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This paper is dedicated to the memory of 95-th birthday
and 10-th death anniversaries of the mathematics and physics luminary
of the former century academician **Nikolay Nikolayevich Bogoliubov**

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THE GENERAL DIFFERENTIAL-GEOMETRIC STRUCTURE OF MULTIDIMENSIONAL DELSARTE TRANSMUTATION OPERATORS IN PARAMETRIC FUNCTIONAL SPACES AND THEIR APPLICATIONS IN SOLITON THEORY. PART 2

Abstract. The structure properties of multidimensional Delsarte transmutation operators in parametric functional spaces are studied by means of differential-geometric tools. It is shown that kernels of the corresponding integral operator expressions depend on the topological structure of related homological cycles in the coordinate space. As a natural realization of the construction presented we build pairs of Lax type commutive differential operator expressions related via a Darboux–Backlund transformation having a lot of applications in soliton theory. Some results are also sketched concerning theory of Delsarte transmutation operators for affine polynomial pencils of multidimensional differential operators.

Keywords: Delsarte transmutation operators, parametric functional spaces, Darboux transformations, inverse spectral transform problem, soliton equations, Zakharov–Shabat equations, polynomial operator pencils.

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1. INTRODUCTION

Consider the Banach space $\mathcal{H} := L_2(l; H)$, $H := L_2(\mathbb{R}^m; \mathbb{C}^N)$, with the natural semi-linear scalar form on $\mathcal{H}^* \times \mathcal{H}$:

$$((\langle \varphi, \psi \rangle))_{\mathcal{H}} := \int_l dt \int_{\mathbb{R}^m} dx \bar{\varphi}^{\intercal}(t; x)\psi(t; x), \tag{1}$$

where $(\varphi, \psi) \in \mathcal{H} \times \mathcal{H}$, $t \in l := [0, T) \in \mathbb{R}_+$ is an evolution parameter, $N \in \mathbb{Z}_+$, "$-$" is the complex conjugation and the sign "$\intercal$" means the usual matrix transposition.

Take now a pair of closed dense subspaces $\mathcal{H}_0$ and $\tilde{\mathcal{H}}_0$ in $\mathcal{H}$ and two linear differential operators of equal order $\mathcal{L} := \frac{\partial}{\partial t} - L$ and $\tilde{\mathcal{L}} := \frac{\partial}{\partial t} - \tilde{L}$ from $\mathcal{H}$ to $\mathcal{H}$, where

$$L := \sum_{|\alpha|=0}^{n(L)} a_\alpha(t;x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \quad \widetilde{L} := \sum_{|\beta|=0}^{n(\tilde{L})} \tilde{a}_\beta(t;x)\frac{\partial^{|\beta|}}{\partial x^\beta}, \tag{2}$$

$(t;x) \in l \times \mathbb{R}^m$ and coefficients a_α, $\tilde{a}_\beta \in C^1(\mathbb{R}; S(\mathbb{R}^m; End\mathbb{C}^N))$ for all $|\alpha|$, $|\beta| = \overline{0,n}$, $n(L) := n =: n(\tilde{L})$.

Definition 1 (J. Delsarte and J. Lions [2]). *A linear invertible operator $\mathbf{\Omega}$ defined on the whole $\mathcal{H}$ and acting from $\mathcal{H}_0$ onto $\tilde{\mathcal{H}}$ is called a Delsarte transmutation operator for the pair of differential operators L and $\tilde{L}$, if the following two conditions hold:*

1) *the operator $\mathbf{\Omega}$ and its inverse $\mathbf{\Omega}^{-1}$ are continuous in $\mathcal{H}$;*
2) *the operator identity $\mathbf{\Omega}L = \tilde{L}\mathbf{\Omega}$ is satisfied.*

Such transmutation operators were for the first time introduced in [1, 2] for the case of one-dimensional second order differential operators. In particular, for the Sturm–Liouville and Dirac operators the complete structure of the corresponding Delsarte transmutation operators was described in [3, 4], where also the extensive applications to spectral theory were given. A special generalization of the Delsarte transmutation operator for two-dimensional Dirac operators was done for the first type in [5], where its applications to inverse scattering theory and solving some nonlinear two-dimensional evolution equations were also presented.

Recently some progress in this direction was made in [6, 7] due to analyzing a special operator structure of Darboux type transformation which appeared in [8]. In this work we describe the general differential-geometric and topological structure of multi-dimensional Delsarte type transmutation operators for differential expressions like (2) acting in parametric functional spaces, by means of the differential-geometric approach devised in [6, 7] and discuss some of their applications to Darboux–Backlund transformations and soliton theory.

2. THE DIFFERENTIAL-GEOMETRIC STRUCTURE OF THE GENERALIZED LAGRANGIAN IDENTITY

Take a multi-dimensional differential operator $\mathcal{L} := L - \partial/\partial t \colon \mathcal{H} \longrightarrow \mathcal{H}$ given above and write down its formally adjoint expression as

$$\mathcal{L}^*(t;x|\partial) := \sum_{|\alpha|=0}^{n(L)} (-1)^{|\alpha|}\frac{\partial^{|\alpha|}}{\partial x^\alpha} \cdot \bar{a}_\alpha^\intercal(t;x) + \partial/\partial t \tag{3}$$

with $(t;x) \in l \times \mathbb{R}^m$.

Consider the following easily derivable generalized Lagrangian identity in:

$$\langle \varphi, \mathcal{L}\psi\rangle - \langle \mathcal{L}^*\varphi, \psi\rangle = \sum_{i=1}^{m}(-1)^{i+1}\frac{\partial}{\partial x_i}Z_i[\varphi,\psi] - \frac{\partial}{\partial t}\big(\bar{\varphi}^{\intercal}(x)\psi(x)\big) \tag{4}$$

where for any pair $(\varphi, \psi) \in D(\mathcal{L}^*) \times D(\mathcal{L})$ from a dense domain $D(\mathcal{L}^*) \times D(\mathcal{L}) \subset \subset \mathcal{H}^* \times \mathcal{H}$ the mappings $Z_i[\varphi,\psi]\colon \mathcal{H}^* \times \mathcal{H} \to \mathbb{C}$, $i = \overline{1,m}$, are semilinear for each $(t;x) \in l \times \mathbb{R}^m$.

The Lagrangian expression (4) can be analyzed effectively by means of the following differential-geometric construction: having multiplied (4) by the oriented Lebesgue measure $dt \wedge dx$, $dx := (\bigwedge_{j=\overrightarrow{1,m}} dx_j)$, we easily obtain that

$$\left(\left\langle \varphi, L\psi - \frac{\partial\varphi}{\partial t}\right\rangle - \left\langle L^*\varphi + \frac{\partial\varphi}{\partial t}, \psi\right\rangle\right) dt \wedge dx = dZ^{(m)}[\varphi,\psi], \tag{5}$$

where $Z^{(m)}[\varphi,\psi] \in \Lambda^m(\mathbb{R}^{1+m};\mathbb{C})$ is a differential m-form on $\mathbb{R} \times \mathbb{R}^m$ given by the expression

$$\begin{aligned} Z^{(m)}[\varphi,\psi] = \sum_{i=1}^{m} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{i-1} \wedge Z_i[\varphi,\psi]dx_{i+1} \wedge \cdots \\ \cdots \wedge dx_m - \bar{\varphi}^{\intercal}(x;t)\psi(x;t)dx. \end{aligned} \tag{6}$$

Take now a pair $\big(\varphi(\lambda), \psi(\mu)\big) \in \mathcal{H}_0^{\circledast} \times \mathcal{H}_0$ with λ, $\mu \in \Sigma$, where $\Sigma \subset \mathbb{C}$ is some "spectral" space of parameters, $\mathcal{H}_0^{\circledast}$ and $\mathcal{H}_0 \subset \mathcal{H}$ are the corresponding closed subspaces of $\mathcal{H}^*$ and $\mathcal{H}$, being defined as solutions to the following evolution equations:

$$\partial\psi/\partial t = L\psi, \qquad \partial\varphi/\partial t = -L^*\varphi \tag{7}$$

with Cauchy data $\psi|_{t=t_0} = \bar{\psi}_\lambda \in L_2(\mathbb{R}^m;\mathbb{C}^N)$ and $\varphi|_{t=t_0} = \bar{\varphi}_\mu \in L_2(\mathbb{R}^m;\mathbb{C}^N)$ for λ, $\mu \in \Sigma$ at $t_0 \in l$, being fixed, $\psi|_\Gamma = 0$ and $\varphi|_\Gamma = 0$ for some chosen piece-wise smooth hypersurface $\Gamma \subset \mathbb{R}^m$.

Having assumed that linear differential equations (7) are solvable for all $t \in \in [t_0, T)$, $t_0 < T \in \mathbb{R}_+$, we can obtain right away from (5) and (6) that the differential m-form $Z^{(m)}[\varphi,\psi](\eta|\xi) \in \Lambda^m(\mathbb{R}^{1+m};\mathbb{C})$ is closed for any η, $\xi \in \Sigma$ and $(\varphi,\psi) \in \in \mathcal{H}_0^{\circledast} \times \mathcal{H}_0$. Thereby, due to the well known Poincare lemma [12] one can state that there exists an $(m-1)$-differential form $\Omega^{(m-1)}[\varphi,\psi](\eta|\xi) \in \Lambda^{m-1}(\mathbb{R}^{1+m};\mathbb{C})$ satisfying the equality

$$Z^{(m)}[\varphi,\psi](\eta|\xi) = d\Omega^{(m-1)}[\varphi,\psi](\eta|\xi) \tag{8}$$

since $d^2 \equiv 0$ on the Grassmann algebra $\Lambda(\mathbb{R}^{1+m};\mathbb{C})$ of differential forms on $\mathbb{R} \times \mathbb{R}^m$. Take now an arbitrary m-dimensional piecewise smooth hyper-surface $S\left(\sigma^{(m-1)}_{(x,t)}, \sigma^{(m-1)}_{(x_0,t_0)}\right) \subset \mathbb{R} \times \mathbb{R}^m$ spanning some two $(m-1)$-dimensional homological cycles $\sigma^{(m-1)}_{(x,t)}$, $\sigma^{(m-1)}_{(x_0,t_0)}$ marked by points (x,t) and $(x_0,t_0) \in \mathbb{R}^m \times l$, in such a way that $\partial S\left(\sigma^{(m-1)}_{(x,t)},\sigma^{(m-1)}_{(x_0,t_0)}\right)=\sigma^{(m-1)}_{(x,t)}-\sigma^{(m-1)}_{(x_0,t_0)}$ and related in some way with the chosen above hypersurface $\Gamma\subset\mathbb{R}^m$. Then one gets from (8) that due to the Stokes theorem [12]

$$
\begin{aligned}
&\int\limits_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)},\sigma^{(m-1)}_{(x_0,t_0)}\right)} Z^{(m)}[\varphi,\psi](\eta|\xi)=\\
&\qquad=\int\limits_{\sigma^{(m-1)}_{(x,t)}}\Omega^{(m-1)}[\varphi,\psi](\eta|\xi)-\int\limits_{\sigma^{(m-1)}_{(x_0,t_0)}}\Omega^{(m-1)}[\varphi,\psi](\eta|\xi):=\\
&\qquad:=\Omega_{(x,t)}[\varphi,\psi](\eta|\xi)-\Omega_{(x_0,t_0)}[\varphi,\psi](\eta|\xi),\\
&\int\limits_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)},\sigma^{(m-1)}_{(x_0,t_0)}\right)} \bar{Z}^{(m),\intercal}[\varphi,\psi](\eta|\xi)=\\
&\qquad=\int\limits_{\sigma^{(m-1)}_{(x,t)}}\bar{\Omega}^{(m-1),\intercal}[\varphi,\psi](\eta|\xi)-\int\limits_{\sigma^{(m-1)}_{(x_0,t_0)}}\bar{\Omega}^{(m-1),\intercal}[\varphi,\psi](\eta|\xi):=\\
&\qquad:=\Omega^{\circledast}_{(x,t)}[\varphi,\psi](\eta|\xi)-\Omega^{\circledast}_{(x_0,t_0)}[\varphi,\psi](\eta|\xi),
\end{aligned}
\tag{9}
$$

where the expressions

$$
\Omega_{(x,t)}[\varphi,\psi](\eta|\xi),\quad \Omega_{(x_0,t_0)}[\varphi,\psi](\eta|\xi),\quad \Omega^{\circledast}_{(x,t)}[\varphi,\psi](\eta|\xi),\quad \Omega^{\circledast}_{(x_0,t_0)}[\varphi,\psi](\eta|\xi)
$$

with $\eta,\ \xi\in\Sigma$ are also considered as the corresponding kernels of invertible integral operators $\Omega_{(x,t)}[\varphi,\psi]$, $\Omega_{(x_0,t_0)}[\varphi,\psi]$, $\Omega^{\circledast}_{(x,t)}[\varphi,\psi]$, $\Omega^{\circledast}_{(x_0,t_0)}[\varphi,\psi]$ in $L_2^{(\rho)}(\Sigma;\mathbb{C})$ of measured functions on Σ with respect to a finite Borel measure ρ on Borel subsets from Σ for any $(x,t)\in\mathbb{R}^m\times l$, considered here as parameters. Moreover, the homotopy conditions in the space $L_2^{(\rho)}(\Sigma;\mathbb{C})$

$$
\begin{aligned}
\lim_{(x,t)\to(x_0,t_0)}\Omega_{(x,t)}[\varphi,\psi]&=\Omega_{(x_0,t_0)}[\varphi,\psi],\\
\lim_{(x,t)\to(x_0,t_0)}\Omega^{\circledast}_{(x,t)}[\varphi,\psi]&=\Omega^{\circledast}_{(x_0,t_0)}[\varphi,\psi]
\end{aligned}
\tag{10}
$$

are assumed to be satisfied for all $(\varphi,\psi)\in\mathcal{H}_0^{\circledast}\times\mathcal{H}_0$.

3. THE MULTIDIMENSIONAL DELSARTE TRANSMUTATION OPERATORS AND THEIR m-DIMENSIONAL TOPOLOGICAL STRUCTURE

For a Delsarte transmutation operators $\mathbf{\Omega}\colon\mathcal{H}\to\mathcal{H}$ and $\mathbf{\Omega}^{\circledast}\colon\mathcal{H}^*\to\mathcal{H}^*$ to be constructed ab initio, it is necessary in accordance with Def. 1 to define the corresponding closed two subspaces $\tilde{\mathcal{H}}_0\subset\mathcal{H}$ and $\tilde{\mathcal{H}}_0^{\circledast}\subset\mathcal{H}^*$.

Let now

$$\begin{aligned}
\tilde{\mathcal{H}}_0 &:= \left\{ \tilde{\psi} \in \mathcal{H}_0 \colon \tilde{\psi} = \psi \Omega^{-1}_{(x,t)}[\varphi,\psi]\Omega_{(x_0,t_0)}[\varphi,\psi], \quad \tilde{\psi}|_{\tilde{\Gamma}} = 0 \right\}, \\
\tilde{\mathcal{H}}_0^{\circledast} &:= \left\{ \tilde{\varphi} \in \mathcal{H}_0^{*} \colon \tilde{\varphi} = \varphi \left(\Omega^{\circledast}_{(x,t)}[\varphi,\psi] \right)^{-1} \Omega^{\circledast}_{(x_0,t_0)}[\varphi,\psi], \quad \tilde{\varphi}|_{\tilde{\Gamma}} = 0 \right\}
\end{aligned} \tag{11}$$

for some hypersurface $\tilde{\Gamma} \subset \mathbb{R}^m$ related in some way with hypersurfaces Γ and Γ^* chosen before, where the operators $\Omega^{-1}_{(x,t)}[\varphi,\psi]$, $(\Omega^{\circledast}_{(x,t)}[\varphi,\psi])^{-1} \colon L_2^{(\rho)}(\Sigma;\mathbb{C}) \longrightarrow$ $\longrightarrow L_2^{(\rho)}(\Sigma;\mathbb{C})$ are correspondingly inverse to the scalar operators $\Omega_{(x,t)}[\varphi,\psi]$, $\Omega^{\circledast}[\varphi,\psi] \colon L_2^{(\rho)}(\Sigma;\mathbb{C}) \longrightarrow L_2^{(\rho)}(\Sigma;\mathbb{C})$, parametrized by variables $(x,t) \in \mathbb{R}^m \times l$. Due to the properties of operators

$$\Omega_{(x,t)}[\varphi,\psi], \quad \Omega_{(x_0,t_0)}[\varphi,\psi], \quad \Omega^{\circledast}_{(x,t)}[\varphi,\psi], \quad \Omega^{\circledast}_{(x_0,t_0)}[\varphi,\psi]$$

in the space $L_2^{(\rho)}(\Sigma;\mathbb{C})$, the spaces (11) are also closed in $\mathcal{H}$ and $\mathcal{H}^*$, correspondingly. Expressions (11) define the following actions

$$\boldsymbol{\Omega} \colon \psi \to \tilde{\psi}, \qquad \boldsymbol{\Omega}^{\circledast} \colon \varphi \to \tilde{\varphi} \tag{12}$$

for any arbitrary but fixed (!) pair of functions $(\varphi,\psi) \in \mathcal{H}_0^{\circledast} \times \mathcal{H}_0$. For retrieving these actions upon the whole space $\mathcal{H}^* \times \mathcal{H}$ at a fixed pair of functions $(\varphi,\psi) \in \mathcal{H}_0^{\circledast} \times \mathcal{H}_0$, let us make use of the well known method of variation of constant:

$$\begin{aligned}
&\boldsymbol{\Omega} \cdot \psi := \tilde{\psi} = \\
&= \psi \Omega^{-1}_{(x,t)}[\varphi,\psi] \left(- \int_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)}, \sigma^{(m-1)}_{(x_0,t_0)}\right)} Z^{(m)}[\varphi,\psi] + \Omega_{(x,t)}[\varphi,\psi] \right) = \\
&= \psi - \psi \Omega^{-1}_{(x,t)}[\varphi,\psi]\Omega_{(x_0,t_0)}[\varphi,\psi]\Omega^{-1}_{(x_0,t_0)}[\varphi,\psi] \int_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)}, \sigma^{(m-1)}_{(x_0,t_0)}\right)} Z^{(m)}[\varphi,\psi] = \\
&= \psi - \tilde{\psi} \Omega^{-1}_{(x_0,t_0)}[\varphi,\psi] \int_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)}, \sigma^{(m-1)}_{(x_0,t_0)}\right)} Z^{(m)}[\varphi,\psi] = \\
&= \left(1 - \tilde{\psi} \Omega^{-1}_{(x_0,t_0)}[\varphi,\psi] \int_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)}, \sigma^{(m-1)}_{(x_0,t_0)}\right)} Z^{(m)}[\varphi,\cdot] \right) \psi;
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{\Omega}^{\circledast} \cdot \varphi := \tilde{\varphi} = \\
&= \varphi \left(\Omega^{\circledast}_{(x,t)}[\varphi,\psi] \right)^{-1} \left(- \int_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)}, \sigma^{(m-1)}_{(x_0,t_0)}\right)} \bar{Z}^{(m),\intercal}[\varphi,\psi] + \Omega^{\circledast}_{(x,t)}[\varphi,\psi] \right) \\
&= \varphi - \varphi \left(\Omega^{\circledast}_{(x,t)}[\varphi,\psi] \right)^{-1} \Omega^{\circledast}_{(x_0,t_0)}[\varphi,\psi] \left(\Omega^{\circledast}_{(x_0,t_0)}[\varphi,\psi] \right)^{-1} \int_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)}, \sigma^{(m-1)}_{(x_0,t_0)}\right)} \bar{Z}^{(m),\intercal}[\varphi,\psi] \\
&= \left(1 - \tilde{\varphi} \left(\Omega^{\circledast}_{(x_0,t_0)}[\varphi,\psi] \right)^{-1} \int_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)}, \sigma^{(m-1)}_{(x_0,t_0)}\right)} \bar{Z}^{(m),\intercal}[\cdot,\psi] \right) \varphi,
\end{aligned} \tag{13}$$

where $(\varphi, \psi) \in \mathcal{H}_0^{\circledast} \times \mathcal{H}_0$ and parameters $(x, t) \in \mathbb{R}^m \times (t_0, T)$ are arbitrary. Thereby, due to 13 one can define invertible extended Delsarte transmutation operators

$$\begin{aligned} \mathbf{\Omega} &:= 1 - \tilde{\psi}\left(\Omega_{(x_0,t_0)}[\varphi,\psi]\right)^{-1} \int\limits_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)},\sigma^{(m-1)}_{(x_0,t_0)}\right)} Z^{(m)}[\varphi,\cdot], \\ \mathbf{\Omega}^{\circledast} &:= 1 - \tilde{\varphi}\left(\Omega^{\circledast}_{(x_0,t_0)}[\varphi,\psi]\right)^{-1} \int\limits_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)},\sigma^{(m-1)}_{(x_0,t_0)}\right)} \bar{Z}^{(m),\intercal}[\cdot,\psi], \end{aligned} \tag{14}$$

acting, correspondingly, in the whole spaces $\mathcal{H}$ and $\mathcal{H}^*$.

Consider now the following commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\frac{\partial}{\partial t} - L} & \mathcal{H} \\ \mathbf{\Omega}\big\downarrow & & \big\downarrow\mathbf{\Omega} \\ \mathcal{H} & \xrightarrow{\frac{\partial}{\partial t} - \tilde{L}} & \mathcal{H}, \end{array}$$

which defines the transformed operator $(\tilde{L} - \frac{\partial}{\partial t})\colon \mathcal{H} \longrightarrow \mathcal{H}$ by means of the Delsarte transmutation expression $\frac{\partial}{\partial t} - \tilde{L} = \mathbf{\Omega}\left(\frac{\partial}{\partial t} - L\right)\mathbf{\Omega}^{-1}$. The pair of functions $\left(\tilde{\varphi}, \tilde{\psi}\right) \in \in \tilde{\mathcal{H}}_0^{\circledast} \times \tilde{\mathcal{H}}_0$ and the operator (15) are described by the following proposition.

Proposition 1. *The pair of transformed functions* $(\tilde{\varphi}, \tilde{\psi}) \in \tilde{\mathcal{H}}_0^{\circledast} \times \tilde{\mathcal{H}}_0$ *solves, correspondingly, the evolution equations*

$$\partial\tilde{\psi}/\partial t = \tilde{L}\tilde{\psi}, \qquad \partial\tilde{\varphi}/\partial t = -\tilde{L}^*\tilde{\varphi} \tag{15}$$

for all $t \in (t_0, T)$.

Proof. It is enough to consider for any $\tilde{\psi} \in \tilde{\mathcal{H}}_0$ the expressions

$$\left(\frac{\partial}{\partial t} - \tilde{L}\right)\tilde{\psi} = \mathbf{\Omega}\left(\frac{\partial}{\partial t} - L\right)\mathbf{\Omega}^{-1}\tilde{\psi} = \mathbf{\Omega}\left(\frac{\partial}{\partial t} - L\right)\psi = 0$$

which holds due to the definition of the closed subspace $\tilde{\mathcal{H}}_0$. The equality $(\partial\tilde{\varphi}/\partial t + +\tilde{L}^*)\tilde{\varphi} = 0$ follows the same as above way. □

It is easy now, due to the symmetry between pairs of functional subspaces $\mathcal{H}_0^{\circledast} \times \mathcal{H}_0$ and $\tilde{\mathcal{H}}_0^{\circledast} \times \tilde{\mathcal{H}}_0$, to construct the inverse operators to (14) and (15):

$$\begin{aligned} \mathbf{\Omega}^{-1} &:= 1 - \psi\Omega^{-1}_{(x_0,t_0)}[\tilde{\varphi},\tilde{\psi}] \int\limits_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)},\sigma^{(m-1)}_{(x_0,t_0)}\right)} \tilde{Z}^{(m)}[\tilde{\varphi},\cdot], \\ \mathbf{\Omega}^{\circledast,-1} &:= 1 - \varphi\left(\tilde{\Omega}^{\circledast}_{(x_0,t_0)}[\tilde{\varphi},\tilde{\psi}]\right)^{-1} \int\limits_{\mathcal{S}\left(\sigma^{(m-1)}_{(x,t)},\sigma^{(m-1)}_{(x_0,t_0)}\right)} \overline{\tilde{Z}}^{(m),\intercal}[\cdot,\tilde{\psi}], \end{aligned} \tag{16}$$

where by definition,

$$\left[\left\langle \tilde{\varphi}, \tilde{L}\tilde{\psi} - \frac{\partial \tilde{\psi}}{\partial t}\right\rangle - \left\langle \tilde{L}^{*}\tilde{\varphi} + \frac{\partial \tilde{\varphi}}{\partial t}, \tilde{\psi}\right\rangle\right] dt \wedge dx = d\tilde{Z}^{(m)}\left[\tilde{\varphi}, \tilde{\psi}\right],$$

$\tilde{Z}^{(m)}[\tilde{\varphi}, \tilde{\psi}] := d\tilde{\Omega}^{(m-1)}[\tilde{\varphi}, \tilde{\psi}] \in \Lambda(\mathbb{R}^{m+1}; \mathbb{C})$, $(\tilde{\varphi}, \tilde{\psi}) \in \tilde{\mathcal{H}}_0^{\circledast} \times \tilde{\mathcal{H}}_0$ and the pair of functions $(\varphi, \psi) \in \mathcal{H}_0^{\circledast} \times \mathcal{H}_0$ satisfies the necessary inverse mappings conditions:

$$\psi = \mathbf{\Omega}^{-1}(\tilde{\psi}), \qquad \varphi = \mathbf{\Omega}^{\circledast,-1}(\tilde{\varphi}), \tag{17}$$

which can be checked easily by simple calculations.

For the construction of the Delsarte transformed operator $\tilde{L} \colon \mathcal{H} \to \mathcal{H}$ to be finished, it is necessary to state that this operator is differential too. The following theorem holds.

Theorem 1. *The Delsarte transformed operator* $\frac{\partial}{\partial t} - \tilde{L} = \mathbf{\Omega}\left(\frac{\partial}{\partial t} - L\right)\mathbf{\Omega}^{-1} \colon \mathcal{H} \to \mathcal{H}$ *is purely differential on the whole space* $\mathcal{H}$ *for any suitably chosen hypersurface* $\mathcal{S}\left(\sigma^{(n-1)}_{(x,t)}, \sigma^{(n-1)}_{(x_0,t_0)}\right) \subset l \times R^n$.

For proving the theorem one needs to show that the formal pseudo-differential expression corresponding to the operator $\tilde{L} \colon \mathcal{H} \to \mathcal{H}$ defined by (15) contains no integral element. Making use of an idea devised in [5, 10], one can formulate such a lemma.

Lemma 1. *A multidimensional pseudo-differential operator* $L \colon L_{2,-}(\mathbb{R}^m; \mathbb{C}^N) \to$ $\to L_{2,-}(\mathbb{R}^m; \mathbb{C}^N)$ *is purely differential iff the following equality*

$$\left(\left\langle h, \left(L\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}\right)_{+} f\right\rangle\right) = \left(\left\langle h, L_{+}\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} f\right\rangle\right) \tag{18}$$

holds for any $|\alpha| \in Z_+$ *and all* $(h, f) \in L_{2,-}(\mathbb{R}^m; \mathbb{C}^N) \times L_{2,-}(\mathbb{R}^m; \mathbb{C}^N)$, *that is the condition* (20) *is equivalent to the equality* $L_+ = L$, *where as usually, the sign "*$(\ldots)_+$*" means the purely differential part of the corresponding expression inside the brackets.*

Proof. (of Theorem 1) Based on Lemma 1 and the exact expression (15) of the reduced on $L_2(\mathbb{R}^m; \mathbb{C}^N)$ the operator $\tilde{L}$, similarly to calculations in [10], one finds right away that the reduced on $L_2\left(\mathbb{R}^m; \mathbb{C}^N\right)$ operator $\tilde{L}$, depending only on a pair of homological cycles $\sigma^{(m-1)}_{(x,t)}$ and $\sigma^{(m-1)}_{(x_0,t_0)}$ marked by points (x,t) and $(x_0,t_0) \in \mathbb{R}^m \times l$, is purely differential in $L_2(\mathbb{R}^m; \mathbb{C}^N)$, thereby proving the theorem. ☐

It is natural to consider now a degenerate case when the operator $L \colon \mathcal{H} \to \mathcal{H}$ doesn't depend on the evolution parameter $t \in l$. Then one can construct closed subspace $\mathcal{H}_0 \subset \mathcal{H}_- := L_{2,-}\left(l; L_2(\mathbb{R}^m; \mathbb{C}^N)\right)$ as follows:

$$\begin{aligned}\mathcal{H}_0 = \big\{&\psi \in \mathcal{H}_- \colon \psi(t; x|\lambda, \xi) = e^{\lambda t}\psi_\lambda(x; \xi),\ \psi_\lambda \in L_{2,0}\big(\mathbb{R}^m; \mathbb{C}^N\big) \colon \psi_\lambda|_\Gamma = 0,\\ &\lambda \in \sigma(L),\ \xi \in \Sigma_\sigma\big\},\end{aligned} \tag{19}$$

where $\sigma(L) \subset \mathbb{C}$ is the generalized spectrum of the extended operator $L\colon L_{2,-}(\mathbb{R}^m;\mathbb{C}^N) \to L_{2,-}(\mathbb{R}^m;\mathbb{C}^N)$ in a suitably Hilbert–Schmidt rigged [13, 14] Hilbert space $L_{2,-}(\mathbb{R}^m;\mathbb{C}^N)$, $L\psi_\lambda = \lambda\psi_\lambda$, $\Sigma_\sigma \subset \Sigma$ is some subset, and $t \in l$ is considered as a parameter. Correspondingly, the conjugated space $\mathcal{H}_0^{\circledast}$ is defined as

$$\mathcal{H}_0^{\circledast} = \big\{\varphi \in \mathcal{H}^*\colon \varphi(t;x|\lambda,\xi) = e^{-\bar{\lambda}t}\varphi_\lambda(x;\xi),\ \varphi_\lambda \in L_{2,0}^{\circledast}\left(\mathbb{R}^m;\mathbb{C}^N\right) : \varphi_\lambda|_\Gamma = 0, \\ \bar{\lambda} \in \sigma(L^*),\ \xi \in \Sigma_\sigma\big\}. \tag{20}$$

Moreover, we can here identify the ρ-measured set Σ with the product $\Sigma = \big(\bar{\sigma}(L^*) \cap \cap \sigma(L)\big) \times \Sigma_\sigma$ and take, correspondingly, $d\rho(\lambda;\xi) = d\rho_\sigma(\lambda) \odot d\rho_{\Sigma_\sigma}$ with $\lambda \in \big(\bar{\sigma}(L^*) \cap \cap\sigma(L)\big)$ and $\xi \in \Sigma_\sigma$. If now to choose a pair of homologically conjugated cycles $\sigma_{(x,t_0)}^{(m-1)}$, $\sigma_{(x_0,t_0)}^{(m-1)}$ lying in the space $\mathbb{R}^m$ for any $t = t_0 \in \mathbb{R}$ being fixed, one easily finds that the corresponding Delsarte transmutation operator $\mathbf{\Omega}\colon \mathcal{H} \to \mathcal{H}$ reduces to the operator $\mathbf{\Omega}\colon L_2(\mathbb{R}^m;\mathbb{C}^N) \to L_2(\mathbb{R}^m;\mathbb{C}^N)$, not depending on the parameter $t \in l$. Thus, we can write down now, that this operator in $L_2\big(\mathbb{R}^m;\mathbb{C}^N\big)$ is given as follows:

$$\begin{aligned}\mathbf{\Omega} = 1 - \int\limits_{\left(\bar{\sigma}(L^*)\cap\sigma(L)\right)} d\rho_{(\sigma)}(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{(\Sigma_\sigma)}(\xi) d\rho_{(\Sigma_\sigma)}(\eta)\tilde{\psi}_\lambda(x;\xi)\times \\ \times \left(\Omega_{(x_0,t_0)}[\varphi_\lambda,\psi_\lambda]\right)^{-1}(\xi,\eta) \int\limits_{\mathcal{S}\left(\sigma_{(x,t_0)}^{(m-1)},\sigma_{(x_0,t_0)}^{(m-1)}\right)} Z^{(m)}[\varphi_\lambda,\cdot](\eta)\end{aligned} \tag{21}$$

and, correspondingly, the operator $\mathbf{\Omega}^{\circledast}\colon L_2^*(\mathbb{R}^m;\mathbb{C}^N) \to L_2^*(\mathbb{R}^m;\mathbb{C}^N)$ is given as

$$\begin{aligned}\mathbf{\Omega}^{\circledast} = 1 - \int\limits_{\left(\bar{\sigma}(L^*)\cap\sigma(L)\right)} d\rho_{(\sigma)}(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{(\Sigma_\sigma)}(\xi) d\rho_{(\Sigma_\sigma)}(\eta)\tilde{\varphi}_\lambda(x;\xi)\times \\ \times \left(\mathbf{\Omega}^*_{(x_0,t_0)}[\varphi_\lambda,\psi_\lambda]\right)^{-1}(\xi,\eta) \int\limits_{\mathcal{S}\left(\sigma_{(x,t_0)}^{(m-1)},\sigma_{(x_0,t_0)}^{(m-1)}\right)} \bar{Z}^{(m),\intercal}[\cdot,\psi_\lambda](\eta)\end{aligned} \tag{22}$$

where $(\varphi_\lambda,\psi_\nu) \in L_{2,0}^{\circledast}(\mathbb{R}^m;\mathbb{C}^N) \times L_{2,0}(\mathbb{R}^m;\mathbb{C}^N)$ are generalized eigenfunctions with the generalized eigenvalues $\lambda,\nu \in \bar{\sigma}(L^*) \cap \sigma(L)$ of the corresponding pair of operators $L^*\colon L_{2,-}^*(\mathbb{R}^m;\mathbb{C}^N) \to L_{2,-}^*(\mathbb{R}^m;\mathbb{C}^N)$ and $L\colon L_{2,-}(\mathbb{R}^m;\mathbb{C}^N) \to L_{2,-}(\mathbb{R}^m;\mathbb{C}^N)$. Since the differential $dt = 0$ in the case (21) and (22), for the differential m-form $Z^{(m)}[\varphi_\lambda,\psi_\nu] \in \Lambda^m(\mathbb{R}^m;\mathbb{C})$ one gets the simple expression

$$Z^{(m)}[\varphi_\lambda,\psi_\nu](\xi,\eta) = -dx\bar{\varphi}_\lambda^\intercal(x;\xi)\psi_\nu(x;\eta) \tag{23}$$

with $\lambda,\,\nu \in \bar{\sigma}(L^*)\cap\sigma(L)$ and $(\xi,\eta) \in \Sigma_\sigma \times \Sigma_\sigma$. Thus the corresponding operator (21) in $L_2(\mathbb{R}^m;\mathbb{C}^N)$ takes the form

$$\mathbf{\Omega} = 1 + \int\limits_{\mathcal{S}\left(\sigma_{(x,t_0)}^{(m-1)},\sigma_{(x_0,t_0)}^{(m-1)}\right)} dyK(x;y)(\cdot), \tag{24}$$

where for a fixed set of functions $(\varphi_\lambda, \psi_\lambda) \in L^{\circledast}_{2,0}(\mathbb{R}^m;\mathbb{C}^N) \times L_{2,0}(\mathbb{R}^m;\mathbb{C}^N)$, $\lambda \in \bar{\sigma}(L^*) \cap \sigma(L)$, the kernel $K(x;y)$, x, $y \in \mathbb{R}^m$, is given as follows:

$$K(x,y) = -\int\limits_{\bar{\sigma}(L^*)\cap\sigma(L)} d\rho_{(\sigma)}(\lambda) \int\limits_{\Sigma_\sigma\times\Sigma_\sigma} d\rho_{(\Sigma_\sigma)}(\xi) d\rho_{(\Sigma_\sigma)}(\eta)\tilde{\psi}_\lambda(x;\xi)\times \\ \times \Omega^{-1}_{(x_0,t_0)}[\varphi_\lambda,\psi_\lambda](\xi,\eta)\bar{\varphi}^{\intercal}_\lambda(y;\eta), \tag{25}$$

being, evidently, of Volterra type and completely similar to that obtained in [14] in the case of selfadjoint operators $L^* = L$ in a Hilbert-Schmidt rigged Hilbert space $L_2(\mathbb{R}^m;\mathbb{C}^N)$. The constant operator $\Omega_{(x_0,t_0)}[\varphi_\lambda,\psi_\lambda]\colon L^\rho_2(\Sigma_\sigma;\mathbb{C}) \to L^\rho_2(\Sigma_\sigma;\mathbb{C})$, is defined naturally by the topological structure of the homological hypercycle $\sigma^{(m-1)}_{(x_0,t_0)} \subset \mathbb{R}^m$, in particular, by asymptotic properties of the generalized eigenfunctions $\varphi_\lambda \in L^{\circledast}_{2,0}(\mathbb{R}^m;\mathbb{C}^N)$ and $\psi_\lambda \in L_{2,0}(\mathbb{R}^m;\mathbb{C}^N)$, $\lambda \in \bar{\sigma}(L^*) \cap \sigma(L)$, as $|x| \to \infty$. Another useful equation on the kerenel (25) based only on its form looks as follows:

$$\bar{\tilde{L}}_{(x)}\,\bar{K}(x,y) = \left(L^*_{(y)}\bar{K}^{\intercal}(x,y)\right)^{\intercal} \tag{26}$$

for all x, $y \in \mathbb{R}^m$. It is completely analogous to the equations which were before derived in the one- and two-dimensional cases in [14] and [3–5].

4. APPLICATIONS TO SPECTRAL AND SOLITON THEORIES: A SHORT SKETCH

Take a differential operator $L\colon L_2(\mathbb{R}^m;\mathbb{C}^N) \to L_2(\mathbb{R}^m;\mathbb{C}^N)$ like (2) and construct its Delsarte transformation $\tilde{L}\colon L_2(\mathbb{R}^m;\mathbb{C}^N) \to L_2(\mathbb{R}^m;\mathbb{C}^N)$ via the expression

$$\tilde{L} = \mathbf{\Omega} L \mathbf{\Omega}^{-1}, \tag{27}$$

being of the same form a differential operator in $L_2(\mathbb{R}^m;\mathbb{C}^N)$. Assuming that the spectral properties of the operator L are known and simplier, one can try to study the corresponding spectral properties of the operator $\tilde{L}$, being more complicated than L. Under such transformations, as is well known, the spectrum of the operator $\tilde{L}$ can change significally, for instance, the discrete spectrum of $\tilde{L}$ can appear, leaving the essential continuous spectrum $\sigma_c(\tilde{L})$ of the transformed operator $\tilde{L}$ unchangeable. An approach realizing in part this idea was before developed in [4, 5] for the case of one and two-dimensional Dirac and Laplace operators.

Subject to soliton theory, it is necessary to take two a priori commuting differential operators $\left(\frac{\partial}{\partial t} - L\right)$ and $\left(\frac{\partial}{\partial y} - M\right)\colon \mathcal{H} \to \mathcal{H}$ with $\mathcal{H} \subset L_2\left(\mathbb{R}^2; L_2(\mathbb{R}^m;\mathbb{C}^N)\right)$, that is

$$\left[\frac{\partial}{\partial t} - L, \frac{\partial}{\partial y} - M\right] = 0. \tag{28}$$

Making use of a fixed Delsarte transmutation constructed for these two operators by means of an invertible operator mapping like (21), (22), one gets two differential operators $\frac{\partial}{\partial t} - \tilde{L}$ and $\frac{\partial}{\partial y} - \tilde{M} \colon \mathcal{H} \to \mathcal{H}$, generated by closed subspaces $\mathcal{H}_0^{\circledast}$ and $\mathcal{H}_0$, where, by definition,

$$\begin{aligned}
\mathcal{H}_0 := \underset{L_{2,-}(\mathbb{R}^m;\mathbb{C}^N)}{\text{closure}} \big\{ \underset{\mathbb{C}}{span} \{ \psi \in \mathcal{H}_- \colon \partial\psi/\partial t = L\psi,\ \psi|_{t=t_0} = \psi_\lambda \in L_{2,0}(\mathbb{R}^m;\mathbb{C}^N), \\
L\psi_\lambda = \lambda\psi_\lambda,\ \psi_\lambda|_\Gamma = 0,\ \lambda \in \bar{\sigma}(L^*) \cap \sigma(L) \} \big\}, \\
\mathcal{H}_0^{\circledast} := \underset{L_{2,-}(\mathbb{R}^m;\mathbb{C}^N)}{\text{closure}} \big\{ \underset{\mathbb{C}}{span} \{ \varphi \in \mathcal{H}_-^* \colon -\partial\varphi/\partial t = L^*\varphi,\ \varphi|_{t=t_0} = \varphi_\lambda \in L_{2,0}^{\circledast}(\mathbb{R}^m;\mathbb{C}^N), \\
L\varphi_\lambda = \bar{\lambda}\varphi_\lambda,\ \varphi_\lambda|_\Gamma = 0,\ \bar{\lambda} \in \bar{\sigma}(L^*) \cap \sigma(L) \} \big\},
\end{aligned}$$

also commuting in $\mathcal{H}$, that is

$$\left[\frac{\partial}{\partial t} - \tilde{L}, \frac{\partial}{\partial y} - \tilde{M} \right] = 0. \tag{29}$$

The latter, so called a Zakharov–Shabat operator equality in $\mathcal{H}$, is as well known [7, 8], equivalent to some system of compatible nonlinear evolution equations upon the coefficients of the operators $\tilde{L}$ and $\tilde{M}$.

Moreover, since flows $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial y}$ in $\mathcal{H}$ are commuting, the corresponding differential m-form $Z^{(m)}[\varphi,\psi]$, given by (6) and defining the Delsarte transmutation operator $\mathbf{\Omega} \colon \mathcal{H} \to \mathcal{H}$, given by (21), has to be naturally changed by a similar extended differential m-form $Z^{(m)}[\varphi,\psi]$, given by the expression

$$\begin{aligned}
Z^{(m)}[\varphi,\psi] = \sum_{i=\overline{1,m}} dt \wedge dx_1 \wedge dx_2 \wedge \ldots dx_{i-1} Z_i^{(L)}[\varphi,\psi] \wedge dx_{i+1} \wedge \ldots \wedge dx_m + \\
+ \sum_{i=\overline{1,m}} dy \wedge dx_1 \wedge dx_2 \wedge \ldots dx_{i-1} Z_i^{(M)}[\varphi,\psi] \wedge dx_{i+1} \wedge \ldots \wedge dx_m + \\
+ \varphi^{\intercal}(x;t,y)\psi(x;t,y)dx
\end{aligned} \tag{30}$$

for any pair $(\varphi,\psi) \in \mathcal{H}_0^{\circledast} \times \mathcal{H}_0$. It is easily seen that the extended differential $(m+1)$-form $dZ^{(m)}[\varphi,\psi] = 0$ upon the space $\mathcal{H}_0^{\circledast} \times \mathcal{H}_0$, that is due to the Stokes theorem [9] there exists a differential $(m-1)$-form $\Omega^{(m-1)}[\varphi,\psi] \in \Lambda^{m-1}(\mathbb{R}^2 \times \mathbb{R}^m;\mathbb{C})$, such that

$$d\Omega^{(m-1)}[\varphi,\psi] = Z^{(m)}[\varphi,\psi] \tag{31}$$

for all $(\varphi,\psi) \in \mathcal{H}_0^{\circledast} \times \mathcal{H}_0$. Making use of this $(m-1)$-form $\Omega^{(m-1)}[\varphi,\psi] \in \Lambda^{m-1}(\mathbb{R}^2 \times \times\mathbb{R}^m;\mathbb{C})$ one can, similarly the way used before, construct the corresponding invertible Delsarte transmutation operators $\mathbf{\Omega} \colon \mathcal{H} \to \mathcal{H}$ and $\mathbf{\Omega}^{\circledast} \colon \mathcal{H}^* \to \mathcal{H}^*$ in the form like (21) and (22), but depending on hyper-surface $\mathcal{S}\left(\sigma_{(x;t,y)}^{(m-1)}, \sigma_{(x_0;t_0,y_0)}^{(m-1)}\right) \subset \mathbb{R}^2 \times \mathbb{R}^m$, spanned between two $(m-1)$-dimensional homologically conjugated cycles $\sigma_{(x;t,y)}^{(m-1)}$, $\sigma_{(x;t_0,y_0)}^{(m-1)} \subset \mathbb{R}^2 \times \mathbb{R}^m$.

This construction finishes our discussion of Delsarte transmutation operators for a commuting pair of operators $\frac{\partial}{\partial t} - L$ and $\frac{\partial}{\partial y} - M$ acting in a parametrically dependent functional space $\mathcal{H}$. In the case of the measure ρ on Σ chosen discrete, the corresponding Delsarte transmutation operators is often called a Darboux–Backlund transformation [8, 15] of a given pair of operators $\frac{\partial}{\partial t} - L$ and $\frac{\partial}{\partial y} - M$, giving rise to the Darboux type formulas like (12) and operator equalities

$$\tilde{L} = L - \left[\mathbf{\Omega}, \frac{\partial}{\partial t} - L\right]\mathbf{\Omega}^{-1}, \ \tilde{M} = M - \left[\mathbf{\Omega}, \frac{\partial}{\partial y} - M\right]\mathbf{\Omega}^{-1}, \tag{32}$$

giving rise to the corresponding Backlund type expressions for the coefficients of the Delsarte transformed operators $\tilde{L}$ and $\tilde{M}$ in $\mathcal{H}$. The latter, as well known, is of great importance for finding new soliton like solutions to the system of evolution equations, equivalent to the operator equality (29). Some applications of this algorithm to finding exact solutions of the Davey–Stuartson and Nizhnik–Novikov–Veselov equations are done, for instance, in [5, 9]. And the last note concerns the applications of the theory devised above to finding the corresponding Delsarte transmutation operators for multidimensional matrix differential operator pencils rationally depending on a “spectral” parameter $\lambda \in \mathbb{C}$: this case can be treated similarly to that considered above making use inside the operators $\partial/\partial t - L$ and $\partial/\partial y - M$, taken in the form

$$\begin{aligned} \partial/\partial t - L &:= \partial/\partial t - \sum_{|\alpha|=0}^{n(L)} a_\alpha(t;x|\lambda)\frac{\partial^{|\alpha|}}{\partial x^\alpha}, \\ \partial/\partial y - M &:= \partial/\partial y - \sum_{|\beta|=0}^{n(M)} b_\beta(t;x|\lambda)\frac{\partial^{|\beta|}}{\partial x^\beta}, \end{aligned} \tag{33}$$

where $a_\alpha,\ b_\beta \in C^1\left(\mathbb{R}^2_{(t,y)}; \mathcal{S}(\mathbb{R}^m; End\mathbb{C}^N)\right) \otimes \mathbb{C}_\lambda$ for all $|\alpha| = \overline{0, n(L)}$, $|\beta| = \overline{0, n(M)}$, $n(L),\ n(M) \in \mathbb{Z}_+$, of the change of the variable $\lambda \in \mathbb{C}$ by the operation of differentiation $\partial/\partial\tau$, $\tau \in \mathbb{R}$, and next applying the developed before approach to constructing the corresponding Delsarte transmutation operators in the functional space $C^1\left(\mathbb{R}_\tau \times \mathbb{R}^2_{(t,y)}; L_2(\mathbb{R}^m; H)\right)$, and at the end returning back to the starting picture putting, correspondingly, the closed subspaces

$$\begin{aligned} \mathcal{H}_0 = \big\{&\psi \in \mathcal{H}\colon \psi(\tau; x; y, t|\lambda; \xi) = e^{\lambda\tau}\psi_\lambda(x; y, t; \xi),\ \psi_\lambda \in L_2\left(\mathbb{R}^2; L_{2,0}(\mathbb{R}^m; \mathbb{C}^N)\right), \\ &\psi_\lambda(x, y; \xi)|_\Gamma = 0,\ \xi \in \Sigma_\sigma,\ \lambda \in \sigma(L)\big\}, \\ \mathcal{H}_0^{\circledast} = \big\{&\varphi \in \mathcal{H}^*\colon \varphi(\tau; x; t|\lambda; \xi) = e^{-\bar{\lambda}t}\varphi_\lambda(x; y; \xi),\ \varphi_\lambda \in L_2\left(\mathbb{R}^2; L^{\circledast}_{2,0}(\mathbb{R}^m; \mathbb{C}^N)\right), \\ &\varphi_\lambda|_\Gamma = 0,\ \xi \in \Sigma_\sigma,\ \bar{\lambda} \in \sigma(L^*), \end{aligned}$$

thereby getting the corresponding two conjucated Delsarte transmutation operators like (14), acting now in the spaces $L_2(\mathbb{R}^m; \mathbb{C}^N)$ and $L_2^*(\mathbb{R}^m; \mathbb{C}^N)$, correspondingly. On these aspects of this technique and on its applications we plan to stop in more detail in another place.

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