Danuta Jaruszewska-Walczak

DIFFERENCE METHODS FOR INFINITE SYSTEMS OF HYPERBOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS ON THE HAAR PYRAMID

Abstract. We consider the Cauchy problem for infinite system of differential functional equations

$$\partial_t z_k(t,x) = f_k(t,x,z,\partial_x z_k(t,x)),\ k \in \mathbf{N}.$$

In the paper we consider a general class of difference methods for this problem. We prove the convergence of methods under the assumptions that given functions satisfy the nonlinear estimates of the Perron type with respect to functional variables. The proof is based on functional difference inequalities. We constructed the Euler method as an example of difference method.

Keywords: initial problems, infinite systems of differential functional equations, difference functional inequalities, nonlinear estimates of the Perron type.

Mathematics Subject Classification: 65M10, 65M15, 35R10.

1. INTRODUCTION

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. Let $\mathbf{N}$ and $\mathbf{Z}$ denote the sets of natural numbers and integers respectively. Denote by $\mathcal{S}^\infty$ the set of all real sequences $p = (p_k)_{k\in\mathbf{N}}$. For $p = (p_k)_{k\in\mathbf{N}} \in \mathcal{S}^\infty$, $\bar{p} = (\bar{p}_k)_{k\in\mathbf{N}} \in \mathcal{S}^\infty$ we write $|p| = (|p_k|)_{k\in\mathbf{N}}$ and $p \le \bar{p}$ if $p_k \le \bar{p}_k$ for $k \in \mathbf{N}$. For $p^{(m)} = (p_k^{(m)})_{k\in\mathbf{N}} \in \mathcal{S}^\infty$, $m \in \mathbf{N}$ and $p = (p_k)_{k\in\mathbf{N}} \in \mathcal{S}^\infty$ we put $\lim_{m\to\infty} p^{(m)} = p$ if $\lim_{m\to\infty} p_k^{(m)} = p_k$ for all $k \in \mathbf{N}$. Let E be the Haar pyramid

$$E = \big\{(t,x) \in R^{1+n} \colon t \in [0,a),\ x \in [-b+Mt,\, b-Mt]\big\}$$

where $a > 0$, $b = (b_1, \dots, b_n)$, $M = (M_1, \dots, M_n) \in R_+^n$, $R_+ = [0, +\infty)$ and $b_i > M_i a$, $i = 1, \dots, n$. Write

$$E_t = \big(([-r_0, 0] \times [-b, b]) \cup E \big) \cap ([-r_0, t] \times R^n), \quad 0 \le t < a,$$

and

$$S_t = [-b, b] \text{ for } t \in [-r_0, 0], \quad S_t = [-b + Mt,\, b - Mt] \text{ for } t \in (0, a),$$

where $r_0 \in R_+$. Let $\Omega = E \times C(E_0 \cup E, \mathcal{S}^\infty) \times R^n$ and suppose that $f \colon \Omega \to \mathcal{S}^\infty$, $f = (f_k)_{k \in \mathbf{N}}$, and $\varphi \colon E_0 \to \mathcal{S}^\infty$ are given functions. For a function $z \colon E_0 \cup E \to \mathcal{S}^\infty$, $z = (z_k)_{k \in \mathbf{N}}$, and for a point $(t, x) \in E$ we write

$$\partial_t z(t, x) = \big(\partial_t z_k(t, x) \big)_{k \in \mathbf{N}}, \quad f(t, x, z, \partial_x z(t, x)) = \big(f_k(t, x, z, \partial_x z_k(t, x)) \big)_{k \in \mathbf{N}}$$

where $\partial_x z_k = (\partial_{x_1} z_k, \dots, \partial_{x_n} z_k)$, $k \in \mathbf{N}$. We consider the Cauchy problem

$$\partial_t z(t, x) = f(t, x, z, \partial_x z(t, x)), \tag{1}$$

$$z(t, x) = \varphi(t, x) \text{ on } E_0. \tag{2}$$

A function $u \colon E_0 \cup E \to \mathcal{S}^\infty$, $u = (u_k)_{k \in \mathbf{N}}$, is called a classical solution of problem (1), (2) if:

(i) u_k is continuous on $E_0 \cup E$ and it is class C^1 on E for all $k \in \mathbf{N}$,

(ii) u satisfies (1) on E and initial condition (2) holds.

The function $f \colon \Omega \to \mathcal{S}^\infty$ is said to satisfy the Volterra condition if for each $(t, x) \in E$, $q = (q_1, \dots, q_n) \in R^n$ and for z, $\bar{z} \in C(E_0 \cup E, \mathcal{S}^\infty)$ such that $z(\tau, s) = \bar{z}(\tau, s)$ on E_t we have $f(t, x, z, q) = f(t, x, \bar{z}, q)$. Note that the Volterra condition means that the value of f at the point (t, x, z, q) depends on (t, x, q) and on the restriction of z to the set E_t. We assume that f satisfies the Volterra condition and we consider classical solutions of (1), (2).

A review of existence results for hyperbolic differential functional equations is given in [2]. Infinite systems of hyperbolic functional inequalities and conditions on uniqueness of classical solutions of problem (1), (2) are considered in [3].

In this paper we give a theorem on the convergence of the general one-step difference methods for the problem (1), (2). It is a generalization of the methods used in [4] and [5] on the case of infinite systems of differential functional equations.

2. FUNCTIONAL DIFFERENCE PROBLEMS

We formulate a difference problem corresponding to (1), (2). We denote by $\mathbf{F}(A, B)$ the class of all functions defined on A and taking values in B, where A and B are arbitrary sets. For x, $\bar{x} \in R^n$, $x = (x_1, \dots, x_n)$, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$, we write $x * \bar{x} = (x_1 \bar{x}_1, \dots, x_n \bar{x}_n)$. We define a mesh on the set $E_0 \cup E$ in the following way.

Suppose that $(h_0, h') = (h_0, h_1, \ldots, h_n)$ stand for steps of the mesh. For $h = (h_0, h')$ and $(i, m) \in \mathbf{Z}^{1+n}$ where $m = (m_1, \ldots, m_n)$ we define nodal points as follows:

$$t^{(i)} = ih_0, \quad x^{(m)} = \left(x_1^{(m_1)}, \ldots, x_n^{(m_n)}\right) = m * h'.$$

Denote by Δ the set of all $h = (h_0, h')$, $h_i > 0$, $0 \le i \le n$, such that there are $N = (N_1, \ldots, N_n) \in \mathbf{N}^n$, $\tilde{N}_0 \in \mathbf{Z}$ with the properties: $\tilde{N}_0 h_0 = r_0$ and $N * h' = b$. We assume that $\Delta \neq \emptyset$ and that there is a sequence $\{h^{(j)}\}$, $h^{(j)} \in \Delta$, and $\lim_{j\to\infty} h^{(j)} = 0$.

Let us fix $h \in \Delta$. There is $N_0 \in \mathbf{N}$ such that $N_0 h_0 < a \le (N_0 + 1)h_0$. Let

$$R_h^{1+n} = \left\{ \left(t^{(i)}, x^{(m)}\right) : (i, m) \in \mathbf{Z}^{1+n} \right\}$$

and $E_{0.h} = E_0 \cap R_h^{1+n}$, $E_h = E \cap R_h^{1+n}$. We assume that $h' \le h_0 M$. For a function $z\colon E_{0.h} \cup E_h \to \mathcal{S}^\infty$ or for a function $z\colon E_0 \cup E \to \mathcal{S}^\infty$ we write $z^{(i,m)} = z\left(t^{(i)}, x^{(m)}\right)$. Put

$$E_{i.h} = \left\{ \left(t^{(j)}, x^{(m)}\right) \in E_{0.h} \cup E_h : j \le i \right\}.$$

and

$$E'_h = \left\{ \left(t^{(i)}, x^{(m)}\right) \in E_h : \left(t^{(i)} + h_0, x^{(m)}\right) \in E_h \right\}.$$

The motivation for the definition of the set E'_h is the following. Approximate solutions of problem (1), (2) are functions u_h defined on E_h. We will write a difference system generated by (1) at each point of the set E'_h. It follows from condition $h' \le h_0 M$ that we calculate all the values of u_h on E_h.

For $1 \le j \le n$ we write $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n$, 1 standing on j-th place. We define the difference operators δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ in the following way:

$$\delta_0 w^{(i,m)} = \frac{1}{h_0}\left(w^{(i+1,m)} - w^{(i,m)}\right), \tag{3}$$

$$\delta_j w^{(i,m)} = \frac{1}{h_j}\left(w^{(i,m)} - w^{(i,m-e_j)}\right) \text{ for } 1 \le j \le \kappa, \tag{4}$$

$$\delta_j w^{(i,m)} = \frac{1}{h_j}\left(w^{(i,m+e_j)} - w^{(i,m)}\right) \text{ for } \kappa + 1 \le j \le n, \tag{5}$$

where $1 \le \kappa \le n$ is a fixed natural number and $w\colon E_{0.h} \cup E_h \to R$. Let $\Omega_h = E'_h \times \mathbf{F}(E_{0.h} \cup E_h, \mathcal{S}^\infty) \times R^n$ and suppose that $f_h\colon \Omega_h \to \mathcal{S}^\infty$, $f_h = (f_{h.k})_{k\in\mathbf{N}}$, and $\varphi_h\colon E_{0.h} \to \mathcal{S}^\infty$ are given functions. For a point $\left(t^{(i)}, x^{(m)}\right) \in E'_h$ we define

$$\delta_0 z^{(i,m)} = \left(\delta_0 z_k^{(i,m)}\right)_{k\in\mathbf{N}}, \; f_h\left(t^{(i)}, x^{(m)}, z, \delta z^{(i,m)}\right) = \left(f_{h.k}(t^{(i)}, x^{(m)}, z, \delta z_k^{(i,m)})\right)_{k\in\mathbf{N}}.$$

We will approximate solutions of problem (1), (2) by means of solutions of the problem

$$\delta_0 z^{(i,m)} = f_h\left(t^{(i)}, x^{(m)}, z, \delta z^{(i,m)}\right), \tag{6}$$

$$z^{(i,m)} = \varphi_h^{(i,m)} \text{ on } E_{0.h} \tag{7}$$

where $z = (z_k)_{k\in\mathbf{N}}$. The function f_h is said to satisfy the Volterra condition if for each $(t^{(i)}, x^{(m)}) \in E'_h$, $q \in R^n$ and for $z,\ \bar{z} \in \mathbf{F}(E_{0.h} \cup E_h, \mathcal{S}^\infty)$ such that $z = \bar{z}$ on $E_{i.h}$ we have $f_h\big(t^{(i)}, x^{(m)}, z, q\big) = f_h\big(t^{(i)}, x^{(m)}, \bar{z}, q\big)$.

If f_h satisfies the Volterra condition then relation $h' \leq h_0 M$ implies that there exists exactly one solution $u_h\colon E_{0.h} \cup E_h \to \mathcal{S}^\infty$ of problem (6), (7).

We formulate general conditions for the convergence of the method (6), (7). Let $I = [0, a)$ and

$$I_h = \{t^{(i)}\colon 0 \leq i \leq N_0\},\ I'_h = I_h \setminus \{t^{(N_0)}\},\ I_{0.h} = \{t^{(i)}\colon -\tilde{N}_0 \leq i \leq 0\}.$$

For $\omega \in \mathbf{F}(I_{0.h} \cup I_h, \mathcal{S}^\infty)$ we write $\omega^{(i)} = \omega(t^{(i)})$.

Now we formulate the main assumptions on f_h. First we introduce the following one.

Assumption H $[\sigma_h]$. Suppose that $\sigma_h\colon I'_h \times \mathbf{F}(I_{0.h} \cup I_h, \mathcal{S}^\infty_+) \to \mathcal{S}^\infty_+$ where $\mathcal{S}^\infty_+ = \{p = (p_k)_{k\in\mathbf{N}} \in \mathcal{S}^\infty\colon p_k \geq 0,\ k \in \mathbf{N}\}$, and σ_h satisfies the conditions:

1) σ_h is nondecreasing with respect to the functional variable and fulfils the Volterra condition;

2) $\sigma_h(t, \theta_h) = \mathbf{0}$ for $t \in I'_h$ where $\theta_h^{(i)} = \mathbf{0}$ for $-\tilde{N}_0 \leq i \leq N_0$ and the difference problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0 \sigma_h\left(t^{(i)}, \eta\right) \text{ for } 0 \leq i \leq N_0 - 1, \tag{8}$$

$$\eta^{(i)} = 0 \text{ for } -\tilde{N}_0 \leq i \leq 0, \tag{9}$$

is stable in the following sense: if η_h is a solution of the problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0 \sigma_h\left(t^{(i)}, \eta\right) + h_0 \gamma(h) \quad \text{for} \quad 0 \leq i \leq N_0 - 1, \tag{10}$$

$$\eta^{(i)} = \alpha_0(h) \quad \text{for} \quad -\tilde{N}_0 \leq i \leq 0 \tag{11}$$

where $\gamma,\ \alpha_0\colon \Delta \to \mathcal{S}^\infty_+$ and $\lim_{h\to 0} \gamma(h) = \mathbf{0}$, $\lim_{h\to 0} \alpha_0(h) = \mathbf{0}$, then there exists a function $\beta\colon \Delta \to \mathcal{S}^\infty_+$ such that $\eta_h^{(i)} \leq \beta(h)$ for $0 \leq i \leq N_0$ and $\lim_{h\to 0} \beta(h) = \mathbf{0}$.

In the sequel we will need the following operator

$$V_h\colon \mathbf{F}(E_{0.h} \cup E_h, \mathcal{S}^\infty) \to \mathbf{F}(I_{0.h} \cup I_h, \mathcal{S}^\infty_+).$$

If $z\colon E_{0.h}\cup E_h \to \mathcal{S}^\infty$, $z=(z_k)_{k\in\mathbf{N}}$, then $V_h z=(V_h z_k)_{k\in\mathbf{N}}$ and

$$(V_h z_k)(t^{(i)}) = \max\left\{\left|z_k^{(i,m)}\right| : \left(t^{(i)}, x^{(m)}\right) \in E_{0.h}\cup E_h\right\}, \quad -\tilde{N}_0 \le i \le N_0.$$

Assumption H $[f_h, \sigma_h]$. Suppose that $f_h = (f_{h.k})_{k\in\mathbf{N}}$ satisfies the Volterra condition and:

1) the derivatives $\partial_q f_{h.k} = (\partial_{q_1} f_{h.k}, \ldots, \partial_{q_n} f_{h.k})$ exist on Ω_h and $\partial_q f_{h.k}(t,x,z,\cdot) \in C(R^n, R^n)$ where $k\in\mathbf{N}$;
2) the estimates

$$\partial_{q_j} f_{h.k}(P) \le 0 \quad \text{for} \quad 1\le j\le \kappa, \qquad \partial_{q_j} f_{h.k}(P) \ge 0 \quad \text{for} \quad \kappa+1\le j\le n$$

and

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} \left|\partial_{q_j} f_{h.k}(P)\right| \ge 0,$$

are satisfied for $P=(t,x,z,q)\in\Omega_h$, $h\in\Delta$, $k\in\mathbf{N}$;
3) Assumption H $[\sigma_h]$ is satisfied and

$$|f_h(t,x,z,q) - f_h(t,x,\bar{z},q)| \le \sigma_h(t, V_h(z-\bar{z})) \quad \text{on} \quad \Omega_h. \tag{12}$$

Now we formulate a general theorem on the convergence of the method (6), (7).

Theorem 1. *Suppose that for all $h\in\Delta$ Assumption* H $[f_h,\sigma_h]$ *is satisfied and:*

1) *the function $u_h\colon E_{0.h}\cup E_h \to \mathcal{S}^\infty$ is a solution of problem* (6), (7) *and there exists $\alpha_0\colon \Delta\to\mathcal{S}_+^\infty$ such that $\left|\varphi^{(i,m)} - \varphi_h^{(i,m)}\right| \le \alpha_0(h)$ on $E_{0.h}$ and $\lim_{h\to 0}\alpha_0(h) = \mathbf{0}$;*
2) *the function $v\colon E_0\cup E\to\mathcal{S}^\infty$ is a classical solution of problem* (1), (2)*;*
3) *there exists a function $\tilde{\beta}\colon \Delta\to\mathcal{S}_+^\infty$ such that*

$$\left|f_h(t^{(i)}, x^{(m)}, v_h, \delta v_k^{(i,m)}) - f(t^{(i)}, x^{(m)}, v, \delta v_k^{(i,m)})\right| \le \tilde{\beta}(h) \quad \textit{on} \quad E_h', \tag{13}$$

and $\lim_{h\to 0}\tilde{\beta}(h) = \mathbf{0}$ where v_h is the restriction of the function v to the set $E_{0.h}\cup E_h$.

Then there exists a function $\tilde{\gamma}\colon\Delta\to\mathcal{S}_+^\infty$ such that

$$\left|u_h^{(i,m)} - v^{(i,m)}\right| \le \tilde{\gamma}(h) \quad \textit{on} \quad E_h \quad \textit{and} \quad \lim_{h\to 0}\tilde{\gamma}(h) = \mathbf{0}. \tag{14}$$

Proof. Let the function $\Gamma_h\colon E_h' \to \mathcal{S}^\infty$, $\Gamma_h = (\Gamma_{h.k})_{k\in\mathbf{N}}$, be defined by

$$v_k^{(i+1,m)} = v_k^{(i,m)} + h_0 f_{h.k}\left(t^{(i)}, x^{(m)}, v_h, \delta v_k^{(i,m)}\right) + h_0 \Gamma_{h.k}^{(i,m)} \quad \text{on} \quad E_h'.$$

It follows from (13) that there exists a function $\gamma\colon \Delta \to \mathcal{S}_+^\infty$, $\gamma = (\gamma_k)_{k\in\mathbf{N}}$, such that $\left|\Gamma_h^{(i,m)}\right| \le \gamma(h)$ on E'_h and $\lim_{h\to 0}\gamma(h) = \mathbf{0}$. We define $\omega_h\colon I_h \to \mathcal{S}_+^\infty$ by

$$\omega_h = V_h(v_h - u_h)$$

We prove that ω_h satisfies the difference functional inequality

$$\omega_h^{(i+1)} \le \omega_h^{(i)} + h_0\sigma_h(t^{(i)},\omega_h) + h_0\gamma(h), \quad 0 \le i \le N_0 - 1. \tag{15}$$

The following estimates hold for $(t^{(i)}, x^{(m)}) \in E'_h$

$$\left|(v_j - u_{h.j})^{(i+1,m)}\right| \le$$
$$\le \left|(v_j - u_{h.j})^{(i,m)} + h_0\Big[f_{h.j}(t^{(i)}, x^{(m)}, v_h, \delta v_j^{(i,m)}) - f_{h.j}(t^{(i)}, x^{(m)}, v_h, \delta u_{h.j}^{(i,m)})\Big]\right| +$$
$$+ h_0\left|f_{h.j}(t^{(i)}, x^{(m)}, v_h, \delta u_{h.j}^{(i,m)}) - f_{h.j}(t^{(i)}, x^{(m)}, u_h, \delta u_{h.j}^{(i,m)})\right| + h_0\gamma_j(h) =$$
$$= \Big|(v_j - u_{h.j})^{(i,m)} + h_0\sum_{\nu=1}^{\kappa}\partial_{q_\nu}f_{h.j}(P_{h.j}^{(i,m)})\frac{1}{h_\nu}[(v_j - u_{h.j})^{(i,m)} - (v_j - u_{h.j})^{(i,m-e_\nu)}] +$$
$$+ h_0\sum_{\nu=\kappa+1}^{n}\partial_{q_\nu}f_{h.j}(P_{h.j}^{(i,m)})\frac{1}{h_\nu}[(v_j - u_{h.j})^{(i,m+e_\nu)} - (v_j - u_{h.j})^{(i,m)}]\Big| +$$
$$+ h_0\sigma_{h.j}(t^{(i)}, \omega_h) + h_0\gamma_j(h), \quad j \in \mathbf{N}$$

where $P_{h.j}^{(i,m)} \in \Omega_h$ are intermediate points. It follows from condition 2) of Assumptions H $[f_h, \sigma_h]$ that

$$\left|(v_j - u_{h.j})^{(i+1,m)}\right| \le$$
$$\le \left|(v_j - u_{h.j})^{(i,m)}\left[1 - h_0\sum_{\nu=1}^{n}\left|\partial_{q_\nu}f_{h.j}(P_{h.j}^{(i,m)})\right|\frac{1}{h_\nu}\right]\right| +$$
$$+ h_0\omega_{h.j}^{(i)}\sum_{\nu=1}^{n}\frac{1}{h_\nu}\left|\partial_{q_\nu}f_{h.j}(P_{h.j}^{(i,m)})\right| + h_0\sigma_{h.j}\left(t^{(i)}, \omega_h\right) + h_0\gamma_j(h) \le$$
$$\le \omega_{h.j}^{(i)} + h_0\sigma_{h.j}(t^{(i)}, \omega_h) + h_0\gamma_j(h), \quad j \in \mathbf{N}.$$

The above estimates imply (15). It follows that the initial inequality $\omega_h^{(i)} \le \alpha_0(h)$, $-\tilde{N}_0 \le i \le 0$, holds.

Let $\eta_h\colon I_{o.h} \cup I_h \to \mathcal{S}_+^\infty$ be the solution of (10), (11). It follows from Assumption H $[\sigma_h]$ that $\omega_h^{(i)} \le \eta_h^{(i)}$ on I_h. Now we obtain the assertion of Theorem 1 from the stability of problem (8), (9). □

In Theorem 1 we have assumed that the functions

$$\mathrm{sign}\partial_q f_{h.k}(P) = \big(\mathrm{sign}\partial_{q_1}f_{h.k}(P), \ldots, \mathrm{sign}\partial_{q_n}f_{h.k}(P)\big),\ k \in \mathbf{N},$$

are constant on Ω_h. This condition can be omited if we define difference operators δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ in the following way:

$$\delta_0 w^{(i,m)} = \frac{1}{h_0}\left(w^{(i+1,m)} - \frac{1}{2n}\sum_{j=1}^{n}\Big(w^{(i,m+e_j)} + w^{(i,m-e_j)} \Big) \right) \tag{16}$$

and

$$\delta_j w^{(i,m)} = \frac{1}{2h_j}\Big(w^{(i,m+e_j)} - w^{(i,m-e_j)} \Big),\ 1 \le j \le n, \tag{17}$$

where $w\colon E_{0.h} \cup E_h \to R$. Consider the difference method (6), (7) with the above given δ_0 and δ.

Assumption $\tilde{\mathrm{H}}\ [f_h, \sigma_h]$. Suppose that the functions f_h and σ_h satisfy conditions 1) and 3) of Assumption H $[f_h, \sigma_h]$ and

$$1 - nh_0 \frac{1}{h_j} \left| \partial_{q_j} f_{h.k}(t, x, z, q) \right| \ge 0 \quad \text{on} \quad \Omega_h$$

where $1 \le j \le n$, $k \in \mathbf{N}$.

Theorem 2. *Suppose that for all* $h \in \Delta$ *Assumption* $\tilde{\mathrm{H}}[f_h, \sigma_h]$ *is satisfied and:*

1) *the function* $u_h\colon E_{0.h} \cup E_h \to \mathcal{S}^\infty$ *is a solution of problem* (6), (7) *with* δ_0 *and* δ *given by* (16), (17)*, there exists* $\alpha_0\colon \Delta \to \mathcal{S}^\infty_+$ *such that* $\left|\varphi^{(i,m)} - \varphi_h^{(i,m)}\right| \le \alpha_0(h)$ *on* $E_{0.h}$ *and* $\lim_{h\to 0} \alpha_0(h) = \mathbf{0}$*;*
2) *the function* $v\colon E_0 \cup E \to \mathcal{S}^\infty$ *is a classical solution of problem* (1), (2)*;*
3) *there exists a function* $\tilde{\beta}\colon \Delta \to \mathcal{S}^\infty_+$ *such that estimate* (13) *is satisfied and* $\lim_{h\to 0} \tilde{\beta}(h) = \mathbf{0}$.

Then there exists a function $\tilde{\gamma}\colon \Delta \to \mathcal{S}^\infty_+$ *such that conditions* (14) *are satisfied.*

Proof. Let the function $\Gamma_h\colon E'_h \to \mathcal{S}^\infty$, $\Gamma_h = (\Gamma_{h.k})_{k\in\mathbf{N}}$, be defined by

$$v_k^{(i+1,m)} = \frac{1}{2n}\sum_{j=1}^{n}\left(v_k^{(i,m+e_j)} + v_k^{(i,m-e_j)} \right) + \\ + h_0 f_{h.k}\left(t^{(i)}, x^{(m)}, v_h, \delta v_k^{(i,m)} \right) + h_0\, \Gamma_{h.k}^{(i,m)} \quad \text{on} \quad E'_h.$$

It follows from (13) that there exists a function $\gamma\colon \Delta \to \mathcal{S}^\infty_+$, $\gamma = (\gamma_k)_{k\in\mathbf{N}}$, such that $\left|\Gamma_h^{(i,m)}\right| \le \gamma(h)$ on E'_h and $\lim_{h\to 0}\gamma(h) = \mathbf{0}$. If $z_h = v_h - u_h$, $z_h = (z_{h.k})_{k\in\mathbf{N}}$ and $\omega_h = V_h(z_h)$ then the following inequalities are true on E'_h

$$\left| z_{h.k}^{(i+1,m)} \right| \le \left| \frac{1}{2n}\sum_{j=1}^{n}\left(z_{h.k}^{(i,m+e_j)} + z_{h.k}^{(i,m-e_j)} \right) + \right.$$

$$+h_0 \sum_{j=1}^{n} \partial_{q_j} f_{h.k} \left(Q_{h.k}^{(i,m)} \right) \frac{1}{2h_j} \left[z_{h.k}^{(i,m+e_j)} - z_{h.k})^{(i,m-e_j)} \right] \Bigg| +$$

$$+h_0 \sigma_{h.k}(t^{(i)}, \omega_h) + h_0 \gamma_k(h) =$$

$$= \frac{1}{2n} \Bigg| \sum_{j=1}^{n} (z_{h.k}^{(i,m+e_j)} \left(1 + nh_0 \frac{1}{h_j} \partial_{q_j} f_{h.k} \left(Q_{h.k}^{(i,m)} \right) \right) +$$

$$+ \sum_{j=1}^{n} z_{h.k}^{(i,m-e_j)} \left(1 - nh_0 \frac{1}{h_j} \partial_{q_j} f_{h.k} \left(Q_{h.k}^{(i,m)} \right) \right) \Bigg| + h_0 \sigma_{h.k}(t^{(i)}, \omega_h) + h_0 \gamma_k(h) \le$$

$$\le \omega_{h.k}^{(i)} + h_0 \sigma_{h.k}(t^{(i)}, \omega_h) + h_0 \gamma_k(h), \quad k \in \mathbf{N}$$

where $Q_{h.k}^{(i,m)} \in \Omega_h$ are intermediate points. Thus

$$\omega_{h.k}^{(i+1)} \le \omega_{h.k}^{(i)} + h_0 \sigma_{h.k}(t^{(i)}, \omega_h) + h_0 \gamma_k(h), \quad k \in \mathbf{N}$$

and the assertion of Theorem 2 follows in this same way as in the proof of Theorem 1. □

3. THE EXAMPLE OF THE DIFFERENCE SCHEME

Now we assume that $h' = h_0 M$ and we consider functional differential problem (1), (2) and the difference system

$$\delta_0 z^{(i,m)} = f \left(t^{(i)}, x^{(m)}, T_h z, \delta z^{(i,m)} \right), \tag{18}$$

with the initial condition

$$z^{(i,m)} = \varphi_h^{(i,m)} \quad \text{on} \quad E_{0.h} \tag{19}$$

where $z = (z_k)_{k \in \mathbf{N}}$, $T_h z = (T_h z_k)_{k \in \mathbf{N}}$ and $T_h \colon \mathbf{F}(E_{0.h} \cup E_h, R) \to C(E_0 \cup E, R)$ is the interpolating operator given in [1]. The operators δ_0, δ are defined by (3)–(5) where $1 \le \kappa \le n$ is given integer.

We will need the following operator $V \colon C(E_0 \cup E, \mathcal{S}^\infty) \to C([-r_0, a), \mathcal{S}_+^\infty)$. If $z \in C(E_0 \cup E, \mathcal{S}^\infty)$, $z = (z_k)_{k \in \mathbf{N}}$ then $Vz = (Vz_k)_{k \in \mathbf{N}}$ and

$$(Vz_k)(t) = \max\{|z_k(t,x)| : (t,x) \in E_0 \cup E\}, \ -r_0 \le t < a, \ k \in \mathbf{N}.$$

Assumption H $[f, \sigma]$. Suppose that the function $f \colon \Omega \to \mathcal{S}^\infty$, $f = (f_k)_{k \in \mathbf{N}}$, is continuous, it satisfies the Volterra condition and:

1) there exists a continuous function $\sigma \colon R_+ \times C([-r_0, a), \mathcal{S}_+^\infty) \to \mathcal{S}_+^\infty$ such that:
   (i) σ is nondecreasing with respect to both variables, satisfies the Volterra condition and $\sigma(t, \theta) = \mathbf{0}$ for $t \in R_+$ where $\theta(t) = \mathbf{0}$ on $[-r_0, a)$,

(ii) the problem

$$\omega'(t) = \sigma(t, \omega), \quad \omega(t) = \mathbf{0} \quad \text{on} \quad [-r_0, 0]$$

is stable and $\bar{\omega}(t) = \mathbf{0}$ for $t \in R_+$ is the maximum solution of it;

2) for $(t, x, q) \in E \times R^n$, z, $\bar{z} \in C(E_0 \cup E, \mathcal{S}^\infty)$ we have

$$|f(t, x, z, q) - f(t, x, \bar{z}, q)| \leq \sigma\big(t, V(z - \bar{z})\big); \tag{20}$$

3) the derivatives $\partial_q f_k = (\partial_{q_1} f_k, \ldots, \partial_{q_n} f_k)$ exist on Ω and $\partial_q f_k(t, x, z, \cdot) \in \in C(R^n, R^n)$ where $k \in \mathbf{N}$;

4) the estimates

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} \left|\partial_{q_j} f_k(P)\right| \geq 0 \tag{21}$$

and

$$\partial_{q_j} f_k(P) \leq 0 \quad \text{for } 1 \leq j \leq \kappa, \tag{22}$$

$$\partial_{q_j} f_k(P) \geq 0 \quad \text{for } \kappa + 1 \leq j \leq n \tag{23}$$

are satisfied for $P = (t, x, z, q) \in \Omega$, $h \in \Delta$ and $k \in \mathbf{N}$.

Theorem 3. *Suppose that Assumption* H $[f, \sigma]$ *is satisfied and:*

1) *for $h \in \Delta$ the function $u_h \colon E_{0.h} \cup E_h \to \mathcal{S}^\infty$ is a solution of problem* (18), (19) *with δ_0, δ given by* (3)–(5)*;*

2) *$v \colon E_0 \cup E \to \mathcal{S}^\infty$, $v = (v_k)_{k \in \mathbf{N}}$, is a solution of* (1), (2), *v_k is of class C^1 on $E_0 \cup E$ and it is of class C^2 on E for all $k \in \mathbf{N}$;*

3) *the derivatives of the second order of v_k are bounded on E, $k \in \mathbf{N}$;*

4) *there exists a function $\alpha_0 \colon \Delta \to \mathcal{S}^\infty_+$ such that $\left|\varphi_h^{(i,m)} - \varphi^{(i,m)}\right| \leq \alpha_0(h)$ on $E_{0.h}$ and $\lim_{h \to 0} \alpha_0(h) = \mathbf{0}$.*

Then there is a number $\varepsilon_0 > 0$ and a function $\tilde{\gamma} \colon \Delta \to \mathcal{S}^\infty_+$ such that we have for $|h| < \varepsilon_0$

$$\left|u_h^{(i,m)} - v^{(i,m)}\right| \leq \tilde{\gamma}(h) \quad \textit{on } E_h \tag{24}$$

and $\lim_{h \to 0} \tilde{\gamma}(h) = \mathbf{0}$.

Proof. We prove that the function

$$f_h(t, x, z, q) = f(t, x, T_h z, q), \quad (t, x, z, q) \in \Omega_h,$$

satisfies all the assumptions of Theorem 1.

Let $L_{h_0}\colon \mathbf{F}(I_{0.h}\cup I_h,\mathcal{S}^\infty)\to C([-r_0,a),\mathcal{S}^\infty)$ be the operator given for $\eta\colon I_{0.h}\cup \cup I_h\to\mathcal{S}^\infty$, $\eta=(\eta_k)_{k\in\mathbf{N}}$, by $L_{h_0}\eta=(L_{h_0}\eta_k)_{k\in\mathbf{N}}$ and

$$(L_{h_0}\eta_k)(t)=\eta_k^{(i+1)}\frac{t-t^{(i)}}{h_0}+\eta_k^{(i)}\left(1-\frac{t-t^{(i)}}{h_0}\right)\quad\text{for } t^{(i)}\le t\le t^{(i+1)}$$

where $-\tilde{N}_0\le i\le N_0-1$ and

$$(L_{h_0}\eta_k)(t)=(L_{h_0}\eta_k)(t^{(N_0)})\quad\text{for}\quad t^{(N_0)}\le t<a.$$

Define $\sigma_h\colon I'_h\times\mathbf{F}(I_{0.h}\cup I_h,\mathcal{S}^\infty_+)\to\mathcal{S}^\infty_+$ by

$$\sigma_h(t,w)=\sigma(t,L_{h_0}w). \tag{25}$$

Assumption (20) implies the estimates

$$\begin{aligned}|f_h(t,x,z,q)-f_h(t,x,\bar{z},q)|&=|f(t,x,T_hz,q)-f(t,x,T_h\bar{z},q)|\le\\ &\le\sigma\big(t,V(T_hz-T_h\bar{z})\big)=\sigma_h(t,V_h(z-\bar{z}))\quad\text{on }\Omega_h,\end{aligned}$$

which proves condition (12). It follows that the consistency condition (13) is satisfied (see Lemma 3.5 in [1]). We prove that the difference problem

$$\eta^{(i+1)}=\eta^{(i)}+h_0\sigma_h(t^{(i)},\eta)\quad\text{for}\quad 0\le i\le N_0-1, \tag{26}$$

$$\eta^{(i)}=0\quad\text{for}\quad-\tilde{N}_0\le i\le 0, \tag{27}$$

is stable in the sense of Assumption H $[\sigma_h]$. Let $\eta_h\colon I_{0.h}\cup I_h\to\mathcal{S}^\infty_+$ be a solution of problem

$$\eta^{(i+1)}=\eta^{(i)}+h_0\sigma_h(t^i,\eta)+h_0\gamma(h)\quad\text{for } 0\le i\le N_0-1, \tag{28}$$

$$\eta^{(i)}=\alpha_0(h)\quad\text{for }-\tilde{N}_0\le i\le 0, \tag{29}$$

where γ, $\alpha_0\colon\Delta\to\mathcal{S}^\infty_+$ and $\lim_{h\to 0}\gamma(h)=\mathbf{0}$, $\lim_{h\to 0}\alpha_0(h)=\mathbf{0}$. The above problem is equivalent to

$$\eta^{(i+1)}=\eta^{(i)}+h_0\sigma(t^i,L_{h_0}\eta)+h_0\gamma(h)\quad\text{for } 0\le i\le N_0-1, \tag{30}$$

$$\eta^{(i)}=\alpha_0(h)\quad\text{for }-\tilde{N}_0\le i\le 0. \tag{31}$$

Let $\omega_h\colon[-r_0,a)\to\mathcal{S}^\infty_+$, $\omega_h=(\omega_{h.k})_{k\in\mathbf{N}}$, be the maximum solution of the problem

$$\omega'(t)=\sigma(t,\omega)+\gamma(h),\quad\omega(t)=\alpha_0(h)\quad\text{for } t\in[-r_0,0].$$

There exists $\varepsilon_0>0$ such that the solution ω_h is defined on $[0,a)$ for $|h|<\varepsilon_0$ and

$$\lim_{h\to 0}\omega_h(t)=\mathbf{0}\quad\text{uniformly on }[0,a).$$

The functions $\omega_{h.k}$, $k \in \mathbf{N}$, are convex on $[0,a)$, therefore we have

$$\omega_h^{(i+1)} \geq \omega_h^{(i)} + h_0\sigma\left(t^{(i)}, \omega_h\right) + h_0\gamma(h) \quad \text{for } 0 \leq i \leq N_0 - 1.$$

Since η_h satisfies conditions (30), (31) and $\eta_h^{(i)} = \omega_h^{(i)}$, $-\tilde{N}_0 \leq i \leq 0$ then we have $\eta_h^{(i)} \leq \omega_h^{(i)}$ for $0 \leq i \leq N_0$, which completes the proof of the stability of problem (26), (27). It follows from Theorem 1 that there is $\tilde{\gamma}\colon \Delta \to \mathcal{S}_+^\infty$ such that estimation (24) is satisfied and $\lim_{h\to 0}\tilde{\gamma}(h) = \mathbf{0}$. This proves the Theorem 3. □

Now we consider the equation 18 with the initial condition 19 where the operators δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ are given by (16), (17).

Assumption H' $[f,\sigma]$. Suppose that conditions 1)–3) of Assumption H $[f,\sigma]$ are satisfied and the estimates

$$1 - nh_0\frac{1}{h_j}\left|\partial_{q_j} f_k(t,x,z,q)\right| \geq 0, \quad \text{for } 1 \leq j \leq n,\ k \in \mathbf{N},\ h \in \Delta, \tag{32}$$

hold on Ω.

Theorem 4. *Suppose that Assumption* H' $[f,\sigma]$ *is satisfied and:*

1) *the function* $u_h\colon E_{0.h} \cup E_h \to \mathcal{S}^\infty$ *is a solution of problem* (18), (19) *with* δ_0, δ *given by* (16), (17)*;*
2) *conditions 2)–4) of Theorem 3 hold.*

Then there is a number $\varepsilon_0 > 0$ *and a function* $\tilde{\gamma}\colon \Delta \to \mathcal{S}_+^\infty$ *such that we have for* $|h| < \varepsilon_0$

$$\left|u_h^{(i,m)} - v^{(i,m)}\right| \leq \tilde{\gamma}(h) \quad \text{on } E_h \tag{33}$$

and $\lim_{h\to 0}\tilde{\gamma}(h) = \mathbf{0}$.

The proof of the above Theorem is analogous to Theorem 3. Details are omitted.

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Danuta Jaruszewska-Walczak
dana@math.univ.gda.pl

Uniwersytet Gdański
Instytut Matematyki
ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

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