

Ewa Frankiewicz

## PRICING OF A DEFAULTABLE COUPON BOND IN AN EXTENDED MERTON'S MODEL

**Abstract.** Three alternative approaches to the valuation of a defaultable coupon bond in an extended Merton's model are given. Probabilistic approach yields a closed-form expression for the arbitrage price of this bond. A boundary value problem method is based on the concept of an CD-extended generator for Markov processes. The third approach relies on a recursive procedure method in which at every step a suitable Cauchy problem is solved.

**Keywords:** arbitrage valuation, Markov processes, contraction semigroup, generators.

**Mathematics Subject Classification 2000:** 60J25, 60J35, 91B70.

### 1. INTRODUCTION

The paper presents three methods of pricing defaultable coupon bond. The considered model is an extension of the classic Merton's approach to corporate debt. Recall that Merton's model (see, e.g., [1, 4, 9]) assumes that a firm's debt is in the form of zero-coupon bonds, and the firm's default may only occur at the debt's maturity. By contrast, in the present paper we consider a coupon bond issued by a firm with the firm's value process given by a geometric Brownian motion. We postulate that the default event may only occur at the coupon payment moments. The last date of the coupon payment is also the bond's maturity, when a notional value is received by a bondholder.

A closed-form expression for the price of a defaultable coupon bond in some model being an extension of the Merton's approach is given, e.g., by Geske in [8]. Geske (see also [5]) considers the option that stockholders have at each coupon payment of buying the next option or not.

In our paper, the derivation of a closed-form expression is based on the risk-neutral valuation formula. For the sake of conciseness, we state here this result without proof; for the detailed calculations the reader is referred to [7]. The default time is expressed in the terms of characteristic function; subsequently, the arbitrage price of the coupon bond is represented in terms of conditional expectations.

In the second part of the paper, we present the PDE approach to valuation of the defaultable coupon bond. In the present setup, it is based on the concept of the CD-extended generator for Markov processes, first introduced in [2] and later analysed in [3] and [7]. This generator appears to be a better tool to analyze the behavior of the semigroup of linear contractions in various time scales (discrete and continuous) than the “classical” generator. In the PDE approach, the default time is conveniently represented as the first exit time from a certain domain. We formulate the boundary value problem, and we prove that the price of the defaultable coupon bond is a solution to this problem.

Finally, we show that the considered valuation problem can be resolved through a recursive procedure, in which at every step we solve a specific Cauchy problem. We use the fact that between the dates of coupon payments, the price of the bond satisfies the usual Black-Scholes partial differential equation with a suitable terminal condition (for the Black-Scholes PDE see, e.g., [10]).

The paper is organized as follows. In Section 2, we describe the framework in which the defaultable coupon bond will be priced.

In the next section, we derive the closed-form formula for the arbitrage price of the bond via the probabilistic approach. The bond’s price is expressed here in terms of the multivariate Gaussian distribution.

Section 4 presents the PDE approach: the boundary value problem and the recursive procedure. In this section we also quote the definition of the CD-extended generator and we discuss some martingale process related to the Markov process in various time scales via the corresponding CD-extended generator.

#### Notation

The following notation is used throughout the text:

- $C_b^k(U)$  – the space of all continuous, bounded functions on  $U$  with continuous and bounded derivatives up to order  $k$ ;
- $C^{k,l}(U)$  – the space of all functions of two variables on  $U$ , with continuous derivatives up to order  $k$  w.r.t. the first variable, and with continuous derivatives up to order  $l$  w.r.t. the second variable,

$$[t]_\rho = \begin{cases} 0, & 0 \leq t < \rho, \\ k\rho, & k\rho \leq t < (k+1)\rho, \text{ for } k = 1, 2, \dots \end{cases}$$

$$\int_{s+}^t = \int_{(s,t]}$$

$$T_t^- f = \lim_{\epsilon \downarrow 0} T_{(t-\epsilon)} f \text{ (if the limit exists);}$$

—  $\frac{\partial_+ f}{\partial x}$  — the right-hand side partial derivative of the function  $f$  with respect to  $x$ .

## 2. DEFAULTABLE COUPON BOND

We will consider a model being an extension of Merton's approach to corporate debt. We work under the assumption that the standard conditions in continuous-time Black-Scholes-type market are satisfied. We begin with the following. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be an underlying probability space, and let  $(V_t)_{0 \leq t \leq T^*}$  be the process on this space given by

$$dV_t = V_t((r - \kappa)dt + \sigma dW_t) \quad (1)$$

where  $r$  is the constant short term interest rate,  $\sigma$  is the constant volatility coefficient,  $\kappa$  is a constant, and  $T^* > 0$  is the finite horizon date. The process  $(W_t)_{0 \leq t \leq T^*}$  is the one-dimensional standard Brownian motion under a martingale measure  $\mathbb{P}^*$  equivalent to  $\mathbb{P}$ . We take  $\mathbb{F}$  to be the filtration generated by the process  $W_t$ , i.e.  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T^*}$ , where  $\mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t)$ .

We assume that the firm, which value is described by the process  $V$ , issues at the moment  $T_0 = 0$  a coupon bond with the maturity date  $T \leq T^*$ . The constant  $\kappa$  in (1) represents, depending on its sign, either the inflow or the outflow of capital to (or from) the firm. Further, we denote by  $T_1, T_2, \dots, T_n = T$  (where  $T_i = iT_1$ ,  $i = 2, \dots, n$ ) the dates of payments of the promised coupons, with equal deterministic value  $c$ . Finally, at the maturity  $T$  the bondholder is promised to be paid the notional value  $L$ .

We postulate that the default event may only occur at the one of the dates  $T_1, T_2, \dots, T$ . Specifically, the firm defaults at the moment  $T_k$ ,  $k = 1, \dots, n - 1$  if  $T_k$  is the first moment when the total value of the firm is less than  $c$ . In addition, the default occurs at maturity, if there was no default at any of the previous coupon dates, and the value of the firm at time  $T$  is less than  $L + c$ . In the case of default at time  $T_k$ , the bondholder receives the amount  $V_{T_k} - (k - 1)c$ .

It is convenient to introduce on the underlying probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  an auxiliary process  $\bar{V}_t$ , which is specified as follows:

$$d\bar{V}_t = dV_t - c d\gamma(t, 0), \quad \forall t \in [0, \infty), \quad (2)$$

where

$$\gamma(t, 0) = \begin{cases} 0, & 0 \leq t < T_1, \\ i, & T_1 + (i - 1)T_1 \leq t < T_1 + iT_1, \text{ for } i = 1, 2, \dots \end{cases} \quad (3)$$

Then the default time  $\tau$  can be represented as follows:

$$\begin{aligned} \tau = & \sum_{k=1}^{n-1} T_k \mathbb{1}_{\{\bar{V}_{T_k} \leq 0, \bar{V}_{T_i} > 0, i=1, \dots, k-1\}} + T \mathbb{1}_{\{\bar{V}_T < L, \bar{V}_{T_i} > 0, i=1, \dots, n-1\}} + \\ & + \infty \mathbb{1}_{\{\bar{V}_T \geq L, \bar{V}_{T_i} > 0, i=1, \dots, n-1\}} \end{aligned}$$

and the coupon bond is formally equivalent to a single cash flow  $D_\tau^c$  at time  $\tau \wedge T$ , where

$$\begin{aligned} D_\tau^c = & \sum_{k=1}^n \left( \sum_{i=1}^{k-1} cB^{-1}(T_i, T_k) + \bar{V}_{T_k} + c \right) \mathbb{1}_{\{\tau=T_k\}} + \\ & + \left( \sum_{k=1}^n cB^{-1}(T_k, T) + L \right) \mathbb{1}_{\{\tau=\infty\}} \end{aligned} \quad (4)$$

and  $B(T_i, T_k) = e^{-r(T_k - T_i)}$  is the price at time  $T_i$  of a risk-free zero coupon bond with maturity  $T_k$ .

### 3. PROBABILISTIC APPROACH

Our first goal is to derive a closed-form expression for the arbitrage price of the defaultable coupon bond. Let us notice that the price at time  $t$  of the unit default-free zero-coupon bond with maturity  $T$  equals  $B(t, T) = e^{-r(T-t)}$ . We begin with the following definition.

**Definition 1.** *The arbitrage price at time  $t$  of the defaultable coupon bond is given, on the set  $\{\tau > t\}$ , as*

$$D(t, T, T_1, \dots, T_{n-1}) = D(t, T) + \sum_{i=1}^{n-1} \mathbb{1}_{\{T_{i-1} \leq t < T_i\}} \sum_{k=i}^{n-1} D_k(t, T_k)$$

where

$$D(t, T) = \mathbb{E}_{\mathbb{P}^*} \left[ B(t, T) \left( (L + c) \mathbb{1}_{\{\tau=\infty\}} + (\bar{V}_T + c) \mathbb{1}_{\{\tau=T\}} \right) \middle| \mathcal{F}_t \right]$$

and for every  $k = 1, 2, \dots, n-1$

$$D_k(t, T_k) = \mathbb{E}_{\mathbb{P}^*} \left[ B(t, T_k) \mathbb{1}_{\{t < T_k\}} \left( c \mathbb{1}_{\{\tau > T_k\}} + (\bar{V}_{T_k} + c) \mathbb{1}_{\{\tau=T_k\}} \right) \middle| \mathcal{F}_t \right].$$

The next result gives the closed-form expression for the arbitrage price of the bond. We will use the following notation:  $k(t) = t + T$  if  $t = [t]_{T_1}$ , and  $[t + T_1]_{T_1}$ , otherwise. In addition,  $K(t) = \frac{k(t)}{T_1}$ . Finally,  $C^i = [c_{j,l}^i]$ ,  $i = 1, \dots, 4$  are the covariance matrices with

$$c_{j,l}^1 = \frac{\min(T_j, T_l) - t}{\sqrt{(T_j - t)(T_l - t)}} \quad j, l = 1, \dots, n,$$

$$c_{j,l}^2 = \begin{cases} \frac{\min(T_j, T_l) - t}{\sqrt{(T_j - t)(T_l - t)}}, & j = 1, \dots, n, l = 1, \dots, n-1, \\ -\frac{\sqrt{T_j - t}}{\sqrt{T - t}}, & j = 1, \dots, n, l = n, \end{cases}$$

$$c_{j,l}^3 = \frac{\min(T_j, T_l) - t}{\sqrt{(T_j - t)(T_l - t)}} \quad j, l = 1, \dots, k,$$

$$c_{j,l}^4 = \begin{cases} \frac{\min(T_j, T_l) - t}{\sqrt{(T_j - t)(T_l - t)}}, & j = 1, \dots, k, l = 1, \dots, k-1, \\ -\frac{\sqrt{T_j - t}}{\sqrt{T - t}}, & j = 1, \dots, k, l = k, \end{cases}$$

for fixed  $n$  and  $k$ . Moreover,

$$d^+ = \frac{\ln\left(\frac{V_t}{L+nc}\right) + (r - \kappa + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

$$d^- = \frac{\ln\left(\frac{V_t}{L+nc}\right) + (r - \kappa - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

$$d_{K(t)}^+ = \frac{\ln\left(\frac{V_t}{cK(t)}\right) + (r - \kappa + \frac{1}{2}\sigma^2)(k(t) - t)}{\sigma\sqrt{k(t) - t}},$$

$$d_{K(t)}^- = \frac{\ln\left(\frac{V_t}{cK(t)}\right) + (r - \kappa - \frac{1}{2}\sigma^2)(k(t) - t)}{\sigma\sqrt{k(t) - t}},$$

$$d_i^+ = \frac{\ln\left(\frac{V_t}{ic}\right) + (r - \kappa + \frac{1}{2}\sigma^2)(T_i - t)}{\sigma\sqrt{T_i - t}},$$

$$d_i^- = \frac{\ln\left(\frac{V_t}{ic}\right) + (r - \kappa - \frac{1}{2}\sigma^2)(T_i - t)}{\sigma\sqrt{T_i - t}}.$$

For the proof of the following proposition the reader is referred to [7].

**Proposition 1.** *The arbitrage price at time  $t$  of the defaultable coupon bond in the extended Merton's model is given, on the set  $\{\tau > t\}$ , by the following expression*

$$\begin{aligned} D(t, T, T_1, \dots, T_{n-1}) &= \\ &= B(t, T) \left[ (L + c) \mathbb{P}^*(\tau = \infty | \mathcal{F}_t) + \mathbb{E}_{\mathbb{P}^*}(\bar{V}_T \mathbf{1}_{\{\tau=T\}} | \mathcal{F}_t) + c \mathbb{P}^*(\tau = T | \mathcal{F}_t) \right] = \\ &= B(t, T) \left[ (L + c) N_{n+1-K(t)} \left( d_{K(t)}^-, \dots, d_{n-2}^-, d_{n-1}^-, d^-, C^1 \right) + \right. \\ &\quad \left. + (\bar{V}_t + cK(t)) \exp((r - \kappa)(T - t)) N_{n+1-K(t)} \left( d_{K(t)}^+, \dots, d_{n-2}^+, d_{n-1}^+, -d^+, C^2 \right) - \right. \\ &\quad \left. - nc N_{n+1-K(t)} \left( d_{K(t)}^-, \dots, d_{n-2}^-, d_{n-1}^-, -d^-, C^2 \right) + \right. \end{aligned}$$

$$\begin{aligned}
& + cN_{n+1-K(t)} \left( d_{K(t)}^-, \dots, d_{n-2}^-, d_{n-1}^-, -d^-, C^2 \right) \Big] + \\
& + \sum_{i=1}^{n-1} \mathbb{1}_{\{T_{i-1} \leq t < T_i\}} \sum_{k=i}^{n-1} B(t, T_k) \mathbb{1}_{\{t < T_k\}} \left[ cN_{k+1-K(t)} \left( d_{K(t)}^-, \dots, d_{k-2}^-, d_{k-1}^-, d_k^-, C^3 \right) + \right. \\
& + (\bar{V}_t + cK(t)) \exp((r - \kappa)(T_k - t)) N_{k+1-K(t)} \left( d_{K(t)}^+, \dots, d_{k-2}^+, d_{k-1}^+, -d_k^+, C^4 \right) - \\
& - ncN_{k+1-K(t)} \left( d_{K(t)}^-, \dots, d_{k-2}^-, d_{k-1}^-, -d_k^-, C^4 \right) + \\
& \left. + cN_{k+1-K(t)} \left( d_{K(t)}^-, \dots, d_{k-2}^-, d_{k-1}^-, -d_k^-, C^4 \right) \right],
\end{aligned}$$

where  $N_n(x_1, \dots, x_n, C)$  is the centered  $n$ -dimensional Gaussian distribution with the covariance matrix  $C$ .

#### 4. PDE APPROACH

In the second part of the paper, we present an alternative approach to the pricing of the defaultable coupon bond. We will show that the price  $D(t, T, T_1, \dots, T_{n-1}) = \bar{u}(\bar{V}_t, t)$ , where the function  $\bar{u}$  solves a suitable boundary value problem. Finally, we will show that the problem of the pricing may be reduced to a recursive procedure, in which at every step we solve an adequate Cauchy problem.

We start with the formulation of the boundary value problem. To this end, it will be convenient to make use of the concept of the CD-extended generator for Markov processes, which was first introduced in [2]. The CD-extended generator appears to be a good tool to analyse the behaviour of the semigroup of linear contractions corresponding to Markov processes in various time scales (discrete and continuous). Before giving the definition of the CD-extended generator, we introduce the following notation.  $\bar{E}$  denotes the state space of the process  $(\bar{V}_t, t)$ ,  $B(\bar{E})$  is the space of all real-valued, bounded and measurable functions on  $\bar{E}$ ,

$$\gamma(r, s) = \begin{cases} 0, & 0 \leq r < \delta(s), \\ n, & \delta(s) + (n-1)\rho \leq r < \delta(s) + n\rho \text{ for } n = 1, 2, \dots, \end{cases} \quad (5)$$

and  $\delta(s) = \rho - s + [s]_\rho$ , where  $\rho > 0$ . The constant  $\rho$  is treated as a moment of the first jump of the process  $(\bar{V}_t, t)$ .

**Definition 2.** Let  $T_t$  be a semigroup of linear contractions corresponding to the process  $(\bar{V}_t, t)$ . Let  $D_{0,1}$  be the set of all elements  $f \in B(\bar{E})$  such that  $D(\tilde{A}) \subseteq D_{0,1}$  (where  $\tilde{A}$  is the weak infinitesimal operator corresponding to semigroup  $T_t$ , with domain  $D(\tilde{A})$ ) and the following equality holds

$$T_t f(\bar{v}, s) = f(\bar{v}, s) + \int_0^t T_r A_0 f(\bar{v}, s) dr + \int_{0+}^t T_{r-} A_1 f(\bar{v}, s) d\gamma(r, s), \quad (6)$$

where  $A_0$  has the same form as the weak infinitesimal operator corresponding to  $T_t$ , and  $A_1$  is a bounded operator on  $B(\bar{E})$  such that  $A_1 T_t f = T_t A_1 f$ . Then the pair

$(A_0, A_1)$  is called the CD-extended generator for  $T_t$  corresponding to the family of times scales  $(t, \gamma)$  with the domain  $D_{0,1}$ .

**Remark 1.** The definition and some properties of the extended generator for a more general case are given in [2, 3] and [7].

It is easily seen that for the process  $(\bar{V}_t, t)$  given by (2),  $\bar{E} = \mathbb{R} \times [0, \infty)$ ,  $\rho = T_1$  and the semigroup corresponding to it can be written in the form (6) with

$$\begin{aligned} A_0 f(\bar{v}, t) &= \frac{\partial_+ f}{\partial t}(\bar{v}, t) + (r - \kappa)\bar{v} \frac{\partial f}{\partial \bar{v}}(\bar{v}, t) + \frac{1}{2} \sigma^2 \bar{v}^2 \frac{\partial^2 f}{\partial \bar{v}^2}(\bar{v}, t), \\ A_1 f(\bar{v}, t) &= f(\bar{v} - c, t) - f(\bar{v}, t), \end{aligned}$$

and  $\delta(s) = T_1 - s + [s]_{T_1}$ , for all  $f$  belonging to the domain  $D_{0,1}$  of the CD-extended generator. In our case if we denote by  $S$  the class of all functions  $f \in B(\mathbb{R} \times [0, \infty))$  which satisfy the following three conditions: (i)  $f$  is of class  $C_b^2(\bar{E})$  w.r.t.  $\bar{v}$ , (ii)  $f$  has the right-hand side derivative w.r.t.  $t$ , (iii)  $f(\bar{v}, t) - f(\bar{v}, t-) \neq 0$  implies that  $t \in I$ , where  $I = \{T_1, T_2, \dots\}$ , then  $S \subseteq D_{0,1}$ .

#### 4.1. AUXILIARY RESULT

The following lemma will be useful in the further part of the paper.

**Lemma 1.** For any  $f \in D_{0,1}$  and  $r \geq 0$ , we define the process  $(\eta_t^{s,f}, t \geq s)$  by setting

$$\begin{aligned} \eta_t^{s,f} &= e^{-r(t-s)} f(\bar{V}_t, t) - f(\bar{V}_s, s) - \int_s^t e^{-r(u-s)} (A_0 f(\bar{V}_u, u) - r f(\bar{V}_u, u)) du \\ &\quad - \int_{s+}^t e^{-r(u-s)} A_1 f(\bar{V}_{u-}, u-) d\gamma(u, 0). \end{aligned}$$

Then the process  $\eta_t^{s,f}$  is an  $\mathbb{F}$ -martingale under  $\mathbb{P}^*$ .

*Proof.* In order to simplify the notation, we shall write  $\eta_t$  instead of  $\eta_t^{r,f}$ . Since  $\eta$  is an adapted and integrable process, it suffices to check that

$$\mathbb{E}_{\mathbb{P}^*} [\eta_{t+h} | \mathcal{F}_t] = \eta_t \tag{7}$$

for all  $h \geq s$  and  $t \geq 0$ . Towards this end, we first observe that

$$\mathbb{E}_{\mathbb{P}^*} \left[ e^{-r(t+h-s)} f(\bar{V}_{t+h}, t+h) | \mathcal{F}_t \right] = e^{-r(t+h-s)} T_t f(\bar{V}_h, h). \tag{8}$$

Moreover,

$$\mathbb{E}_{\mathbb{P}^*} \left[ \int_s^{t+h} e^{-r(u-s)} (A_0 f(\bar{V}_u, u) - r f(\bar{V}_u, u)) du | \mathcal{F}_t \right] =$$

$$\begin{aligned}
&= \int_s^{t+h} e^{-r(u-s)} \mathbb{E} [A_0 f(\bar{V}_u, u) - r f(\bar{V}_u, u) \mid \mathcal{F}_t] du = \\
&= \int_s^h e^{-r(u-s)} (A_0 f(\bar{V}_u, u) - r f(\bar{V}_u, u)) du + \\
&\quad + \int_h^{t+h} e^{-r(u-s)} (T_{u-h} A_0 f(\bar{V}_h, h) - r T_{u-h} f(\bar{V}_h, h)) du = \\
&= \int_s^h e^{-r(u-s)} (A_0 f(\bar{V}_u, u) - r f(\bar{V}_u, u)) du + \\
&\quad + \int_0^t e^{-r(u+h-s)} (T_u A_0 f(\bar{V}_h, h) - r T_u f(\bar{V}_h, h)) du.
\end{aligned}$$

Upon integrating by parts, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^*} \left[ \int_s^{t+h} e^{-r(u-s)} (A_0 f(\bar{V}_u, u) - r f(\bar{V}_u, u)) du \mid \mathcal{F}_t \right] = \\
&= \int_s^h e^{-r(u-s)} (A_0 f(\bar{V}_u, u) - r f(\bar{V}_u, u)) du + \\
&\quad + e^{-r(\delta(h)+h-s)} T_{\delta(h)-} f(\bar{V}_h, h) - e^{-r(h-s)} f(\bar{V}_h, h) + \\
&\quad + e^{-r(\delta(h)+\rho+h-s)} T_{(\delta(h)+\rho)-} f(\bar{V}_h, h) - e^{-r(\delta(h)+h-s)} T_{\delta(h)} f(\bar{V}_h, h) + \\
&\quad + e^{-r(\delta(h)+n\rho+h-s)} T_{(\delta(h)+n\rho)-} f(\bar{V}_h, h) - \\
&\quad - e^{-r(\delta(h)+(n-1)\rho+h-s)} T_{\delta(h)+(n-1)\rho} f(\bar{V}_h, h) + \\
&\quad + \dots + e^{-r(t+h-s)} T_t f(\bar{V}_h, h) - e^{-r(\delta(h)+n\rho+h-s)} T_{\delta(h)+n\rho} f(\bar{V}_h, h).
\end{aligned} \tag{9}$$

We also have that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^*} \left[ \int_{s+}^{t+h} e^{-r(u-s)} A_1 f(\bar{V}_{u-}, u-) d\gamma(u, 0) \mid \mathcal{F}_h \right] = \\
&= \int_{s+}^h e^{-r(u-s)} A_1 f(\bar{V}_{u-}, u-) d\gamma(u, 0) + \int_{h+}^{t+h} e^{-r(u-s)} T_{(u-h)-} A_1 f(\bar{V}_h, h) d\gamma(u, 0).
\end{aligned}$$



In view of Corollary 3.1 in [3],  $\gamma(u+h, 0) = \gamma(u, h)$ , and thus we have

$$\int_{h+}^{t+h} e^{-r(u-s)} T_{(u-h)-} A_1 f(\bar{V}_h, h) d\gamma(u, 0) = \int_{0+}^t e^{-r(u+h-s)} T_{u-} A_1 f(\bar{V}_h, h) d\gamma(u, h).$$

Moreover, by Lemma 2.2 in [3]

$$A_1 T_{(\delta(s)+\rho n)-} f(x, s) = T_{\delta(s)+\rho n} f(x, s) - T_{(\delta(s)+\rho n)-} f(x, s),$$

so that we obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^*} \left[ \int_{s+}^{t+h} e^{-r(u-s)} A_1 f(\bar{V}_{u-}, u-) d\gamma(u, 0) \mid \mathcal{F}_h \right] &= \\ &= \int_{0+}^t e^{-r(u+h-s)} T_{u-} A_1 f(\bar{V}_h, h) d\gamma(u, h) = \\ &= e^{-r(\delta(h)+h-s)} (T_{\delta(h)} f(\bar{V}_h, h) - T_{\delta(h)-} f(\bar{V}_h, h)) + \\ &+ e^{-r(\delta(h)+\rho+h-s)} (T_{\delta(h)+\rho} f(\bar{V}_h, h) - T_{(\delta(h)+\rho)-} f(\bar{V}_h, h)) + \dots + \\ &+ e^{-r(\delta(h)+(n-1)\rho+h-s)} (T_{\delta(h)+(n-1)\rho} f(\bar{V}_h, h) - T_{(\delta(h)(n-1)\rho)-} f(\bar{V}_h, h)) + \\ &+ e^{-r(\delta(h)+n\rho+h-s)} (T_{\delta(h)+n\rho} f(\bar{V}_h, h) - T_{(\delta(h)+n\rho)-} f(\bar{V}_h, h)). \end{aligned} \quad (10)$$

Finally, combining (8), (9), (10), and using Definition 2, we obtain (7).  $\square$

#### 4.2. BOUNDARY VALUE PROBLEM

Let  $D$  be the domain of the form

$$D = \{(\bar{v}, t) : (\bar{v}, t) \in (0, \infty) \times [0, T], i = 1, \dots, n-1\}$$

with  $\partial D$  given by

$$\partial D = \{(0, T_i), i = 1, \dots, n-1; (\bar{v}, T) \in (-c, \infty) \times \{T\}\}.$$

Let us introduce the  $\mathbb{F}$ -stopping time

$$\tau_D = \inf\{t \in I : (\bar{V}_t, t) \in \partial D\},$$

that is, the first exit time from  $D$ . Using  $\tau_D$ , we can represent the bond as a single cash flow:

$$\begin{aligned} D_{\tau_D}^c &= \sum_{k=1}^{n-1} \left[ \sum_{i=1}^{k-1} c e^{r(T_k - T_i)} + \bar{V}_{T_k} + c \right] \mathbf{1}_{\{\tau_D = T_k\}} + \\ &+ \left( \sum_{k=1}^{n-1} c e^{r(T - T_k)} + \min(L + c, \bar{V}_T + c) \right) \mathbf{1}_{\{\tau_D = T\}}. \end{aligned}$$

We are in the position to formulate the boundary value problem associated with the valuation of the defaultable bond:

Find a function  $u(\bar{v}, t): (-c, \infty) \times [0, T] \rightarrow [0, \infty)$ , which belongs to the class  $C_b^2(\bar{E})$  w.r.t.  $\bar{v}$ , is right-differentiable w.r.t.  $t$ , and satisfies the following conditions:

$$\begin{aligned} A_0 u(\bar{v}, t) - ru(\bar{v}, t) &= 0 \quad \text{for } (\bar{v}, t) \in D, \\ A_1 u(\bar{v}, T_k-) + c &= 0 \quad \text{for } (\bar{v}, T_k) \in D, \end{aligned}$$

where  $A_1 u(\bar{v}, T_k-) = u(\bar{v} - c, T_k) - u(\bar{v}, T_k-)$ , as well as the boundary condition

$$u(\bar{v}, t) = g(\bar{v}, t) \quad \text{for } (\bar{v}, t) \in \partial D,$$

where  $g(\bar{v}, T) = \min(L + c, \bar{v} + c)$ ,  $g(\bar{v}, T_i) = \bar{v} + c$ .

Before we formulate the main result of this section, let us introduce an equivalent definition of the price of the defaultable coupon bond, based on the representation of the bond by  $D_{\tau_D}^c$ .

**Definition 3.** *The arbitrage price at time  $t$  of the defaultable coupon bond is given, on the set  $\{\tau_D > t\}$ , as*

$$\bar{u}(\bar{v}, t) = \mathbb{E}_{\bar{v}, t} \left( g(\bar{V}_{\tau_D}, \tau_D) e^{-r(\tau_D - t)} + \sum_{t < T_k \leq \tau_D - T_1} c e^{-r(T_k - t)} \right)$$

where  $\mathbb{E}_{\bar{v}, t}$  denotes the conditional expectation under the condition  $\bar{V}_t = \bar{v}$ , and

$$g(\bar{V}_{\tau_D}, \tau_D) = \min(L + c, \bar{V}_{\tau_D} + c) \mathbf{1}_{\{\tau_D = T\}} + \sum_{k=1}^{n-1} (\bar{V}_{\tau_D} + c) \mathbf{1}_{\{\tau_D = T_k\}}.$$

The following theorem is the main result of this section.

**Theorem 1.** *The following condition holds for a fixed  $t < T$  on the set  $\{\tau_D > t\}$ : Let  $u$  be a solution to the boundary value problem such that  $u$  belongs to  $S$ . Then the process*

$$\eta_{s \wedge \tau_D} = u(\bar{V}_{s \wedge \tau_D}, s \wedge \tau_D) e^{-r(s \wedge \tau_D - t)} - u(\bar{V}_t, t) + \int_{t+}^{s \wedge \tau_D} c e^{-r(h-t)} d\gamma(h, 0),$$

where  $s \in [t, \infty)$ ,  $\gamma$  is given by (5) with  $\rho = T_1$ , is an  $\mathbb{F}$ -martingale under  $\mathbb{P}^*$  with the boundary condition

$$\eta_{\tau_D} = g(\bar{V}_{\tau_D}, \tau_D) e^{-r(\tau_D - t)} - u(\bar{V}_t, t) + \int_{t+}^{\tau_D} c e^{-r(h-t)} d\gamma(h, 0).$$

Moreover  $u = \bar{u}$ .

*Proof.* Observe first that if  $u \in S$  then, as a corollary of Lemma 1, we obtain that the process

$$u(\bar{V}_{s \wedge \tau_D}, s \wedge \tau_D) e^{-r(s \wedge \tau_D - t)} - u(\bar{V}_t, t) - \int_t^{s \wedge \tau_D} (A_0 u(\bar{V}_h, h) - ru(\bar{V}_h, h)) e^{-r(h-t)} dh - \int_{t+}^{s \wedge \tau_D} A_1 u(\bar{V}_{h-}, h-) e^{-r(h-t)} d\gamma(h, 0)$$

is an  $\mathbb{F}$ -martingale for  $s \in [t, \infty)$ . Since  $u$  is assumed to be a solution of the boundary value problem, we obtain that the stopped process  $\eta_{s \wedge \tau_D}$  with above mentioned boundary conditions, is an  $\mathbb{F}$ -martingale under  $\mathbb{P}^*$ .

To prove the second part of the theorem let us suppose that  $u$  is a solution to the boundary value problem belonging to  $S$ . Then, by virtue of the first part of the proof, the process  $\eta_{s \wedge \tau_D}$  is an  $\mathbb{F}$ -martingale. Moreover, since  $\tau_D = \tau_D \wedge T$ , we obtain

$$\mathbb{E}_{\bar{v}, t} \eta_{\tau_D} = \mathbb{E}_{\bar{v}, t} \left[ g(\bar{V}_{\tau_D}, \tau_D) e^{-r(\tau_D - t)} + \sum_{t < T_k \leq \tau_D - T_1} c e^{-r(T_k - t)} \right] - u(\bar{V}_t, t) = \eta_t = 0,$$

which immediately implies that  $u = \bar{u}$ .  $\square$

#### 4.3. RECURSIVE APPROACH

The following proposition shows that the problem of the pricing of the defaultable coupon bond can be resolved through a suitable recursive approach.

**Proposition 2.** *Let  $u_1, u_2, \dots, u_n$  be a sequence of functions, of class  $C^{2,1}((0, \infty) \times [T_{k-1}, T_k])$ ,  $k = 1, \dots, n$ , respectively, which satisfy the conditions:*

(a) *For  $t \in [T_{n-1}, T]$  the function  $u_n(\bar{v}, t)$  is a solution of the Cauchy problem:*

$$\begin{aligned} \frac{\partial u_n}{\partial t}(\bar{v}, t) + (r - \kappa) \bar{v} \frac{\partial u_n}{\partial \bar{v}}(\bar{v}, t) + \frac{1}{2} \sigma^2 \bar{v}^2 \frac{\partial^2 u_n}{\partial \bar{v}^2}(\bar{v}, t) - ru_n(\bar{v}, t) &= 0, \\ u_n(\bar{v}, T) &= \min(L + c, \bar{v} + c). \end{aligned} \quad (11)$$

(b) *For  $t \in [T_i, T_{i+1})$ ,  $i = 0, \dots, n-2$  the functions  $u_{i+1}(\bar{v}, t)$  are solutions of the problems:*

$$\begin{aligned} \frac{\partial u_{i+1}}{\partial t}(\bar{v}, t) + (r - \kappa) \bar{v} \frac{\partial u_{i+1}}{\partial \bar{v}}(\bar{v}, t) + \frac{1}{2} \sigma^2 \bar{v}^2 \frac{\partial^2 u_{i+1}}{\partial \bar{v}^2}(\bar{v}, t) - ru_{i+1}(\bar{v}, t) &= 0, \\ u_{i+1}(\bar{v}, T_{i+1}) &= \begin{cases} u_{i+2}(\bar{v} - c, T_{i+1}) + c & \text{for } \bar{v} > c, \\ \bar{v} & \text{for } \bar{v} \leq c. \end{cases} \end{aligned} \quad (12)$$

*Then the pricing function  $D(t, T, T_1, \dots, T_{n-1})$  is determined by the following recursive procedure.*

(a') If  $t \in [T_{n-1}, T]$  then  $D(t, T, T_1, \dots, T_{n-1}) = u_n(\bar{v}, t)$  on the set  $\{\tau > T_{n-1}\}$ .

(b') If  $t \in [T_i, T_{i+1})$ ,  $i = 0, 1, \dots, n-2$  then  $D(t, T, T_1, \dots, T_{n-1}) = u_{i+1}(\bar{v}, t)$  on the set  $\{\tau > T_i\}$ .

*Proof.* Ad (a') According to Definition 1, for  $t \in [T_{n-1}, T]$  we have

$$D(t, T, T_1, \dots, T_{n-1}) = D(t, T) = B(t, T) \mathbb{E}_{\mathbb{P}^*} (\min(L + c, \bar{V}_T + c) | \mathcal{F}_t).$$

Hence, by virtue of Feynman-Kac theorem (see e.g. [6], Theorem 3.3.4, Remark 3.3.5), we have that  $D(t, T, T_1, \dots, T_{n-1}) = u_n(\bar{v}, t)$ .

Ad (b') To prove (b'), let us first observe that

$$\begin{aligned} D(T_{n-1}^-, T, T_1, \dots, T_{n-1}) &= \\ &= B(T_1, T) \mathbb{E}_{\mathbb{P}^*} ((L + c)\mathbf{1}_{\{\tau=\infty\}} + (\bar{V}_T + c)\mathbf{1}_{\{\tau=T\}} | \bar{V}_{T_{n-1}} = \bar{v} - c) + \\ &+ \mathbb{E}_{\mathbb{P}^*} (c\mathbf{1}_{\{\tau>T_{n-1}\}} + (\bar{V}_{T_{n-1}} + c)\mathbf{1}_{\{\tau=T_{n-1}\}} | \bar{V}_{T_{(n-1)}^-} = \bar{v}), \end{aligned}$$

which implies that

$$D(T_{n-1}^-, T, T_1, \dots, T_{n-1}) = \begin{cases} u_{i+2}(\bar{v} - c, T_{i+1}) + c & \text{for } \bar{v} > c, \\ \bar{v} & \text{for } \bar{v} \leq c. \end{cases}$$

If  $t \in [T_{n-2}, T_{n-1})$ , then, by Feynman-Kac theorem,  $D(t, T, T_1, \dots, T_{n-1}) = u_{n-1}(\bar{v}, t)$ , where  $u_{n-1}$  is a solution to the Cauchy problem (12). Reasoning in a similar way, one can show that the statement is true for the remaining intervals.  $\square$

**Proposition 3.** Let  $u_1, u_2, \dots, u_n$  be a sequence of functions which solve (11) and (12). Moreover, let

$$u(\bar{v}, t) = \begin{cases} u_1(\bar{v}, t) & t \in [0, T_1), \\ u_2(\bar{v}, t) & t \in [T_1, T_2), \\ \dots & \dots \\ u_n(\bar{v}, t) & t \in [T_{n-1}, T]. \end{cases} \quad (13)$$

Then  $u \in S$  and  $u = \bar{u}$ .

*Proof.* We first observe that  $u_1, u_2, \dots, u_n$  are the functions of class  $C^{2,1}((0, \infty) \times [T_{k-1}, T_k))$ . Moreover, since  $u$  is given by (13), we have  $u \in S$ . The equality  $u = \bar{u}$  follows from the proof of Proposition 2.  $\square$

## Acknowledgements

Research supported by State Committee for Scientific Research (Komitet Badań Naukowych) Grant PBZ-KBN-016—P03—1999.

## REFERENCES

- [1] Ammann M.: *Credit Risk Valuation. Methods, Models, and Applications*. Springer-Verlag, 2001.
- [2] Bielecki T.: *On Extended Generator for Semigroup of Linear Contractions, Markov Processes and Related Martingales*. Unpublished manuscript, 1995.
- [3] Bielecki T., Frankiewicz E.: *On Extended Generators for Markov Processes and Controlled Martingale Problems*. Submitted to SIAM J. Control Optim., 2003.
- [4] Bielecki T., Rutkowski M.: *Credit Risk: Modeling, Valuation and Hedging*. Springer-Verlag, 2002.
- [5] Cossin D., Pirotte H.: *Advanced Credit Risk Analysis. Financial Approaches and Mathematical Models to Assess, Price and Manage Credit Risk*. John Wiley & Sons, LTD., 2001.
- [6] Dane R.A., Jeanblanc M.: *Financial Markets in Continuous Time*. Springer-Verlag, 2003.
- [7] Frankiewicz E.: *Extended Generators of Markov Processes and Their Applications in the Mathematics of Finance*. Doctoral dissertation, Warsaw University of Technology. In preparation, 2003.
- [8] Geske R.: *The Valuation of Corporate Liabilities as Compound Options*. J. Finan. Quant. Anal. **12** (1977), 541–552.
- [9] Merton R.C.: *On the Pricing of Corporate Debt: The Risk Structure of Interest Rates*. J. Finan. Econom. **3** (1974), 125–144.
- [10] Musiela M., Rutkowski M.: *Martingale Methods in Financial Modelling*. Springer-Verlag, 1997.

Ewa Frankiewicz  
efrankiewicz@prioris.mini.pw.edu.pl

Warsaw University of Technology  
Faculty of Mathematics and Information Science  
Pl. Politechniki 1, 00-661 Warsaw, Poland

*Received: May 11, 2004.*