem in case $H = P(k,l)$, a multipath with two edges whose multiplicities are k and l, $k \geq l \geq 1$.

Problem $\mathcal{P}_H$:
INSTANCE: a multigraph M.
QUESTION: Is there an H-decomposition of M?

In case that H is a simple graph we use the symbol $\tilde{\mathcal{P}}_H$ to denote the restriction of problem $\mathcal{P}_H$ to graphs as instances. Only recently efforts of many researchers have resulted in the following remarkable result.

Theorem 0. *If H is a simple graph and H has component of size at least 3 then the problem $\tilde{\mathcal{P}}_H$ is NP-complete. Otherwise it is polynomial.*

Holyer is the first who showed the NPC part of this theorem for special H. His conjecture [8] that size of H makes the problem hard (i.e. NPC) is false. Nevertheless, he claims that NP-completeness "seems difficult to prove" if , e.g., $H = 3K_2$. This, if at all possible, is really so because polynomiality for $H = 3K_2$ is proved by Bialostocki and Roditty [5]. The NPC part is eventually completed by Dor and Tarsi [7].

The first contribution to the polynomial part is perhaps an old result by Kotzig [13] of 1957, published in Slovak and only recently referred to by Asratian and Oksimets [1].

Theorem (of Kotzig; see also [11, 12, 6]). *A connected simple graph is P_3--decomposable iff the graph has even number of edges.*

The result was rediscovered over 20 years later in late seventies [11, 12, 6]. Surprisingly enough in Bosák's monograph [4] the result is not accredited to Kotzig though Kotzig's article [13] is referred to in another context. At the end of the paper we give a brief sketch of Kotzig's elegant proof of his result.

Concluding is just the recent contribution by Bryś and Lonc [3] (announced in [2]) which establishes polynomiality of $\tilde{\mathcal{P}_H}$ in case H is a disjoint union of a matching and two or more paths P_3 on 3 vertices, $H = sP_3 \cup tK_2$, with $st > t \geq 1$ just to complete former contributions, see references in [7, 3] for such contributions.

The problem of establishing the complexity status of $\mathcal{P}_H$ (for multigraphs) remains unsolved. The polynomiality of $\mathcal{P}_H$ is known when H has two edges and is (besides C_2) either the 2-matching $2K_2$ (Skupień [18]) or the path P_3 (Ivančo *et al.* [9]), or H is the 3-matching $3K_2$ (Lonc *et al.* [15]). In all these cases as well as if mixed decomposition parts are allowed as in [10] (where single H is replaced by a subset $\mathcal{H}$ of $\{C_2, 2K_2, P_3\}$) good characterizations of decomposable multigraphs are provided. On the other hand, NP-completeness for $H = P(2, 1)$, the simplest of 'true' multipaths $P(k, l)$, was proved by the first author (J.K.) in June 1996 a few days after the question [17] was posed. The general result, presented below, was essentially proved in 1998. In the meantime a generalization has been found by Priesler [16].

By $P(k, l)$ we mean a multipath of length 2 whose multiplicities of edges are k and l. We show that the problem $\mathcal{P}_{P(k,l)}$ is polynomial if $k = l$ and is NPC otherwise. We include our proof as it is rather short, different from Priesler's, having some history, and relying on (generalized) matchings. The next step in studying $\mathcal{P}_H$ is just to deal with multimatchings H.

2. MULTIPATHS OF LENGTH 2

We consider the problem $\mathcal{P}_H$ for $H = P(k, l)$, a multipath of length two with edge multiplicities k and l. Note that $P(1, 1) = P_3$, the simple path on three vertices. The first related result (motivated by a problem and infinitely many examples in Skupień [18]) was shown by Ivančo, Meszka and Skupień who in [9] (submitted, revised, and accepted in 1995) proved polynomiality of the problem $\mathcal{P}_{P(1,1)}$ in the class of all multigraphs (see also Kouider *et al.* [14] for some related results). Due to Theorem of Kotzig, the problem is trivial when restricted to simple graphs. The main result of the paper is establishing the complexity status of the problem $\mathcal{P}_{P(k,l)}$.

Theorem 1. *For $H = P(k,l)$, a multipath, the problem $\mathcal{P}_H$ is NP-complete if $k \neq l$. Otherwise it is polynomial.*

Proof. If $H = P(k,k)$ then a multigraph M has an H-decomposition iff the multiplicity of every edge in M is divisible by k and M', the multigraph obtained from M by dividing the multiplicities of the edges by k, has a $P(1,1)$-decomposition. The last problem is polynomial by Kotzig's theorem.

Let $k > l$ and assume first that k and l are relatively prime. We prove NP-completeness by reduction from the following problem dDM (d-dimensional matching) which is NP-complete precisely when $d \geq 3$. □

Problem dDM:
INSTANCE: an integer q and a subset $U \subseteq X_1 \times X_2 \times \cdots \times X_d$ where X_i's are disjoint sets of cardinality q each.
QUESTION: Is there a subset $W \subseteq U$ of cardinality q such that distinct elements of W do not share any coordinate?

Let $d = k + l$. We construct a multigraph M in the following way. The set of vertices contains the set $X = \bigcup_{i=1}^{d} X_i$ together with its disjoint copy $X' = \{x' \colon x \in X\}$. For every $x \in X$ join x and x' with an edge of multiplicity k. For every $u = (x_1, x_2, \ldots, x_d) \in U$ construct a submultigraph M_u with $2d + 1$ vertices of which d are components x_i of u and remaining $d+1$ are new vertices $y_1^u, y_2^u, \ldots, y_d^u, z^u$; moreover, the edges of M_u are $x_1y_1^u, x_2y_2^u, \ldots, x_dy_d^u$ with multiplicity l and $y_1^uz^u, y_2^uz^u, \ldots, y_d^uz^u$ with multiplicity $t = l(k + l - 1)$.

Clearly this construction of M requires polynomially many steps. We prove that U contains a matching if and only if M has an H-decomposition.

Suppose there is a matching $U' \subseteq U$, $|U'| = q$. The set of edges of M can be partitioned into subsets inducing either the multigraphs M_u when $u \notin U'$ or M'_u, $M'_u := M_u + \{\text{all edges } x_ix'_i\}$, when $u = (x_1, x_2, \ldots, x_d) \in U'$.

First, observe that a multistar different from a multiple edge and with multiplicities of edges equal to $k + l$ can be decomposed into copies of $P(k,l)$.

Then every multigraph M_u has a $P(k,l)$-decomposition because after removing the copies of $P(k,l)$ containing the edges $x_iy_i^u$ of multiplicities l we end up with a multistar whose multiplicities of edges are equal to $t - k = (k + l)(l - 1)$. This multistar can be decomposed into $l - 1$ multistars with multiplicities of edges equal to $k + l$. They are $P(k,l)$-decomposable as we noticed in the preceding paragraph.

To show that each multigraph M'_u is $P(k,l)$-decomposable we first remove from M'_u all copies of $P(k,l)$ with central vertices x_i. It suffices to prove that the resulting multistar with $k + l$ edges of multiplicity t all incident to the vertex z^u is $P(k,l)$-decomposable. To this end, for each $i = 1, 2, \ldots, d-1$, form a pair of edges comprising the edge $y_i^uz^u$ with multiplicity k only and the edge $y_d^uz^u$ with multiplicity l. Each of these $d-1$ pairs of edges constitutes a copy of $P(k,l)$. The remaining edges form a multistar with $d-1$ edges, each of multiplicity $t-k=(k+l)(l-1)$. As before it is $P(k,l)$-decomposable. Thus we have shown that M has a $P(k,l)$-decomposition.

Conversely, assume that M has a $P(k,l)$-decomposition π. For any $u \in U$, the edge $y_i^u x_i$ of multiplicity l forms a copy of $P(k,l)$ in π either with $x_i x_i'$ (whose multiplicity is k) or with $y_i^u z^u$ (taken with multiplicity k). Suppose that for s values of i the first case takes place and for $d-s$ values of i the second does. Removing these members of π we obtain a connected component which is a $P(k,l)$-decomposable multistar centered at the vertex z^u. Hence its size is divisible by d $(=k+l)$, and equals $ts+(t-k)(d-s)=d(t-k)+ks \equiv ks \equiv 0 \pmod d$. However, integers $d(=k+l)$ and k are relatively prime whence $s \equiv 0 \pmod d$, and consequently $s=0$ or $s=d$. The d-tuples u for which $s=d$ form a required matching in U.

If $k>l$ and k and l have the greatest common divisor $a>1$ then M has a $P(k,l)$-decomposition if and only if the multiplicity of every edge in M is divisible by a and the multigraph obtained from M by dividing the multiplicities of edges by a has a $P(\frac{k}{a},\frac{l}{a})$-decomposition. Since $\frac{k}{a}$ and $\frac{l}{a}$ are relatively prime, this case is reducible to the previous one.

3. CONCLUDING REMARKS

Sketch of Kotzig's proof of his Theorem follows. Theorem 3 in [13] shows that a (simple) connected graph of even size admits an orientation such that each vertex has even indegree (easy; given any orientation, choose recursively a chain connecting a pair of bad vertices and reverse arc orientations on the chain). Then (Theorem 4 in [13]), because the graph is simple, we get a required edge decomposition into adjacency pairs by arbitrarily creating disjoint pairs out of all incoming edges at each vertex.

Some more historical remarks follow. The first proof of 1996 that the specialized problem $\mathcal{P}_H$ is NPC for $H=P(2,1)$, the simplest multipath, (even when restricted to planar input multigraphs) is by reduction from 1-in-3 SATISFIABILITY, a restricted 3-SATISFIABILITY. For $H=P(k,l)$, the above Theorem 1 and its proof (almost the same as stated above) was proposed and e-mailed by the 2nd author (Z.L.) in September 1998. A joint article containing also a polynomial characterization of $3K_2$-decomposable multigraphs (with rather lengthy detailed proof), completed in mid 2002, was submitted first to Graphs Combin. This has resulted in accepting an excerpt [15]. The present paper includes the remaining part together with historical complements.

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